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PROPERTIES OF CONTINUOUS FUNCTIONS ON A COMPACT

В роботі представлено визначення неперервної функції, визначення неперервної функції за Коші, за Гейне, на мові приростів. Детально вивчені властивості функцій неперервних на компакті (відрізку). Представлені 1-а, 2-а теореми Вейрштрасса, 1-а, 2-а теорема Коші, а також основні наслідки з них. Покроково представлені докази теорем і наслідків

Ключові слова: безперервні функції, компактність, теорема Вейрштрасса, теорема Коші

В работе представлено определение непрерывной функции, определение непрерывной функции по Коши, по Гейне, на языке приращений. Подробно изучены свойства функций непрерывных на компакте (отрезке). Представлены 1-я, 2-я теоремы Вейрштрасса, 1-я, 2-я теоремы Коши, а также основные следствия из них. Пошагово представлены доказательства теорем и следствий

Ключевые слова: непрерывные функции, компактность, теорема Вейрштрасса, теорема Коши

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1. Introduction

The theory of functions is a branch of mathematics that studies the properties of various functions. The theory of functions is divided into two areas: the theory of functions of a real variable and the theory functions of a complex variable, the difference between them is so great that they are usually treated separately. Without going into details, we can say that on the merits the distinction lies, on the one hand, in a detailed study of the basic concepts of mathematical analysis (such as continuity, differentiation, integration, etc.), on the other hand, in the theoretical analysis of the development of specific functions represented by sedate rows. One of the achievements of the theory of functions of actual variable was the creation of the theory of integration.

In mathematics, a function f is uniformly continuous if, roughly speaking, it is possible to guarantee that f(x) and f(y) is as close to each other as we please by requiring only that x and y are sufficiently close to each other; unlike ordinary continuity, the maximum distance between f(x) and f(y) cannot depend on x and y themselves. For instance, any isometry (distance-preserving map) between metric spaces is uniformly continuous.

The image of a totally bounded subset under a uniformly continuous function is totally bounded. However, the image of a bounded subset of an arbitrary metric space under a uniformly continuous function should not to be bounded: as a counterexample, consider the identity function from the integers endowed with the discrete metric to the integers endowed with the usual Euclidean metric.

The Heine–Cantor theorem asserts that every continuous function on a compact set is uniformly continuous. In particular, if a function is continuous on a closed bounded interval of the real line, it is uniformly continuous on that interval. The Darboux integrability of continuous functions follows almost immediately from the uniform continuity theorem.

2. Definition of a continuous function

The basic definition of a continuous function [1 – 3]:

The function f(x) is continuous at some point x_0, if $\lim_{x \to x_0} f(x) = f(x_0)$.

Cauchy's definition of a continuous function:

f(x) is continuous at the point x_0, if

$$\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0 : \forall x : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon .$$

Heine's definition of a continuous function:

f(x) is continuous at the point x_0, if

$$\forall \{x_n\} : \lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0) .$$

The definition of a continuous function (in the increment language):

f(x) is continuous at the point x_0, if

$\lim_{\Delta x \to 0} \Delta f(x) = 0$, i.e. the infinitesimal increment of the function corresponds to the infinitesimal increment of the argument.

3. Properties of continuous functions on a compact (on an interval)

The function is said to be continuous on a set, i.e. $f(x) \in C[a,b]$, if it is continuous at every point of this set.

3.1. The 1st Weierstrass theorem [4, 5]

Every continuous function on the interval is limited on this interval, i.e. if $f(x) \in C[a,b]$, then f(x) is limited on [a, b].

Proof

By contradiction: let f(x) be unlimited on [a, b], i.e. $\forall M > 0 \exists x_M \in [a,b] : |f(x_M)| > M$.

Let M = 1, then $\exists x_1 \in [a,b] : |f(x_1)| > 1$

M = 2 $\Rightarrow$ $\exists x_2 \in [a,b] : |f(x_2)| > 2$, ...

M = n $\Rightarrow \exists x_n \in [a,b] : |f(x_n)| > n$, ...

We obtain a sequence $\{x_n\} : a < x_n < b, \forall n \in N \Rightarrow |f(x_n)| > n$.

Since $\{x_n\}$ is limited, we can distinguish a convergent subsequence from it by the Bolzano-Weierstrass theorem, i.e. $\exists \{x_{nk}\} \subset \{x_n\} : \lim_{n\to\infty} x_{nk} = c, c \in [a,b]$.

Since a subsequence has all properties of a sequence,

$$|f(x_{nk})| > n_k, \forall k = 1,2,... \quad (1)$$

Since $c \in [a,b]$, then f(x) is continuous at the point c. Using the definition of a continuous function at the point c in the increment language by Heine

$$\lim_{n\to\infty} f(x_{nk}) = f(c) . \quad (2)$$

It turns out that (1) and (2) are in contrast: out of (1) $\Rightarrow \lim_{n\to\infty} f(x_{nk}) = \infty$. It means that the assumption is false.

The theorem is proved.

Note! The theorem becomes false if we substitute an interval with an open interval (a, b) in it. For example, $f(x) = \frac{1}{x} \in C[0,1]$, but unlimited $\lim_{x\to 0} \frac{1}{x} = \infty$.

3.2. The 2nd Weierstrass theorem

If $f(x) \in C[a,b]$, it reaches sup and in f on this interval, i.e [6, 7].

$$\exists x_1 \in [a,b] : f(x_1) = \sup_{[a,b]} f(x), \exists x_2 \in [a,b] : f(x_2) = \inf_{[a,b]} f(x) .$$

Proof

By the 1st Weierstrasss theorem, the function f(x) is limited on [a, b], so by the theorem of the existence of sup and inf [*if a set is limited from above (below), it has sup (inf)*] $\exists \sup_{[a,b]} f(x) = M, \exists \inf_{[a,b]} f(x) = m$.

It is necessary to show: $\exists x_1 \in [a,b] : f(x_1) = M$.

Proof by contradiction: let not $\exists x_1 \in [a,b] : f(x_1) = M$.

Let us introduce the auxiliary function $\phi(x) = \frac{1}{M - f(x)}$, it is defined and continuous on [a,b], so, according to the 1st Weierstrass theorem, $\phi(x)$ is limited from below 0, and from above $\exists c > 0 : 0 < \phi \le c, \forall x \in [a,b]$

$$\frac{1}{M - f(x)} \le c, \forall x \in [a,b] \Rightarrow M - f(x) \ge \frac{1}{c} \Rightarrow f(x) \le M - \frac{1}{c} .$$

I.e. $M - \frac{1}{c} < M$, then $M - \frac{1}{c}$ cannot be the superior, thus $\sup_{[a,b]} f(x) = M$, it means that there is at least one point $x_1 \in [a,b] : f(x_1) = M$.

The theorem is proved.

3.3. The 1st Cauchy theorem (vanishing theorem)

If $f(x) \in C[a,b]$ and f(a)f(b)<0 (at the ends of an interval the function possesses values of different signs), then $\exists c \in [a,b] : f(c) = 0$ [8, 9].

Proof (constructive)

Let f(a)<0, f(b)>0. We divide [a, b] into two. If at the point of division $f(a_1) = 0$, the theorem is proved. If $f(a_1) \ne 0$, we select such an interval $[a_1, b_1]$, at the ends of which $f(a_1) < 0, f(b_1) > 0$ and keep dividing it. ... At the k step:

$$[a_k, b_k] : f(a_k) < 0, f(b_k) > 0 .$$

We have obtained the sequence of intervals, nested into each other:

$$[a,b] \supset [a_1,b_1] \supset [a_2,b_2] \supset ... \supset [a_k,b_k] \supset ...$$

$$\|[a_k,b_k]\| = \frac{b-a}{2^k} \underset{k\to\infty}{\to} 0 .$$

Then by the Cantor's Nested Interval Theorem

$$\exists! c \in [a_k,b_k], \forall k = 1,2,... : \lim_{k\to\infty} a_k = c, \lim_{k\to\infty} b_k = c .$$

Since $c \in [a,b]$, $f(x) \in C[a,b]$ and $\lim_{k\to\infty} a_k = c$, then the function is continuous at the point c too, i.e. $\lim_{k\to\infty} f(a_k) = f(c) \le 0, f(a_k) < 0$, then, passing to the limit in the inequality $f(c) \le 0$.

Similarly $\lim_{k\to\infty} f(b_k) = f(c) \geq 0$, since $f(b_k) > 0$, then

$$f(c) \geq 0, 0 \leq f(c) \leq 0 \Rightarrow f(c) = 0 .$$

The theorem is proved.

Corollary: If $f(x) \in C[a,b]$ and $f(x) \neq 0, \forall x \in (a,b)$, the function maintains a sign on (a,b).

Proof

By contradiction: let

$$\exists x_1 \in (a,b) : f(x_1) < 0, \exists x_2 \in (a,b) : f(x_2) > 0.$$

Let $x_1 < x_2$, then by the Cauchy theorem on $[x_1, x_2]$ we obtain $\exists c \in [x_1, x_2] \subset [a,b] : f(c) = 0$, which contradicts conditions of the theorem.

The corollary is proved.

3.4. The 2nd Cauchy theorem (on intermediate value)

If $f(x) \in C[a,b]$, $f(a) = A, f(b) = B, A \neq B$, then $\forall C$, situated between A and B $\exists \xi \in [a,b] : f(\xi) = C$ [1, 10].

Proof

Let A<B and $\forall c : A < C < B$.

Let us introduce $\phi(x) = f(x) - C, \forall x \in [a,b]$. $\phi(x) \in C[a,b]$.

$$\phi(a) = f(a) - C = A - C < 0 ,$$

$$\phi(b) = f(b) - C = B - C > 0 .$$

Then by the 1st Cauchy theorem

$$\exists \xi \in [a,b] : \phi(\xi) = 0, \phi(\xi) = f(\xi) - C = 0 \Rightarrow f(\xi) = C .$$

The theorem is proved.

Corollary 1

If $f(x) \in C[a,b]$ and $m = \inf_{[a,b]} f(x), M = \sup_{[a,b]} f(x)$, then f(x) on [a, b] possesses all values between m and M, i.e. a set of values of a continuous function on an interval is an interval.

Proof

Since $f(x) \in C[a,b]$, then by the 2nd Weierstrass theorem:

$$\exists x_1 \in [a,b] : f(x_1) = \inf_{[a,b]} f(x) = m .$$

$$\exists x_2 \in [a,b] : f(x_2) = \sup_{[a,b]} f(x) = M .$$

If $x_1 < x_2$, then $[x_1, x_2] \subset [a,b] \Rightarrow f(x) \in C[x_1, x_2]$ by the Cauchy theorem

$$\forall c : m \leq c \leq M \exists \xi \in [x_1, x_2] : f(\xi) = c .$$

The corollary 1 is proved.

Corollary 2

If f(x) is defined and steady on [a, b] and takes on all values between f(a) and f(b), then $f(x) \in C[a,b]$.

Proof

By contradiction: let $f(x) \notin C[a,b]$, i.e. $\exists x_0 \in [a,b]$ to be a point of discontinuity. Let f(x) to increase monotonically for definiteness, then by the theorem on the limit of monotonic sequence:

on $[a, x_0) \exists f(x_0 - 0)$,

on $(x_0, a] \exists f(x_0 + 0)$.

Let us analyze $[a, x_0)$; $f(x)$ strictly increases, i.e.

$$\forall x \in [a, x_0) \Rightarrow f(a) \leq f(x) < f(x_0 - 0) ,$$

On $(x_0, b]$: $\forall x \in (x_0, b] \Rightarrow f(x_0 + 0) < f(x) \leq f(b)$.

Since $f(x_0 - 0) \neq f(x_0 + 0)$, none value between $f(x_0 - 0)$ and $f(x_0 + 0)$ is possessed by the function. $[f(x_0 - 0), f(x_0 + 0)] \subset [f(a), f(b)]$, which contradicts $f(x) \in C[a,b]$.

The corollary 2 is proved.

4. Conclusion

The behavior of continuous functions on a compact is studied in this paper. The following theorems and their corollaries are presented: the 1st and the 2nd Weierstrass theorems, the 1st and the 2nd Cauchy theorems, and their proofs are provided.

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