Most illegal immigrants come to Ukraine from the Russian Federation and Belarus, so it would be reasonable to arrange their first eastern and northern borders.

Purpose. *With this in mind, this study aims to analyze the current situation of economic security of Ukraine in the light of the features of its participation in international migration as a transit country and country of illegal accumulation.*

Methods. *Methods used in the article: theoretical analysis and synthesis of the test material, social and qualitative research methods, analytical - statistical method.*

Results. *Illegal transit migrants at border crossings using known forms and methods of infiltration desired country, through the green zone border; through legal customs checkpoints using forged or foreign documents; using large vehicles for industrial use.*

Originality. *A number of reasonable and effective ways to counter the threat to economic security from Ukraine and transit of illegal migration and the negative effects that caused this process.*

Conclusion. *Revealed that the increase of illegal migrants in Ukraine contributing factors such as "softness" Ukraine visa policy; lack of control over the activities of businesses and individuals that invite and take in Ukraine foreigners; employability and uncontrolled movement of illegal immigrants; the lack of an effective mechanism expulsion and deportation of foreigners from Ukraine.*

Keywords: *economic security of Ukraine, a transit country, transit migration, illegal migration, criminalization of society.*

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KALASHNIKOV Vyacheslav V.,
Tecnológico de Monterrey (ITESM), Monterrey, Mexico Central Economics and Mathematics Institute (CEMI), Moscow, Russia
Sumy State University, Sumy, Ukraine
KALASHNYKOVA Nataliya I.,
Sumy State University, Sumy, Ukraine
Universidad Autónoma de Nuevo León, San Nicolás de los Garza, Mexico
LEAL-CORONADO Mariel A.,
Tecnológico de Monterrey (ITESM), Monterrey, Mexico

SOLUTION OF THE PORTFOLIO OPTIMIZATION MODEL AS A BILEVEL PROGRAMMING PROBLEM

In this paper, we consider a mixed-integer bi-level linear programming (or a leader's) problem with one parameter in the right-hand side of the constraints in the lower level (or a follower's) problem. Motivated by the application to a fuzzy portfolio optimization model, we consider a particular case that consists in maximizing the investor's expected return. The functions are linear at the upper level and quadratic at the lower level, and the proposed algorithm is based upon an approximation of the optimal value function using the branch-and-bound method. Therefore, at every node of this branch-and-bound structure, we apply a new branch-and-bound technique to process the integrity condition.

Keywords: *fuzzy portfolio optimization, integer programming, parametric programming, branch-and-bound approach.*

Introduction (Formulation of the problem). The portfolio theory was developed to support decision making for allocation of financial assets (securities, bounds) traded at the stock exchange [1]. This allocation is known as “investment" decision making. The investor considers the asset as a matter of future income. The better combination of financial assets (securities) of the portfolio leads to better return for the investor. The portfolio contains a set of securities, and the problem of portfolio optimization targets the optimal resource allocation in investment process of trading financial assets [2]. The resource allocation means investing capital in financial assets (or goods), which gives return to the investor after certain period of time. For the investment process, the aim is to maximize the return while the investment risk has to be minimal [3].

Harry Markowitz suggested a powerful approach for quantifying the risk in 1952. The analytical relations among the portfolio risk V_p, portfolio return E_p, and the values of the investment per type of assets x_i, according to the portfolio theory [1], are:

$$E_p := \sum_{i=1}^{n} E_i x_i = E^T x; \tag{1}$$

$$V_p := \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} x_i Q_{ij} x_j = \frac{1}{2} x^T Q x, \tag{2}$$

where E_i is the average value of the return for asset i; $E^T = (E_1, \ldots, E_n) \in R^n$ is a vector of dimension $1 \times n$, and $Q \in R^{n \times n}, Q \geq 0$, is a symmetric positive semi-definite assets variance-covariance matrix. This scheme reflects the elements of fuzziness in the proposed construction. The portfolio theory introduces the so called "standard" optimization problem as follows:

$$z := \frac{1}{2} x^T Q x - \sigma E^T x \to \min_x , \tag{3}$$

subject to

$$\sum_{i=1}^{n} x_i = 1, \tag{4}$$

and

$$x_i \geq 0, \quad i = 1, \ldots, n, \tag{5}$$

where $\sigma \geq 0$ is a parameter of the investor's risk aversion (his/her tolerance to accepting risky investments).

The numerical assessment of σ is a task of the financial analyzer, and it has some subjective meaning. This coefficient strongly influences the definition and hence the solutions of the portfolio problem. As a result, the selected value of σ changes the final investment decision as well.

Because of that, the most appropriate for the investor is the following *fuzzy bilevel programming problem* (FBLP):

$$w := E(\sigma) = E^T y \to \max_{\sigma \geq 0} , \tag{6}$$

subject to the following constraint:

$$y \in \Psi(\sigma) := \{x \in R^n : x \quad solves problem\, (3)-(3)(5)\} \tag{7}$$

This paper proposes an efficient numerical algorithm to solve the fuzzy bilevel programming problem (6)–(7). The method is based upon the lower level objective function optimal value techniques (*cf.*, [4]).

The paper is organized as follows. Preliminaries, a general formulation of the problem and the mathematical model are given in Section 2 and Section 3, respectively. The geometry of the problem is described in Section 4, whereas the approximation algorithm is presented in Section 5. Section 6 illustrates the algorithm by a numerical example, and conclusions are listed in Section 7, with Acknowledgments and References completing the paper.

Preliminaries (Analysis of recent research and publications). Hierarchical decision making is strongly motivated by real-world applications. For example, in engineering design, the main objective of the design engineer may be constrained by the properties inherent in the process (such as minimum energy), which, in turn, may be parametric in decision variables, chosen by the engineer. These problems can be formulated within a bilevel programming problem (BLP) framework, where an upper level (or, outer) optimization problem is constrained by another, lower level (or, inner) optimization problem.

Hierarchical problems also arise in the (non-simultaneous) Stackelberg games [5], in which various decision makers try to maximize their utility functions with delay. Because of that, they are often not able to realize their decisions independently and at the same time, but are forced to act according to certain hierarchy. We will consider the simplest case of such a situation, where there are only two acting decision makers. The leader is the one that can handle the market independently, whereas the follower has to act in a dependent manner.

In mathematical terms, it means that the set of variables is partitioned into two vector variables, x and y, where $y \in R^m$ are the leader's variables and $x \in R^n$ are those governed by the follower. Using y as a parameter, the follower solves a parametric optimization problem, and the values $x = x(y)$ are determined by the follower knowing the selection y of the leader. The leader has to determine the best choice of y knowing the (optimal) reaction $x = x(y)$ of the follower to the leader's decision.

However, important decision making problems may involve decisions both in discrete and continuous variables. For example, a chemical engineering design problem may involve discrete decisions regarding the existence of chemical process units *in addition to* decisions in continuous variables, such as temperatures or pressures. Problems of this class, dealing with both discrete and continuous decision variables, are referred to as mixed-integer BLPs.

A particular case of the mixed-integer bi-level programming problem is presented by the real-world problem of minimizing the cash-out penalty costs of a natural gas shipping company [6]. This problem arises when a (gas) shipper draws a contract with a pipeline company to deliver a certain amount of gas at several delivering meters. What is actually shipped may be higher or lower than the amount that had been originally agreed upon (this phenomenon is called an *imbalance*). When such an imbalance occurs, the pipeline penalizes the shipper by imposing a cash-out penalty policy. As this penalty is a function of the operating daily imbalances, an important problem for the shippers is how to carry out their daily imbalances so as to minimize the incurred penalty. On the other hand, the pipeline (the follower) tries to minimize the absolute values of the cash-outs, which produces the optimal response function taken into account by the leader in order to find the optimal imbalance operating strategy. Integer variables are involved at the lower level problem, and various algorithms to solve the natural gas cash-out problem are described in [6]–[10].

In general, mixed-integer BLPs can be classified into four classes [11]:

(I) **Integer Upper, Continuous Lower**: If the sets of inner (lower level) integer and outer (upper level) continuous variables are empty, and on the contrary, the sets of outer integer and inner continuous variables is nonempty, then the MIBLP is of Type I.

(II) **Purely Integer**: If the sets of inner and outer integer variables are nonempty, and the sets of inner and outer continuous variables are empty, then the problem is a purely integer BLP.

(III) **Continuous Upper, Integer Lower**: When the sets of inner continuous and outer integer variables are empty, and vice versa, the sets of inner integer and outer continuous variables are nonempty, then the problem is a MIBLP of Type III.

(IV) **Mixed-Integer Upper and Lower**: If the sets of both inner and outer continuous and integer variables are nonempty, then the problem is a MIBLP of Type IV.

Advances in the solution of the mixed-integer bilevel programming problems (MIBLP) of all four types can greatly expand the scope of decision making instances that can be modeled and solved within a bilevel optimization framework. However, very little attention has been paid in the literature to both the solution and the application of BLP governing discrete variables. This is mainly because these problems pose major algorithmic challenges in the development of efficient solution strategies.

In the literature, methods developed for the solution of the MIBLP have so far addressed a very restricted class of problems. More attention has been paid to *linear* problems. For instance, for the solution of the purely integer (Type II) linear BLP, a branch-and-bound type of enumerating technique has been proposed by Moore and Bard [12], whereas Nishizaki et al. [13] applied a kind of genetic algorithm to the same problem. For the solution of the mixed-integer BLP of Type I, another branch-and-bounds approach has been developed by Wen and Yang [14]. Cutting plane and parametric solution techniques have been elaborated by Dempe [15] to solve MIBLP, in which the lower level has only one upper level (outer) variable involved into the (lower level) objective function. Dempe [16] also proposed an algorithm to solve the Linear Bilevel Programming Problem (BLPP) using the simplex method with additional variables in the basis set, using the theory of subgradients. Bard [17] obtained upper bounds for the objective functions at both levels. Thus he generated a non-decreasing sequence of lower bounds for the objective function at the upper level, which, under certain conditions, converges to the solution of the general BLPP for continuously differentiable functions. Methods based upon decomposition technique have been proposed by Saharidis and Ierapetritou [18] and Zhang and Wu [19].

Mixed-integer *nonlinear* bilevel programming problems have received even less attention in the literature. The developed methods include an algorithm making use of parametric analysis to solve separable monotone nonlinear MIBLP proposed by Jan and Chern [20], a stochastic simulated annealing method presented by Sahin and Ciric [21], a global optimization approach based on parametric programming technique published by Fasca et al. [22]. Floudas et al. in [11] and [23] developed several algorithms dealing with global optimization of mixed-integer bilevel programming problems of both deterministic and stochastic nature. The sensitivity analysis for MIBLPP also was considered in [24].

In [25]–[26], we already started considering and solving mixed-integer linear BLP of Type I. Somethimes, a BLP can be reduced to solving a multi-objective optimization problem, which is efficiently processed by Liu and Wang in [27]. Bi-level programming problems with discrete variables are also examined in Hu et al. [28].

The purpose and objectives of the study. The main goal of this paper is to propose an efficient algorithm to solve the mixed-integer linear BLP of Type I, because in our bilevel portfolio optimization problem (6)–(7), the upper level variable σ can clearly treated as integer one. Knowing that this problem is hard to solve, we propose an algorithm generating approximations that converge to a global solution. The main novelty of the presented heuristic approach lies in the combination of branch-and-bound (B&B) technique with with the simplicial subdivision algorithms. The numerical experiments demonstrate the robust performance of the developed method for instances of the small and medium size.

General Formulation (Presentation of the main research material). We consider the hierarchical optimization problem in two levels: the decision making at the upper level is governed by the constraints that are defined in part by a lower level (parametric) optimization problem. Let the lower level problem be defined as follows:

$$\min_{x}\{f(x,y)\mid g(x,y)\le 0, h(x,y)=0\}, \tag{8}$$

where $f: R^n\times R^m \to R, g: R^n\times R^m\to R^p$ and $h: R^n\times R^m\to R^q$ with $g(x,y)=-(g_1(x,y),\ldots,g_p(x,y))^T$ and $h(x,y)=(h_1(x,y),\ldots,h_q(x,y))^T$. This problem is called the lower level, or the follower's problem. Let $\psi(y)$ denote the solution set of problem (8) for a fixed parameter $y\in R^m$.

Now we can formulate the bilevel problem as follows:

$$\min_{x,y}\{F(x,y)\mid y\in Y, x\in\psi(y)\}, \tag{9}$$

where $F : R^n \times R^m \to R$, and Y is a closed subset of R^m . This is the upper level or the leader's problem. Problem (8)–(9) is referred to as a *bilevel programming problem*.

In order to guarantee that the bilevel programming problem is well-defined, we assume the following:

1. The set $M = \{(x, y) \mid g(x, y) \le 0, h(x, y) = 0\}$ is nonempty.
2. Both $F(x, y)$ and $f(x, y)$ are bounded from below on M.

Definition 2.1 *A pair* (x, y) *is said to be feasible to the linear bi-level programming problem if it satisfies* $y \in Y$ *and* $x \in \psi(y)$.

Definition 2.2 *A feasible pair* (x', y') *is called an optimal solution to the bi-level programming problem if* $F(x', y') \le F(x, y)$ *for all the feasible solutions* (x, y) .

3 Mathematical Model

The Mixed Integer Bi-Level Linear Programming Problem with a parameter in the objective function of the lower level is formulated as follows:

$$\min_{x,y} \left\{ \langle a, x \rangle + \langle b, y \rangle \mid Gy = d, x \in \psi(y), y \in Z_+^m \right\}, \tag{10}$$

which represents the upper level, where $a, x \in R^n$, $b, y \in R^m, G$ is an $r \times m$ matrix, $d \in R^r$. Note that we use the optimistic version of the bilevel programming problem here, *see* [15]. From now on, $\langle \cdot, \cdot \rangle$ is the inner product. In general, $\psi(y)$ is defined as follows:

$$\psi(y) = \mathrm{Arg} \min_x \left\{ \langle c, x \rangle \mid Ax = y, x \ge 0 \right\}, \tag{11}$$

which describes the set of optimal solution of the lower level problem (the set of rational reactions). Here $c, x \in R^n$, A is an $m \times n$ matrix with $m \le n$.

For our particular problem, for a fixed current approximate solution $x^k \in R_+^n$ of the lower level problem, the set $\psi(\sigma)$ is defined as follows:

$$\psi(\sigma) = \mathrm{Arg} \min_x \left\{ \langle c, x \rangle + \tau \mid Ax = 1, x \ge 0 \right\}, \tag{12}$$

where

$$c^T := \left(x^k\right)^T Q - \sigma E^T, \quad \text{and}\ \tau := 12 \left(x^k\right)^T Q x^k - \sigma E^T x^k, \tag{13}$$

which describes the set of optimal solution of the approximate (linearized) lower level problem (the set of rational reactions). Here $x \in R^n$, Q is an $n \times n$ matrix, A is the $1 \times n$ matrix with $a_{1i} = 1$, $i = 1, \ldots, n$.

Let us determine the optimal value function of the lower level problem as follows:

$$\varphi(y) = \min_x \left\{ \langle c, x \rangle \mid Ax = y, x \ge 0 \right\}. \tag{14}$$

We suppose that the feasible set of problem (11) is non-empty. Again, in the example of the portfolio optimization model, σ is the parameter that can represent the values of different degree of risk aversion on part of the investor. The lower level, depending on our objectives, may try to minimize the risk, the uncertainty of possible returns, etc.

In this paper, we consider a reformulation of (10)–(14) based upon an approach reported in the literature (*see* [29], or [15]) as a classical nondifferentiable optimization problem. If we take into account the lower level optimal value function (14), then problem (10)–(14) can be replaced by:

$$\min_{x,y} \left\{ \langle a, x \rangle + \langle b, y \rangle \mid Gy = d, \langle c, x \rangle \le \varphi(y), Ax = y, x \ge 0, y \in Z_+^m \right\}. \tag{15}$$

Our work is concentrated on the lower level objective value function (14). For this reason, we show some important characteristics (*see* [30] or [31]) that will be helpful for solving problem (15).

4 Problem's Geometry

Consider the parametric linear programming problem (14)

$$\varphi(y) = \min_x \{\langle c, x\rangle \mid Ax = y, x \geq 0\}.$$

In order to solve this problem, we use the dual simplex algorithm, like in [31]. Let us fix $y = y^*$ and let x^* be an optimal basic solution for $y = y^*$ with the corresponding basic matrix B, which is a quadratic submatrix of A having the same rank as A, and such that $x^* = (x_B^*, x_N^*)^T$, with $x_B^* = B^{-1}y$ and $x_N^* = 0$. Moreover, let us fix the upper level variable value $y = y^*$. Then, we can say that $x^*(y^*) = (x_B^*(y^*), x_N^*(y^*))^{\mathrm{T}} = (B^{-1}y^*, 0)^{\mathrm{T}}$ is an optimal basic solution of problem (14) for a fixed parameter y^*. And if the following inequality holds:

$$B^{-1}y \geq 0,$$

then $x^*(y) = (x_B^*(y), x_N^*(y))^{\mathrm{T}} = (B^{-1}y, 0)^{\mathrm{T}}$ is also optimal for the parameter vector y.

It is possible to perturb y^* so that B remains a basic optimal matrix [30]. We denote by $\Re(B)$ a set that we call the *stability region* of B, which is defined as

$$\Re(B) = \{y \mid B^{-1}y \geq 0\}.$$

For all $y \in \Re(B)$, the point $x^*(y) = (x_B^*(y), x_N^*(y))^{\mathrm{T}} = (B^{-1}y, 0)^{\mathrm{T}}$ is an optimal basic solution of the problem (14). This region is nonempty because $y^* \in \Re(B)$. Furthermore, it is closed but not necessarily bounded. If $\Re(B)$ and $\Re(B')$ are two different stability regions with $B \neq B'$, then only one of the following cases is possible.

1. $\Re(B) \cap \Re(B') = \{0\}$.
2. $\Re(B) \cap \Re(B')$ contains the common border of the regions $\Re(B)$ and $\Re(B')$.
3. $\Re(B) = \Re(B')$.

Moreover, $\Re(B)$ is a convex polyhedral set, on which the lower level optimal value function is a finite and linear function. To determine an explicit description of the function φ consider the dual problem to problem(14). If $\varphi(y)$ is finite, then

$$\varphi(y) = \max\{\langle y, u\rangle : A^{\mathrm{T}}u \leq c\}.$$

Let $u^1, u^2, \ldots, u^s$ denote the vertices of the polyhedral set $\{u : A^{\mathrm{T}}u \leq c\}$. Then,

$$\varphi(y) = \max\{\langle y, u^1\rangle, \langle y, u^2\rangle, \ldots, \langle y, u^s\rangle\},$$

whenever $\varphi(y)$ is finite.

By duality, for some basic matrix B_i with $y \in \Re(B_i)$ we have $B_i^{\mathrm{T}}u = c_{B_i}$ or $u = \left(B_i^{\mathrm{T}}\right)^{-1} c_{B_i}$ and, thus,

$$\langle y, u^i\rangle = \langle y, \left(B_i^{\mathrm{T}}\right)^{-1} c_{B_i}\rangle = \langle (B_i)^{-1} y, c_{B_i}\rangle.$$

Setting $x^i(y) = ((B_i)^{-1} y, 0)$ we derive

$$\varphi(y) = \max\{\langle c, x^1(y)\rangle, \langle c, x^2(y)\rangle, \ldots, \langle c, x^q(y)\rangle\}.$$

It is easy to understand that the stability regions are represented by the segments on the y-axis. The function φ is nonsmooth, which makes this kind of problems hard to solve.

Now, we introduce the following definition [29].

Definition 4.1 *Let* (x^*, y^*) *solve problem (15). Then (15) is called partially calm at* (x^*, y^*) *if there exist a constant* $\mu > 0$ *and a neighborhood* U *of* $(x^*, y^*, 0) \in R^n \times R^m \times R$, *such that for all* $(x, y, u) \in U$ *feasible to the problem:*

$$\min_{x,y,u} \left\{ \langle a, x \rangle + \langle b, y \rangle \mid Gy = d,\ Ax = y,\ x \ge 0,\ y \in Z_+^m \right\}, \tag{16}$$

we have

$$\langle a, x \rangle + \langle b, y \rangle - \langle a, x^* \rangle - \langle b, y^* \rangle + \mu \mid u \mid \ge 0.$$

Here $|\ u\ |$ represents the absolute value of u.

Theorem 4.1 *Let* (x^*, y^*) *solve problem (10)–(14), then (15) is partially calm at* (x^*, y^*).

5 An Approximation Algorithm

The basis to start describing the algorithm is given above in this paper. The difficulty in the work with the objective value function (14) is due to the simple fact that we do not have it in an explicit form. This algorithm tries to approximate function (14) with a finite number of iterations. Also (14) is not differentiable: *cf.* [32], [29] working with subdifferential calculus based upon the non-smooth Mangasarian-Fromowitz constraint qualification.

The tools that we use in this paper are mainly based on the fact that (14) is piecewise-linear and convex. Also, the basis for developing a good algorithm is given in the next theorems, important for keeping on the convexity at every level of approximation.

Definition 5.1 *The intersection of all the convex sets containing a given subset* W *of* R^m *is called the convex hull of* W *and is denoted by conv* W.

Theorem 5.1 (Carathadory's Theorem) *Let* W *be any set of points in* R^m, *and let* $C = \text{conv } W$. *Then* $y \in C$ *if and only if* y *can be expressed as a convex combination of* $m+1$ *(not necessarily distinct) points in* W. *In fact,* C *is the union of all the generalized* d*-dimensional simplices whose vertices belong to* W, *where* $d = \dim C$.

Corollary 5.1 *Let* $\{C_i \mid i \in I\}$ *be an arbitrary collection of convex sets in* R^m, *and let* C *be the convex hull of the union of the collection. Then every point of* C *can be expressed as a convex combination of* $m+1$ *or fewer affinely independent points, each belonging to a different* C_i.

The details and proofs of Theorems 5.1 and Corollary 5.1 can be found in [33].

Now, we describe the proposed algorithm as follows:

Step 0. *Initialization.* Let the problems' list initially include only the Approximate Integer Problem (AIP) build as follows:

We consider problem (15):

$$\min_{x,y} \left\{ \langle a, x \rangle + \langle b, y \rangle : Gy = d, \langle c, x \rangle \le \varphi(y),\ Ax = y, x \ge 0,\ y \in Z_+^m \right\}.$$

Now, let us consider the polytop Y composed as a convex hull of the leader's strategies at the upper level: $Y = \{y \mid Gy = d, y \ge 0\}$, and select $\dot{m}+1$ affine independent points y^j such that $Y \subset conv\{y^1, \ldots, y^{\dot{m}+1}\} \subset \{y : |\varphi(y)| < \infty\}$.

Here $\dot{m} = m - rank(G)$, and $y^2 - y^1, y^3 - y^1, \ldots, y^{\dot{m}+1} - y^1$ form a linearly independent system. We denote this set of vertices as $V = \{y^1, \ldots, y^{\dot{m}+1}\}$. Also we consider a tolerance value $\varepsilon > 0$. Then, we solve the lower level linear programming problem (14) at each vertex, i.e., find $\varphi(y^1), \ldots, \varphi(y^{\dot{m}+1})$ and the corresponding solution vectors $(x^1, y^1), \ldots, \left(x^{\dot{m}+1}, y^{\dot{m}+1}\right)$.

Now we build the first approximation of the optimal value function as follows:

$$\Phi(y) = \sum_{i=1}^{\dot{m}+1} \lambda_i \varphi(y^i), \tag{17}$$

defined over

$$y = \sum_{i=1}^{\dot{m}+1} \lambda_i y^i, \tag{18}$$

with $\lambda_i \geq 0$, $i = 1, \ldots, \dot{m}+1$, and

$$\sum_{i=1}^{\dot{m}+1} \lambda_i = 1. \tag{19}$$

In (17) we have an expression with the variable λ, that leads to variable y using (18) and (19). Now since the function φ is convex,

$$\langle c, x \rangle \leq \varphi(y) \leq \Phi(y),$$

our condition $\langle c, x \rangle \leq \varphi(y)$ in (15) can be relaxed to the following explicit inequality:

$$\langle c, x \rangle \leq \Phi(y).$$

Thus we obtain a new optimization problem that can be solved, for example, with a branch-and-bound method. The Approximate Integer Problem (AIP) is described as follows:

$$\min_{x,y} \left\{ \langle a, x \rangle + \langle b, y \rangle : Gy = d, \langle c, x \rangle \leq \Phi(y), Ax = y, x \geq 0, y \in Z_+^m \right\}. \tag{20}$$

Now let $t = 1$, and $z_t = +\infty$, where z_t is the incumbent objective value. Put this problem into the problems list. By definition, this problem corresponds to the convex polyhedron Y. Go to Step 1.

Step 1. *Termination criterion.* Stop if the problems list is empty, or if all the current solutions of problem (5) are close enough:

$$\max_{1 \leq i \neq k \leq \dot{m}+1} \left\| (x^i, y^i) - (x^k, y^k) \right\| < \varepsilon.$$

In these cases, select the point (x^r, y^r), where $\varphi(y^r) = \min \left\{ \varphi(y^1), \ldots, \varphi\left(y^{\dot{m}+1}\right) \right\}$ as the best approximation to the optimal solution of the original problem.

Otherwise, arbitrarily select and remove a program from the problems list. Go to Step 2.

Step 2. Solve the problem taken from the problems list using typical methods for integer programming (e.g., like branch-and-bound) to manage the integrality constraint. Denote the set of optimal solutions as $S = \left\{ (\tilde{x}^1, \tilde{y}^1), \ldots \right\}$ and $\tilde{z}$ the objective function value. If the problem has no feasible solution, or if its objective function value is larger than z_t, then fathom this branch, let $z_{t+1} = z_t$, $t = t+1$ and go to Step 1. Otherwise go to Step 3.

Step 3. If the components y of all the solutions belonging to S are elements of V, then store the solutions, set $z_{t+1} = \tilde{z}$, $t = t+1$ and go to Step 1 (for such values of y, the point (x, y) is feasible for problem(15)). Otherwise, considering the solution $(\tilde{x}^j, \tilde{y}^j)$ from S such that the component $\tilde{y}^j$ is different from all the elements of V, we add $\tilde{y}^j$ to V, set $z_{t+1} = z_t$, $t = t+1$ and go to Step 4.

Step 4. *Subdivision.* Make a subdivision of the set Y corresponding to this problem. By construction, problem (20) corresponds to one set of $m+1$ affine independent points, which without loss of generality are assumed to be the points $y^1,\ldots,y^{m+1}$. Adding the point $\tilde{y}^j$ to this set, it becomes affinely dependent. Excluding one element of the resulting set, affine independence can eventually be obtained (this is guaranteed if some correct element is dropped). When one uses this approach, at most $m+1$ new affine independent sets arise, each corresponding to a new linear approximation of the lower level objective function on the convex hull of these points. If one such simplex T is a subset of some region of stability: $T\subset\Re(B_i)$, the feasible points (x,y) of problem (20) are also feasible for problem (15). Aim of this step is to find these simplices by subsequent subdivisions of the set Y. These problems are then added to the problems list.

To calculate the new approximation of the lower level optimal value function we proceed as follows: First compute $\varphi(\tilde{y}^j)$. Then construct one set of affinely independent points as described above, i.e. delete one of the previous points, say y^ℓ, where $\ell\in\{1,\ldots,m+1\}$, and compute

$$\Phi_\ell(y)=\sum_{i=1,i\neq\ell}^{m+1}\lambda_i\varphi\left(y^i\right)+\mu\varphi\left(\tilde{y}^j\right),$$

defined over

$$y=\sum_{i=1,i\neq\ell}^{m+1}\lambda_i y^i+\mu\tilde{y}^j, \tag{21}$$

with $\lambda_i\geq 0$, $i=1,\ldots,m+1$, $i\neq\ell$, and

$$\sum_{i=1,i\neq\ell}^{m+1}\lambda_i+\mu=1. \tag{22}$$

Thus we construct at most $m+1$ new problems:

$$(P^\ell)\quad \min_{x,y}\left\{\langle a,x\rangle+\langle b,y\rangle : Gy=d, \langle c,x\rangle\leq\Phi_\ell(y), Ax=y, x\geq 0, y\in Z_+^m\right\},$$

and add them to the problems list. Go to Step 1.

Another idea of how to solve problem (15) is to work with the exact penalty function described in [25], [6], [32]). Namely, we deduce a new reformulation of (15) using the facts that the objective value function (14) is piecewise-linear, convex and partially calm, as we showed it in Section 4.

We suppose that there exists a $k_0<\infty$ such that a point (x^0,y^0) is locally optimal for problem (15) if and only if it is locally optimal for the problem:

$$\min_{x,y}\left\{\langle a,x\rangle+\langle b,y\rangle+k\left[\langle c,x\rangle-\varphi(y)\right] \mid Gy=d, Ax=y, x\geq 0, y\in Z_+^m\right\}, \tag{23}$$

for all $k\geq k_0$.

The difficulty in dealing with (23) arises from the fact that the exact penalty function:

$$\langle a,x\rangle+\langle b,y\rangle+k\left[\langle c,x\rangle-\varphi(y)\right] \tag{24}$$

is not explicit due to the nature of the lower level optimal value function (14). Moreover, the penalty function (24) is also nonconvex. For this reason, we propose to use the algorithms presented in [34] and [35].

6 A numerical example

We consider the following bilevel parametric lineal programming problem, where the upper level is described as:

$$\min_{x_1,x_2,x_3,y_1,y_2} \{3x_1 + 2x_2 + 6x_3 + 2y_1 \mid 4y_1 + y_2 = 10,\ x \in \psi(y),\ y_1, y_2 \in Z_+\},$$

where

$$\psi(y_1,y_2) = \operatorname{Argmin}_{x_1,x_2,x_3}\{-5x_1 - 8x_2 - x_3 \mid 4x_1 + 2x_2 \le y_1, 2x_1 + 4x_2 + x_3 \le$$

$$\le y_2, x_1, x_2, x_3 \ge 0\},$$

and the lower level optimal value is given by:

$$\varphi(y_1,y_2) = \min_{x_1,x_2,x_3}\{-5x_1 - 8x_2 - x_3 \mid 4x_1 + 2x_2 \le y_1, 2x_1 + 4x_2 + x_3 \le$$

$$\le y_2, x_1, x_2, x_3 \ge 0\}.$$

The optimal solution of this problem is $(x_1^*, x_2^*, x_3^*; y_1^*, y_2^*) = (1/3, 1/3, 0; 2, 2)$. We start to solve the problem using the proposed algorithm.

Step 0. We choose the vertices $y^1 = (5/2, 0)$ and $y^2 = (0, 10)$ that belong to the convex hull of the leader's strategies at the upper level. Fix the tolerance value $\varepsilon = 0.1$. Now, we calculate $\varphi(y^1) = 0$ and $\varphi(y^2) = -10$, set $z_1 = +\infty$, then the first approximation is build as follows:

$$\Phi(y) = -y_2.$$

The approximate integer problem (AIP) that we add to the problems' list is given as follows:

$$\min_{x,y}\{3x_1 + 2x_2 + 6x_3 + 2y_1 \mid \begin{array}{l} 4y_1 + y_2 = 10, 4x_1 + 2x_2 \le y_1 \\ 2x_1 + 4x_2 + x_3 \le y_2, -5x_1 - 8x_2 - x_3 \le -y_2 \\ x_1, x_2, x_3 \ge 0, y_1, y_2 \in Z_+\} \end{array}$$

Step 1. We select (AIP) from the problems list.

Step 2. We solve problem (AIP) and obtain the (unique) solution $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3; \tilde{y}_1, \tilde{y}_2) = (0, 1/4, 0; 2)$ with $\tilde{z} = 15/4$. Because $\tilde{z}$ is less than $+\infty$, we go to Step 3.

Step 3. As $\tilde{y} = (\tilde{y}_1, \tilde{y}_2) = (2,2)$ is different from the elements of the set V, we add $\tilde{y} = (2,2)$ to V, set $z_2 = +\infty$, $t = 2$ and go to Step 4.

Step 4. Make a subdivision at $\tilde{y} = (2,2)$ thus obtaining two new problems: the first one corresponding to $conv\{y^2 = (0,10), \tilde{y} = (2,2)\}$, and the second one corresponding to $conv\{\tilde{y} = (2,2), y^1 = (5/2,0)\}$. Then we add these two new programs to the problems list, each one described as follows: the first one with the approximation

$$\Phi_1(y) = -17\, y_2/24 - 70/24,$$

and the second one with the approximation

$$\Phi_2(y) = -13\, y_2/6.$$

Finally, the new problems can be specified as follows:

$$(P^1) \quad \min_{x_1,x_2,x_3,y_1,y_2}\{3x_1 + 2x_2 + 6x_3 + 2y_1 \mid 4y_1 + y_2 = 10, 4x_1 + 2x_2 \le$$

$$\le y_1, 2x_1 + 4x_2 + x_3 \le y_2, -5x_1 - 8x_2 - x_3 \le \Phi_1(y), x_1, x_2, x_3 \ge 0, y_1, y_2 \in Z_+\}.$$

(when removing y^1 from V), and

$$(P^2)\quad \min_{x_1,x_2,x_3,y_1,y_2} \{3x_1 + 2x_2 + 6x_3 + 2y_1 \mid 4y_1 + y_2 = 10, 4x_1 + 2x_2 \le$$

$$\le y_1, 2x_1 + 4x_2 + x_3 \le y_2, -5x_1 - 8x_2 - x_3 \le \Phi_2(y), x_1, x_2, x_3 \ge 0, y_1, y_2 \in Z_+\}.$$

(when removing y^2 from V). Go to Step 1.

Step 1. We select (P^1) from the problems list and go to Step 2.

Step 2. We solve (P^1) yielding the (unique) solution $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3;\tilde{y}_1,\tilde{y}_2)=(1/3,1/3,0;2,2)$ with $\tilde{z} = 17/3$. And because $\tilde{z}$ is less than z_2 , then we go to Step 3.

Step 3. As $\tilde{y} = (\tilde{y}_1, \tilde{y}_2) = (2,2)$ coincides with one of the elements of V , we store the solution $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3;\tilde{y}_1,\tilde{y}_2)=(1/3,1/3,0;2,2)$, set $z_3 = 17/3$, $t = 3$, and go to Step 1.

Step 1. We select (P^2) from the problems list and go to Step 2.

Step 2. We solve (P^2) obtaining the (unique) solution $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3;\tilde{y}_1,\tilde{y}_2)=(1/3,1/3,0;2,2)$ with $\tilde{z} = 17/3$. And as $\tilde{z}$ is equal to z_3, then we go to Step 3.

Step 3. Because $\tilde{y} = (\tilde{y}_1, \tilde{y}_2) = (2,2)$ coincides with one of the elements of V , we store the solution $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3;\tilde{y}_1,\tilde{y}_2)=(1/3,1/3,0;2,2)$, set $z_4 = 17/3$, $t = 4$, then go to Step 1.

Step 1. The problems list is empty, so we finish the algorithm.

Therefore, the last stored solution $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3;\tilde{y}_1,\tilde{y}_2)=(1/3,1/3,0;2,2)$ with $z = 17/3$ is the solution obtained with our algorithm, and it coincides with the exact solution of the problem.

Conclusions. In this paper, we propose an approximation algorithm to solve the Mixed-Integer Linear Programming Problem, and at the same time, using the exact penalty function, we provide upper and lower bounds for a feasible solution. This algorithm can be applied to solve numerically the bilevel portfolio optimization problem (6)–(7). In the latter application, $y=\sigma$ is the upper level variable, which uses to have discrete nature. Therefore, after having numerated various possible values of σ, the vector y can be interpreted as belonging to Z_+^m .

The work does not stop here, our goal is to analyze more alternatives such as a convexification of the exact penalty function, or making use of the subdifferential calculus as another alternative. Later, comparing the algorithms, we will choose the best one according to its performance and robustness.

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