МАТЕМАТИЧНА ТА ОБЧИСЛЮВАЛЬНА ФІЗИКА

Miranda Gabelaia

ON DEFLECTIONS OF A PRISMATIC SHELL EXPONENTIALLY CUSPED AT INFINITY IN THE $N = 0$ APPROXIMATION OF HIERARCHICAL MODELS

In the $N = 0$ approximation of hierarchical models the well-posedness of boundary value problems for an equation of deflections of a prismatic shell exponentially cusped at infinity is studied. Static problem of the shell with the thickness as follows

$$h = h_0 e^{-\kappa(x_1^2 + x_2^2)}, \quad h_0 = const > 0, \quad \kappa = const \geq 0, \quad x_1 \in (-\infty, +\infty), \quad x_2 \geq 0,$$

is given and investigated.

The solution of the posed boundary value problem is given in an integral form.

Keywords: Cusped Prismatic Shells, Cusped Plates, Vekuas's Hierarchical Models, Degenerate Partial Differential Equations, Elliptic Equations, Reimann Function

Introduction

The elastic body is called a prismatic shell if it is bounded above and below by the surfaces

$$x_3 = \overset{(+)}{h}(x_1, x_2) \text{ and } x_3 = \overset{(-)}{h}(x_1, x_2)$$

laterally by a cylindrical surface of generatrix parallel to the x_3-axis and its vertical dimension is sufficiently small comparing with other dimensions of the body [1-3].

Vekua's hierarchical models for elastic prismatic shells are the mathematical models [1-10]. Their construction is based on the multiplication of the basic equations of linear elasticity:

Motion Equations

$$X_{ij,j} + \Phi_i = \rho \ddot{u}_i(x_1, x_2, x_3, t), \quad x \in \Omega \subset R^3, \ t > t_0;$$

Generalized Hooke's law (isotropic case)

$$X_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij};$$

Kinematics Relations

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right),$$

by Legendre polynomials $P_r(ax_3 - b)$. After that it is necessary to integrate with respect to x_3 within the limits $\overset{(-)}{h}(x_1, x_2)$ and $\overset{(+)}{h}(x_1, x_2)$. Here Φ_i are the volume force components, X_{ij} are the stress tensors, u_i are the displacements, e_{ij} are the strain tensors, λ and μ are the Lame constants, ρ is the density, δ_{ij} is the Kronecker's symbol. Moreover, repeated indices imply summation (Greek letters run from 1 to 2, and Latin letters run from 1 to 3, unless otherwise stated), subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables. Legendre polynomials has the following form

$$P_n(\tau) = \frac{1}{2^n n!} \frac{d^n\left(\tau^2 - 1\right)^n}{d\tau^n}, \quad n = 0, 1, \ldots,$$

$$P_0(\tau)=1,\ P_1(\tau)=\tau,\ \ P_2(\tau)=\frac{3\tau^2-1}{2}.$$

Let us consider the $N=0$ approximation of Vekua's hierarchical models (for details see, e.g. [2,3]). The equation for the weighted zero moment of the displacement vector component u_i has the following form

$$\mu\left[\left(hv_{\alpha 0,\beta}\right)_{,\beta}+\left(hv_{\beta 0,\alpha}\right)_{,\beta}\right]+\lambda\delta_{\alpha\beta}\left(hv_{\gamma 0,\gamma}\right)_{,\beta}$$
$$+Q_{\overset{(+)}{v}\alpha}\sqrt{\left(\overset{(+)}{h}_{,1}\right)^2+\left(\overset{(+)}{h}_{,2}\right)^2+1}+Q_{\overset{(-)}{v}\alpha}\sqrt{\left(\overset{(-)}{h}_{,1}\right)^2+\left(\overset{(-)}{h}_{,2}\right)^2+1}+\Phi_{\alpha 0}=\rho h\ddot{v}_{\alpha 0}, \quad (1)$$

$$\mu\left(hv_{30,\beta}\right)_{,\beta}+Q_{\overset{(+)}{v}3}\sqrt{\left(\overset{(+)}{h}_{,1}\right)^2+\left(\overset{(+)}{h}_{,2}\right)^2+1}+Q_{\overset{(-)}{v}3}\sqrt{\left(\overset{(-)}{h}_{,1}\right)^2+\left(\overset{(-)}{h}_{,2}\right)^2+1}+\Phi_{30}=\rho h\ddot{v}_{30,\alpha}, \quad (2)$$

where Φ_{i0} is the zero moment of the volume force component Φ_i, $Q_{\overset{(+)}{v}i}$, $Q_{\overset{(-)}{v}i}$ are stresses given on the upper and lower surfaces of the shell, $v_{i0}=\frac{1}{h}u_{i0}$,

$$u_{i0}(x_1,x_2,t):=\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}u_i(x_1,x_2,x_3,t)dx_3,\qquad \Phi_{i0}(x_1,x_2,t):=\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}\Phi_i(x_1,x_2,x_3,t)dx_3;$$

$$\overset{(\pm)}{v}_\beta=\frac{\mp\overset{(\pm)}{h}_{,\beta}}{\sqrt{\left(\overset{(\pm)}{h}_{,1}\right)^2+\left(\overset{(\pm)}{h}_{,2}\right)^2+1}},\qquad \overset{(\pm)}{v}_3=\pm\frac{1}{\sqrt{\left(\overset{(\pm)}{h}_{,1}\right)^2+\left(\overset{(\pm)}{h}_{,2}\right)^2+1}};$$

$$X_{i\beta}\left(x_1,x_2,\overset{(+)}{h}(x_1,x_2),t\right)\overset{(+)}{v}_\beta+X_{i3}\left(x_1,x_2,\overset{(+)}{h}(x_1,x_2)\right)\overset{(+)}{v}_3=Q_{\overset{(+)}{v}i}(x_1,x_2,t),$$

$$X_{i\beta}\left(x_1,x_2,\overset{(-)}{h}(x_1,x_2),t\right)\overset{(-)}{v}_\beta+X_{i3}\left(x_1,x_2,\overset{(-)}{h}(x_1,x_2)\right)\overset{(-)}{v}_3=Q_{\overset{(-)}{v}i}(x_1,x_2,t).$$

1. Statement of the Problem

The system (1)-(2) for plates in static case has the following form

$$\mu\left[\left(hv_{\alpha 0,\beta}\right)_{,\beta}+\left(hv_{\beta 0,\alpha}\right)_{,\beta}\right]+\lambda\delta_{\alpha\beta}\left(hv_{\gamma 0,\gamma}\right)_{,\beta}$$
$$+Q_{\overset{(+)}{v}\alpha}\sqrt{\left(\overset{(+)}{h}_{,1}\right)^2+\left(\overset{(+)}{h}_{,2}\right)^2+1}+Q_{\overset{(-)}{v}\alpha}\sqrt{\left(\overset{(-)}{h}_{,1}\right)^2+\left(\overset{(-)}{h}_{,2}\right)^2+1}+\Phi_{\alpha 0}=0,$$

$$\mu\left(hv_{30,\beta}\right)_{,\beta}+Q_{\overset{(+)}{v}3}\sqrt{\left(\overset{(+)}{h}_{,1}\right)^2+\left(\overset{(+)}{h}_{,2}\right)^2+1}+Q_{\overset{(-)}{v}3}\sqrt{\left(\overset{(-)}{h}_{,1}\right)^2+\left(\overset{(-)}{h}_{,2}\right)^2+1}+\Phi_{30}=0. \quad (3)$$

Let

$$h(x_1,x_2)=h_0e^{-\kappa(x_1^2+x_2^2)},\quad h_0=const>0,\quad \kappa=const\geq 0,\quad x_1\in(-\infty,+\infty),\quad x_2\geq 0. \quad (4)$$

We consider prismatic shell whose projection on Ox_1x_2 is (see, Fig.1)

$$\omega:=\{(x_1,x_2):-\infty<x_1<+\infty;\quad 0\leq x_2<+\infty\}.$$

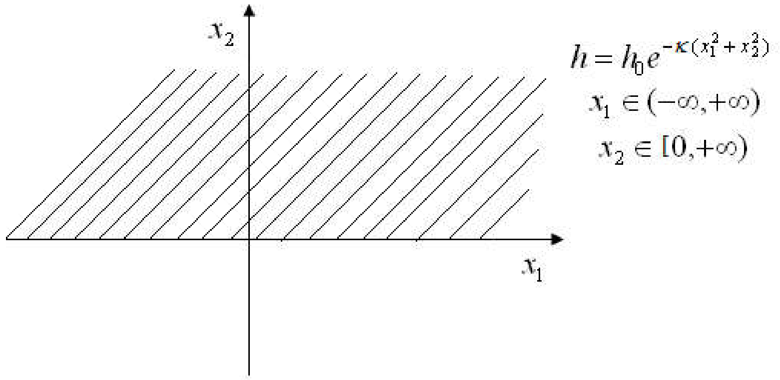

Figure 1. The projection of the plate on Ox_1x_2.

The profiles of the plate on Ox_1x_3 and Ox_2x_3 are given on Fig. 2 and Fig. 3, respectively.

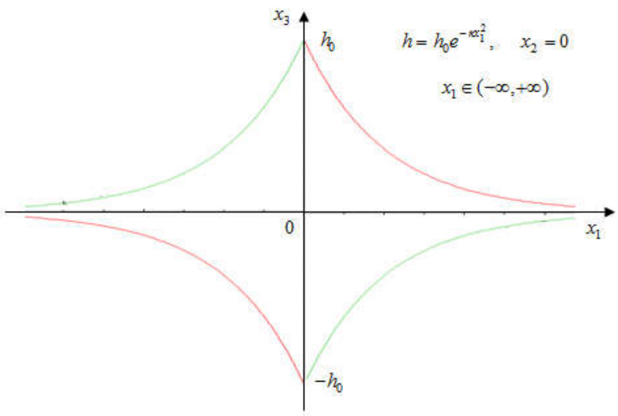

Figure 2. The profile of the plate on Ox_1x_3

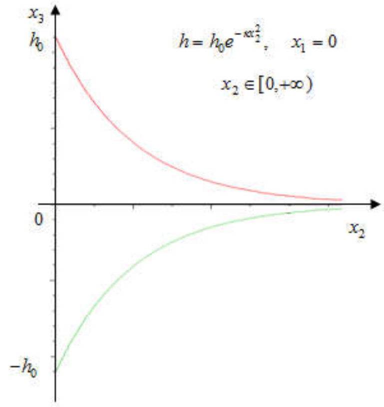

Figure 3. The profile of the plate on Ox_2x_3

Using (4), equation (3) can be written as follows

$$\mu h_0 e^{-\kappa(x_1^2+x_2^2)}(v_{30,11} + v_{30,22} - 2x_1\kappa v_{30,1} - 2x_2\kappa v_{30,2}) + F_{30} = 0, \tag{5}$$

where

$$F_{30}(x_1,x_2) := (\underset{v\,3}{Q^{(+)}}(x_1,x_2) + \underset{v\,3}{Q^{(-)}}(x_1,x_2))\sqrt{\left(h_{,1}\right)^2 + \left(h_{,2}\right)^2 + 1} + \Phi_{30}(x_1,x_2).$$

Let $F_{30} \in C([0,l])$.

We consider the following problem:

Problem. Find the solution v_{30} of the equation (5)

$$v_{30} \in C^2(\omega) \cap C(\overline{\omega}),$$

under following boundary conditions

$$v_{30}(0,x_2) = 0 \tag{6}$$

and condition at infinity

$$v_{30}(x) = O(e^{\kappa(x_1^2+x_2^2)}), \text{ when } |x| \to \infty, \quad x := (x_1,x_2). \tag{7}$$

We use methods proposed in [11,12] for solving Problem. Let us rewrite equation (5) in the following form

$$v_{30,11} + v_{30,22} - 2x_1\kappa v_{30,1} - 2x_2\kappa v_{30,2} = F, \tag{8}$$

where

$$F := -\frac{1}{\mu h_0} F_{30} e^{\kappa(x_1^2 + x_2^2)}.$$

Equation (8) in the complex form can be rewritten as follows

$$L(U(z,\zeta)) := \frac{\partial^2 U(z,\zeta)}{\partial z \partial \zeta} + A\frac{\partial U(z,\zeta)}{\partial z} + B\frac{\partial U(z,\zeta)}{\partial \zeta} - F_1(z,\zeta) = 0, \quad (9)$$

where $z = x_1 + ix_2$, $\zeta = x_1 - ix_2$, $A(z,\zeta) := -\frac{\kappa}{2} z$, $B(z,\zeta) := -\frac{\kappa}{2}\zeta$,

$U := v_{30}\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$, $F_1 := \frac{1}{4} F\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$.

The boundary condition (6) and condition at infinity (7) can be written as follows

$$U(z,\zeta_0) = 0, \qquad U(z_0,\zeta) = 0, \quad (10)$$

$$U(z,\zeta) = O(e^{\kappa z\zeta}), \quad \text{where} \quad |z| \to \infty, \ |\zeta| \to \infty. \quad (11)$$

Riemann function for problem (9)-(11) has the following form

$$R(z,\zeta;t,\tau) = e^{-\frac{\kappa}{2}t(\zeta-\tau) - \frac{\kappa}{2}\tau(z-t)}.$$

Therefore the solution of the boundary value problem (9)-(11) can be written by such a way (see [11])

$$U(z,\zeta) = \int_{z_0}^{z}\int_{\zeta_0}^{\zeta} e^{\frac{\kappa}{2}z(\zeta-\tau) + \frac{\kappa}{2}\zeta(z-t)} \cdot F_1(t,\tau)d\tau\, dt, \quad (12)$$

where $z_0 = x_1^0 + ix_2^0$, $\zeta_0 = x_1^0 - ix_2^0$, $(x_1^0, x_2^0) \in \partial\omega$.

Let at first $F_1(t,\tau) \equiv 1$. From (12) we have

$$U(z,\zeta) = e^{\kappa z\zeta}\int_{z_0}^{z} e^{-\frac{\kappa}{2}\zeta t} dt \int_{\zeta_0}^{\zeta} e^{-\frac{\kappa}{2}z\tau} d\tau =$$

$$= e^{\kappa z\zeta}\left[-\frac{2}{\kappa\zeta}\left(e^{-\frac{\kappa}{2}\zeta z} - e^{-\frac{\kappa}{2}\zeta z_0}\right)\right]\cdot\left[-\frac{2}{\kappa z}\left(e^{-\frac{\kappa}{2}z\zeta} - e^{-\frac{\kappa}{2}z\zeta_0}\right)\right] = \quad (13)$$

$$= \frac{4}{\kappa^2 z\zeta}\left(1 - e^{\frac{\kappa}{2}z\zeta - \frac{\kappa}{2}z\zeta_0} - e^{\frac{\kappa}{2}z\zeta - \frac{\kappa}{2}\zeta z_0} + e^{\kappa z\zeta - \frac{\kappa}{2}\zeta z_0 - \frac{\kappa}{2}z\zeta_0}\right).$$

Thus, due to (13) we obtain the following estimate

$$|v_{30}(x_1,x_2)| \le \frac{4}{\kappa^2(x_1^2+x_2^2)}\Bigg|\left(1-e^{\frac{\kappa}{2}\left(x_1^2+x_2^2-x_1x_{10}-x_2x_{20}+i(x_1x_{20}-x_2x_{10})\right)}\right.$$

$$\left.-e^{\frac{\kappa}{2}\left(x_1^2+x_2^2-x_1x_{10}-x_2x_{20}+i(x_2x_{10}-x_1x_{20})\right)}+e^{\kappa(x_1^2+x_2^2-x_1x_{10}-x_2x_{20})}\right)\Bigg| \le \tag{14}$$

$$\le \frac{4}{\kappa^2(x_1^2+x_2^2)}\Bigg|\left(1-2e^{\frac{\kappa}{2}\left(x_1^2+x_2^2-x_1x_{10}-x_2x_{20}\right)}\cos\left(\frac{\kappa}{2}(x_1x_{20}-x_2x_{10})\right)\right.$$

$$\left.+e^{\kappa(x_1^2+x_2^2-x_1x_{10}-x_2x_{20})}\right)\Bigg|$$

From (14) we get

$$v_{30}(x)=O(e^{\kappa(x_1^2+x_2^2)}),\ \text{ when } |x|\to\infty,\ \ x:=(x_1,x_2).$$

Let further $F_1(t,\tau)$ be an arbitrary continuous bounded function $\left(|F_1(t,\tau)|\le M<const<\infty\right)$. From (12) we have

$$U(z,\zeta)=\left|\int_{z_0}^{z}\int_{\zeta_0}^{\zeta} e^{\frac{\kappa}{2}z(\zeta-\tau)+\frac{\kappa}{2}\zeta(z-t)}\cdot F_1(t,\tau)d\tau\,dt\right| \le$$

$$\le \int_{z_0}^{z}\int_{\zeta_0}^{\zeta}\left|e^{\frac{\kappa}{2}z(\zeta-\tau)+\frac{\kappa}{2}\zeta(z-t)}\right|\cdot|F_1(t,\tau)|d\tau\,dt \le$$

$$\le M\cdot\int_{z_0}^{z}\int_{\zeta_0}^{\zeta} e^{\frac{\kappa}{2}z(\zeta-\tau)+\frac{\kappa}{2}\zeta(z-t)}d\tau\,dt.$$

So, (12) is a solution of the setting problem.

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Анотація. М. Габелая. Про відхилення призматичної оболонки, експоненціально зростаючої на нескінченності, в N=0 апроксимації ієрархічних моделей. *При $N=0$ апроксимації ієрархічних моделей вивчається корректно поставлена крайова задача для рівняння відхилення призматичної оболонки, експоненціально зростаючої на нескінченності,. Сформульовано та досліджено наступну задачу*

$$h = h_0 e^{-\kappa(x_1^2+x_2^2)}, \quad h_0 = const > 0, \quad \kappa = const \geq 0, \quad x_1 \in (-\infty,+\infty), \quad x_2 \geq 0 .$$

Подано явний вигляд розв' язку рівняння в інтегральній формі.

Ключові слова: загострена призматична оболонка, загострена пластинка, ієрархічні моделі, еліптичні рівняння, рівняння в частинних похідних, вироджені диференціальні рівняння в частинних похідних.

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И. В. Атамась

ПЛОЩАДЬ РЕШЕНИЙ ЛИНЕЙНЫХ РАЗНОСТНЫХ УРАВНЕНИЙ ХУКУХАРЫ

Получены явные формулы для вычисления площади решения разностных уравнений Хукухары в пространстве $conv\mathbb{R}^n$.

Ключевые слова: Разность Хукухары, смешанная площадь Минковского, метод сравнения Чаплыгина–Важевского, разностные уравнений, динамические системы, метрика Хаусдорфа.