Розробляються та досліджуються лінійні та нелінійні математичні моделі процесу теплопровідності в однорідних і шаруватих середовищах із включеннями. Наведені неоднорідні системи нагріваються зосередженим тепловим потоком у локальній області межових поверхонь конструкцій. Висвітлено підходи до розв'язування відповідних лінійних та нелінійних крайових задач теплопровідності. Створено алгоритми та розрахункові програми, які дають змогу аналізувати температурні поля в кусково-однорідних середовищах

Ключові слова: теплопровідність, температурне поле, чужорідне наскрізне включення, термочутлива система, тепловий потік

Разрабатываются и исследуются линейные и нелинейные математические модели процесса теплопроводности в однородных и слоистых средах с включениями. Приведенные неоднородные системы нагреваются сосредоточенным тепловым потоком в локальной области граничных поверхностей конструкций. Изложены подходы к решению соответствующих линейных и нелинейных граничных задач теплопроводности. Созданы алгоритмы и расчетные программы, с помощью которых можно анализировать температурные поля в кусочно-однородных средах

Ключевые слова: теплопроводность, температурное поле, инородное сквозное включение, термочувствительная система, тепловой поток

UDC 536.24
DOI: 10.15587/1729-4061.2017.97272

EXAMINING THE TEMPERATURE FIELDS IN FLAT PIECEWISE-UNIFORM STRUCTURES

V. Havrysh
Doctor of Technical Sciences, Professor*
E-mail: ikni.pz@gmail.com

Ya. Pelekh
PhD, Associate Professor
Department of Computational Mathematics and Programming***
E-mail: omp.imfn@gmail.com

L. Kolyasa
PhD**
E-mail: kolyasa.lubov@gmail.com

I. Ovchar
PhD, Associate Professor
Department of Mathematics
Ivano-Frankivsk National Technical University of Oil and Gas
Karpatska str., 15, Ivano-Frankivsk, Ukraine, 76019
E-mail: iovchar@hotmail.com

H. Ivasyk
PhD**
E-mail: Ivasyk-G@yandex.ua

O. Bilas
PhD, Associate Professor*
E-mail: Orest.Y.Bilas@lpnu.ua

*Department of Software***
Department of Mathematics*
***Lviv Polytechnic National University
S. Bandery str., 12, Lviv, Ukraine, 79013

1. Introduction

Efficiency of the processes of heat and mass exchange affects the temperature regime of the environment and living premises, as well as operational processes in various technological installations. That is why, over recent decades, the theory of heat exchange has intensively developed related to the needs of thermal power generation, nuclear energy, space exploration, etc. At present, under development are the methods of thermal protection in high-speed aerial installations, in particular for multi-mission space exploration vehicles. This protection is required also in active areas of reactors, in magnetohydrodynamic energy generators (installations for the direct conversion of heat into electrical energy), gas turbine plants. The processes of heat exchange are studied under low temperature modes, in particular in the installations that employ the effect of superconductivity, for example in magnets, which create large fields. Work continues on creating the cryosurgical instruments for operations involving rapid freezing of separate areas of the tissue. Progress in this field is largely associated with correct organization of the heat exchange processes both in the instrument and in the tissue. There are attempts at creating installations for the freeze-drying of food products, whose successful development depends on correct organization of the sublimation and de-sublimation processes. Methods for exploring the heat exchange processes on the Earth and in its atmosphere are being improved, in particular weather forecasting. Requests from various industries stimulate sustained and rapid development of research into the area of heat exchange [1]. Little studied until now are mathematical models of heat exchange in complex systems where piecewise uniform structure and thermal sensitivity (dependence of thermal-physical parameters on temperature) of their design elements are not considered [2, 3]. This explains the relevance of research into improvement of the existing and

© V. Havrysh, Ya. Pelekh, L. Kolyasa, I. Ovchar, H. Ivasyk, O. Bilas, 2017

creation of new linear and nonlinear mathematical models of heat exchange for the uniform and layered, inclusive of design elements, complex systems and development of new effective methods for solving the boundary problems that match these models.

2. Literature review and problem statement

Determining the temperature regimes in both uniform and non-uniform designs attracts attention of many researchers [4, 5].

Paper [1] developed a mathematical model for calculating the quasi-stationary temperature field in a solid cylinder of rotation. This cylinder is made of composite material. The non-linear boundary conditions are assigned, which take into account a dependence of thermal-physical parameters of materials on the temperature. Analytical expressions obtained for determining the temperature fields make it possible to select the composition of composite materials for the parts of cylindrical type for a purpose of extending their operational lifecycle.

One-dimensional (flat, cylindrical-symmetric and spherically-symmetric) nonlinear problems on thermal conductivity for a given heat flux in the origin of coordinates in the form of a power function dependent on time were explored. Approximate solutions were obtained for the indicated problems with their convergence analyzed [2].

Analytical-numerical solution for a nonlinear problem on thermal conductivity using the integral method of thermal balance was found [3]. In order to improve the accuracy of solution, a temperature function is approximated by the polynomials of high degrees. To determine coefficients of the polynomials, additional boundary conditions are introduced. It is shown that such an approach as fast as in the second approximation leads to a considerable improvement in accuracy of solving the problem.

Paper [6] received an analytical-numerical solution to the axisymmetric problem on thermoelasticity for a thick-wall cylinder under the action of heat flux with the arbitrarily assigned boundary conditions. The resulting solution makes it possible to analyze the effect of thermal and mechanical loads on the thermal-mechanical behavior of the cylinder.

One-dimensional stationary temperature and mechanical problems were solved with a presented interrelation to determine the thermal and mechanical loads in a hollow thick-wall sphere. The distribution of temperature is displayed by a function of the radial coordinate for the given general thermal and mechanical boundary conditions at the inner and outer surfaces of the sphere [7].

Article [8] solved a non-stationary problem on thermal conductivity and thermoelasticity for functional-gradient thick-wall spheres. Thermal-physical and thermoelastic parameters of materials, except for the Poisson coefficient, are arbitrary functions of the radial coordinate.

Axisymmetric stationary problem on thermal conductivity and thermoelasticity of the hollow functionally gradient spheres relative to the heat source was considered. The solutions are obtained as functions of the spatial coordinates for temperature, the displacement component vector and stress tensor by using boundary conditions for the radial and angular coordinates [9].

An overview of main literary sources revealed that the models, which remain insufficiently examined and underdeveloped, are those that would consider the piecewise uniform structure of designs and their thermal sensitivity. As the structures are exposed to temperature influences, then, on certain intervals of temperatures, an impact of thermal sensitivity on the results of calculating the temperature fields becomes vivid. This leads to the development of nonlinear models for the process of thermal conductivity and for their analysis since the solutions of boundary problems that match these models are more precise than the solutions of the appropriate linear boundary problems. Calculations of temperature fields in such systems are applied subsequently when designing sophisticated systems to provide their thermal stability. The accuracy of these computations will affect efficiency of the methods that are used in the process of design.

3. The aim and tasks of the study

The aim of present work is to create linear and nonlinear mathematical models for the process of thermal conductivity for an isotropic plate and a layered plate with a through-inclusion, which are heated by heat flux. This will make it possible to improve accuracy in the calculation of temperature fields in complex systems and effectiveness of the methods for their design.

To accomplish the set aim, the following tasks have to be solved:

– to obtain original equations of thermal conductivity with discontinuous and singular coefficients with boundary conditions and their analytical-numerical solutions. These solutions will allow us to represent a thermal field in arbitrary point of the structures “plate – inclusion” and “layered plate – inclusion”;

– by using the introduced linearizing functions, to linearize the original nonlinear boundary problems on thermal conductivity. To obtain the ratios for determining these functions and, for a linear variable coefficient of thermal conductivity, to derive calculation formulas. These formulas represent a temperature field in an arbitrary point of the thermosensitive structures “plate –inclusion” and “layered plate – inclusion”;

– to devise algorithms and calculation programs for their numerical realization to analyze temperature modes in a plate and in a layered plate with an inclusion.

4. Main results of examining the process of thermal conductivity for piecewise-uniform environments

We shall state the boundary linear and nonlinear problems on thermal conductivity and describe a procedure to solve them.

4. 1. Isotropic plate with a through-inclusion

4. 1. 1. The object of study and its mathematical model

Let us consider a plate of thickness 2δ, isotropic relative to thermal-physical parameters, with thermally insulated face surfaces $|z| = \pm\delta$. This plate contains a foreign through-inclusion of length 2h. It refers to the rectangular Cartesian coordinate system (0xyz) with the origin in the center of inclusion. In region $\Omega_0=\{(x, -l, z): |x|\le h, |y|\le l, |z|\le\delta\}$ of boundary surface $K_-=\{(x, -l, z): |x|<\infty, |z|\le\delta\}$ of the plate, the system is heated with heat flux whose surface density is q_0=const, and the other part of this surface of the plate beyond the inclusion and surface $K_+=\{(x, l, z): |x|<\infty, |z|\le\delta\}$ are thermally insulated. At the boundary surfaces of inclusion $K_{\pm h}=\{(\pm h, y, z): |y|\le l, |z|\le\delta\}$ is a perfect thermal contact:

$$t_0=t_1,\ \ \lambda_0\frac{\partial t_0}{\partial x}=\lambda_1\frac{\partial t_1}{\partial x}$$

for $|x|=h$ (0 – for the inclusion, 1 – for the plate) (Fig. 1).

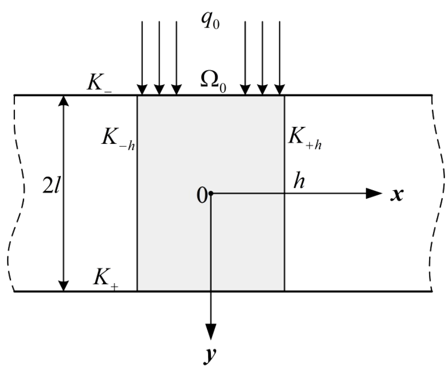

Fig. 1. Cross-section of isotropic plate with a through-inclusion by plane z=0

In the specified structure, it is required to determine the distribution of temperature t(x,y) by spatial coordinates, which we obtain by solving equation of thermal conductivity [10, 11]:

$$\frac{\partial}{\partial x}[\lambda(x)\frac{\partial t}{\partial x}]+\lambda(x)\frac{\partial^2 t}{\partial y^2}=0 \tag{1}$$

with boundary conditions:

$$t\big|_{|x|\to\infty}=t_c,\quad \frac{\partial t}{\partial x}\bigg|_{|x|\to\infty}=0,\quad \frac{\partial t}{\partial y}\bigg|_{y=l}=0,$$

$$\lambda_0\frac{\partial t}{\partial y}\bigg|_{y=-l}=-q_0 S_-\left(h-|x|\right), \tag{2}$$

where $\lambda(x)$ is the coefficient of thermal conductivity of a non-uniform plate,

$$\lambda(x)=\lambda_1+(\lambda_0-\lambda_1)S_-(h-|x|); \tag{3}$$

λ_1 and λ_0 are the coefficients of thermal conductivity of material of the plate and the inclusion, respectively; t_c is the ambient temperature; $S_\pm(\zeta)$ are the asymmetrical single functions [12],

$$S_\pm(\zeta)=\begin{cases}1, & \zeta>0,\\ 0{,}5\mp0{,}5, & \zeta=0,\\ 0, & \zeta<0.\end{cases}$$

Introduce function [13]:

$$T(x,y)=\lambda(x)\theta(x,y) \tag{4}$$

and differentiate it by variables x and y considering expression for the coefficient of thermal conductivity $\lambda(x)$ (3). As a result, we shall obtain:

$$\lambda(x)\frac{\partial\theta}{\partial x}=\frac{\partial T}{\partial x}-(\lambda_0-\lambda_1)[\theta\big|_{x=-h}\delta_-(x+h)-\theta\big|_{x=h}\delta_+(x-h)];$$

$$\lambda(y)\frac{\partial\theta}{\partial y}=\frac{\partial T}{\partial y}. \tag{5}$$

Here

$$\theta(x,y)=t(x,y)-t_c$$

is the excess temperature;

$$\delta_\pm(\zeta)=\frac{dS_\pm(\zeta)}{d\zeta}$$

is the asymmetrical Dirac delta functions [12].

Substituting expressions (5) in relation (1), we arrive at a differential equation with partial derivatives with singular coefficients:

$$\Delta T-(\lambda_0-\lambda_1)[\theta\big|_{x=-h}\delta_-(x+h)-\theta\big|_{x=h}\delta_+(x-h)]=0, \tag{6}$$

where Δ is the Laplace operator in the Cartesian rectangular coordinate system,

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Therefore, the desired temperature field in the presented system is entirely determined by equation (6) with boundary conditions (2).

4. 1. 2. Analytical-numerical solution

Let us approximate function $t(\pm h,y)$ (Fig. 2) by expression

$$t(\pm h,y)=t_1^\pm+\sum_{j=1}^{n-1}(t_{j+1}^\pm-t_j^\pm)S_-(y-y_j), \tag{7}$$

where $y_j\in]-l;l[$, $y_1\le y_2\le\ldots\le y_{n-1}$; $t_j^\pm$ $(j=\overline{1,n})$ are the unknown approximated values of temperature.

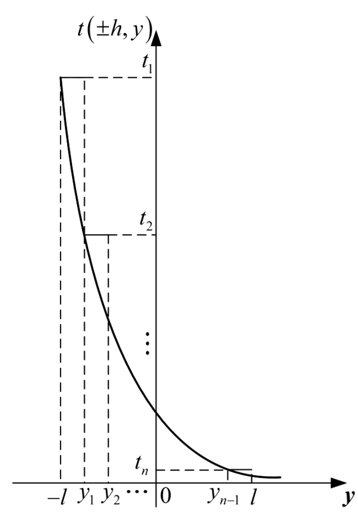

Fig. 2. Approximation of function t (±h,y)

Substituting expression (7) in equation (6), we shall obtain:

$$\Delta T=(\lambda_0-\lambda_1)\Big\{[\theta_1^-+\sum_{j=1}^{n-1}(\theta_{j+1}^--\theta_j^-)S_-(y-y_j)]\delta'_+(x+h)-$$

$$-[\theta_1^++\sum_{j=1}^{n-1}(\theta_{j+1}^+-\theta_j^+)S_-(y-y_j)]\delta'_+(x-h)\Big\}. \qquad (8)$$

Let us apply the integral Fourier transform by the x coordinate to equation (8) and boundary conditions (2) taking into consideration relation (4). Upon solving the obtained boundary problem relative to the representation

$$\bar{T}(\xi,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\xi x}T(x,y)dx$$

of function T(x,y), and then passing over to the original, we shall receive the solution of problem (1), (2) in the form:

$$T(x,y)=\frac{1}{\pi}\int_0^{\infty}\frac{1}{\xi}\{(\lambda_0-\lambda_1)[\sin\xi(x+h)\times$$

$$\times(\theta_1^-+\sum_{j=1}^{n-1}(\theta_{j+1}^--\theta_j^-)(\frac{ch\xi(l+y)}{sh2\xi l}sh\xi(l-y_j)+$$

$$+(1-ch\xi(y-y_j))S_-(y-y_j)))-$$

$$-\sin\xi(x-h)(\theta_1^++\sum_{j=1}^{n-1}(\theta_{j+1}^+-\theta_j^+)$$

$$\times(\frac{ch\xi(l+y)}{sh2\xi l}sh\xi(l-y_j)+$$

$$+(1-ch\xi(y-y_j))S_-(y-y_j)))]+$$

$$+\frac{2q_0}{\xi}\cos\xi x\sin\xi h\frac{ch\xi(y-l)}{sh2\xi l}\}d\xi. \qquad (9)$$

The unknown approximated values $\theta_j^{\pm}$ $(j=\overline{1,n})$ of excess temperature will be found by solving the system of 2n linear algebraic equations obtained from expression (9).

Therefore, the desired temperature field in a plate with a through-inclusion is expressed by formula (9). From this formula we receive temperature value in arbitrary point of the structure "plate – inclusion".

4. 2. Isotropic multi-layer plate with a through-inclusion

4. 2. 1. The object of study and its mathematical model

Let us consider an isotropic layered plate of thickness 2δ with thermally insulated face surfaces $|z|=\delta$. This plate consists of n layers that differ in geometric (width) and thermal-physical (coefficient of thermal conductivity) parameters. It refers to the rectangular Cartesian coordinate system (0xyz) with the origin at one of its boundary surfaces. The plate contains a through-inclusion. At boundary surface $K_0=\{(x, 0, z): |x|<\infty, |z|\le\delta\}$ of the plates in region $\Omega_0=\{(x, 0, z): |x|\le h, |z|\le\delta\}$, the system is heated by the concentrated heat flux with surface density q_0. Another part of this surface beyond the inclusion and boundary surface $K_n=\{(x, y_n, z): |x|<\infty, |z|\le\delta\}$ are thermally insulated. At the surfaces of layers $K_j=\{(x, y_j, z): |x|<\infty, |z|\le\delta\}$ $(j=\overline{1,n-1})$ and inclusion $K_{\pm}=\{(\pm h, y, z): 0\le y\le y_n, |z|\le\delta\}$ is an ideal thermal contact:

$$t_j=t_{j+1},\quad \lambda_j\frac{\partial t_j}{\partial y}=\lambda_{j+1}\frac{\partial t_{j+1}}{\partial y}$$

for $y=y_j$ $(j=\overline{1,n-1})$;

$$t_0=t_j,\quad \lambda_0\frac{\partial t_0}{\partial x}=\lambda_j\frac{\partial t_j}{\partial x}\quad (j=\overline{1,n})$$

for $|x|=h$, where 0 is for the inclusion, j is for the j-th layer of the plate (Fig. 3).

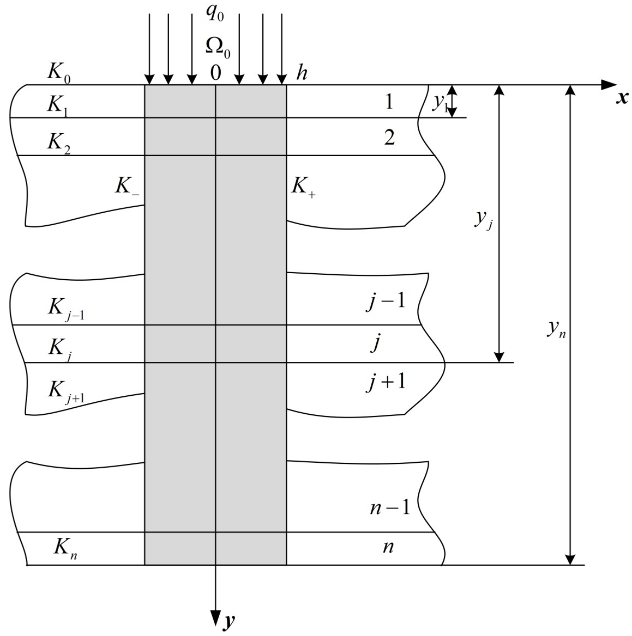

Fig. 3. Cross-section of isotropic multi-layered plate with a through foreign inclusion by plane z=0

In the specified structure, it is necessary to determine the distribution of temperature t(x, y) by spatial coordinates, which we shall receive by solving the equation of thermal conductivity [10, 11]:

$$\frac{\partial}{\partial x}[\lambda(x,y)\frac{\partial t}{\partial x}]+\frac{\partial}{\partial y}[\lambda(x,y)\frac{\partial t}{\partial y}]=0, \qquad (10)$$

$$\theta\Big|_{|x|\to\infty}=0,\ \frac{\partial\theta}{\partial x}\Big|_{|x|\to\infty}=\frac{\partial\theta}{\partial y}\Big|_{y=y_n}=0,$$

$$\lambda_0\frac{\partial\theta}{\partial y}\Big|_{y=0}=-q_0S_-(h-|x|), \qquad (11)$$

where

$$\lambda(x,y)=\sum_{j=1}^{n}\Big[\lambda_j+\big(\lambda_0-\lambda_j\big)S_-(h-|x|)\Big]N(y,y_{j-1}) \qquad (12)$$

is the coefficient of thermal conductivity of a non-uniform plate; λ_j, λ_0 are the coefficients of thermal conductivity of materials of the j-th layer of the plate and the inclusion, respectively;

$$y_0=0;\quad N(y,y_{j-1})=S_-(y-y_{j-1})-S_+(y-y_j).$$

Introduce function [13]:

$$T(x,y)=\lambda(x,y)\theta(x,y) \qquad (13)$$

and differentiate it by variables x and y, considering the expression for coefficient of thermal conductivity (12).

As a result, we shall obtain:

$$\lambda(x,y)\frac{\partial\theta}{\partial x}=\frac{\partial T}{\partial x}+\theta\Big|_{|x|=h}\delta_+\left(|x|-h\right)\sum_{j=1}^{n}\left(\lambda_0-\lambda_j\right)N\left(y,y_{j-1}\right);$$

$$\lambda(x,y)\frac{\partial\theta}{\partial y}=\frac{\partial T}{\partial y}-$$
$$-\sum_{j=1}^{n-1}\left[\lambda_j+\left(\lambda_0-\lambda_j\right)S_-\left(h-|x|\right)\right]\times$$
$$\times\left[\theta\Big|_{y=y_{j-1}}\delta_-\left(y-y_{j-1}\right)-\theta\Big|_{y=y_j}\delta_+\left(y-y_j\right)\right]. \qquad (14)$$

Considering expressions (14), original equation (10) will take the form:

$$\Delta T+\theta\Big|_{|x|=h}\delta'_+\left(|x|-h\right)\sum_{j=1}^{n}\left(\lambda_0-\lambda_j\right)N\left(y,y_{j-1}\right)-$$
$$-\sum_{j=1}^{n-1}\left[\lambda_j+\left(\lambda_0-\lambda_j\right)S_-\left(h-|x|\right)\right]\times$$
$$\times\left[\theta\Big|_{y=y_{j-1}}\delta'_-\left(y-y_{j-1}\right)-\theta\Big|_{y=y_j}\delta'_+\left(y-y_j\right)\right]. \qquad (15)$$

Therefore, the desired temperature field in the presented system is fully determined by equation (15) with boundary conditions (11).

4. 2. 2. Analytical-numerical solution

Approximate functions $\theta(\pm h,y),\theta(x,y_j)$ by expressions [14]:

$$\theta(\pm h,y)=\theta_1^{(jh)}+\sum_{k=1}^{m-1}(\theta_{k+1}^{(jh)}-\theta_k^{(jh)})S_-(y-y_k^{(j)*});$$

$$\theta(x,y_j)=\theta_1^{(j)}+\sum_{l=1}^{p-1}(\theta_{l+1}^{(j)}-\theta_l^{(j)})S_-(x-x_l), \qquad (16)$$

$x\in]-x_*;-h[\cup]h;x_*[;$

$$\theta(x,y_j)=\theta_1^{(j)}+\sum_{l=1}^{t-1}(\theta_{l+1}^{(j)}-\theta_l^{(j)})S_-(x-x_l),\quad x\in\left[-h;h\right];$$

where $y_k^{(j)*}\in]y_{j-1};y_j[$; $y_1^{(j)*}\le y_2^{(j)*}\le\ldots\le y_{m-1}^{(j)*}$; $x_l\in]-x_*;x_*[$, $x_1\le x_2\le\ldots\le x_{p-1}$, $x_1\le x_2\le\ldots\le x_{t-1}$; m, t, p is the number of partitions of intervals $]y_{j-1};y_j[$, $[-h;h]$, $]-x_*;-h[\cup]h;x_*[$ respectively; $\theta_k^{(jh)}$ $(k=\overline{1,m})$, $\theta_l^{(j)}$ $(l=\overline{1,p+t})$, $(j=\overline{1,n})$ are the unknown approximated values of temperature; x_* is the value of the x coordinate, in which temperature t(x, y) is almost equal to ambient temperature t_c.

Substituting expressions (16) in relations (15), we shall receive:

$$\Delta T=\sum_{j=1}^{n}\Bigg\{\Bigg[\lambda_j\left(\theta_1^{(y_{j-1})}+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(y_{j-1})}-\theta_l^{(y_{j-1})}\right)S_-\left(x-x_l\right)\right)+$$
$$+\left(\lambda_0-\lambda_j\right)\left(\theta_1^{(y_{j-1})}S_-\left(h-|x|\right)+\sum_{l=1}^{t-1}\left(\theta_{l+1}^{(y_{j-1})}-\theta_l^{(y_{j-1})}\right)S_-\left(x-x_l\right)\right)\Bigg]\delta'_-\left(y-y_{j-1}\right)-$$
$$-\Bigg[\lambda_j\left(\theta_1^{(y_j)}+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(y_j)}-\theta_l^{(y_j)}\right)S_-\left(x-x_l\right)\right)+$$
$$+\left(\lambda_0-\lambda_j\right)\left(\theta_1^{(y_j)}S_-\left(h-|x|\right)+\sum_{l=1}^{t-1}\left(\theta_{l+1}^{(y_j)}-\theta_l^{(y_j)}\right)S_-\left(x-x_l\right)\right)\Bigg]\delta'_-\left(y-y_j\right)\Bigg\}-$$
$$-\sum_{j=1}^{n}\left(\lambda_0-\lambda_j\right)\left[\theta_1^{(jh)}N\left(y,y_{j-1},y_j\right)+\sum_{k=1}^{m-1}\left(\theta_{k+1}^{(jh)}-\theta_k^{(jh)}\right)N\left(y,y_k^{(j)^*},y_j\right)\right]\times$$
$$\times\left[\delta'_-\left(x+h\right)-\delta'_+\left(x-h\right)\right]. \qquad (17)$$

Let us apply the integral Fourier transform by the *x* coordinate to equation (17) and boundary conditions (11) with regard to ratio (13). Upon solving the obtained boundary problem relative to representation $\overline{T}(\xi,y)$ of function of T(x,y) and then passing over to the original, we shall receive the solution to problem (10), (11) in the form:

$$T=\frac{1}{\pi}\int_0^{\infty}\frac{1}{\xi}\Bigg\langle\Bigg\{\Bigg[\Bigg[\lambda_j\Bigg(2\cos\xi x\sin\xi x^*\theta_1^{(y_{j-1})}+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(y_{j-1})}-\theta_l^{(y_{j-1})}\right)\times$$
$$\times\left(2\cos\xi x\sin\xi x^*+\sin\xi\left(x-x_l\right)-\sin\xi\left(x+x^*\right)\right)\Bigg)+$$
$$+\left(\lambda_0-\lambda_j\right)\Bigg(2\cos\xi x\sin\xi h+\sum_{l=1}^{t-1}\left(\theta_{l+1}^{(y_{j-1})}-\theta_l^{(y_{j-1})}\right)\times$$
$$\times\left(2\cos\xi x\sin\xi x^*+\sin\xi\left(x-x_l\right)-\sin\xi\left(x+x^*\right)\right)\Bigg)\Bigg]\times$$
$$\times\left[ch\xi\left(y-y_{j-1}\right)S_-\left(y-y_{j-1}\right)-\frac{ch\xi y}{sh\xi y_n}sh\xi\left(y_n-y_{j-1}\right)\right]-$$
$$-\Bigg[\lambda_j\Bigg(2\cos\xi x\sin\xi x^*\theta_1^{(y_j)}+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(y_j)}-\theta_l^{(y_j)}\right)+$$
$$+\left(2\cos\xi x\sin\xi x^*+\sin\xi\left(x-x_l\right)-\sin\xi\left(x+x^*\right)\right)\Bigg)+$$
$$+\left(\lambda_0-\lambda_j\right)\Bigg(2\cos\xi x\sin\xi h+\sum_{l=1}^{t-1}\left(\theta_{l+1}^{(y_j)}-\theta_l^{(y_j)}\right)\times$$
$$\times\left(2\cos\xi x\sin\xi x^*+\sin\xi\left(x-x_l\right)-\sin\xi\left(x+x^*\right)\right)\Bigg)\Bigg]\times$$
$$\times\left[ch\xi\left(y-y_j\right)S_+\left(y-y_j\right)-\frac{ch\xi y}{sh\xi y_n}sh\xi\left(y_n-y_j\right)\right]\Bigg]\Bigg\}+$$
$$+2\cos\xi x\sin\xi h\Bigg\{\sum_{i=1}^{n}\left(\lambda_0-\lambda_j\right)\Bigg[\theta_1^{(jh)}\Bigg(\Big(ch\xi\left(y-y_{j-1}\right)S_-\left(y-y_{j-1}\right)-$$
$$-ch\xi\left(y-y_j\right)S_+\left(y-y_j\right)-N\left(y,y_{j-1},y_j\right)\Big)-$$
$$-\frac{ch\xi y}{sh\xi z_n}\left(sh\xi\left(y_n-y_j\right)-sh\xi\left(y_n-y_{j-1}\right)\right)\Bigg)+$$
$$+\sum_{k=1}^{m-1}\left(\theta_{k+1}^{(jh)}-\theta_k^{(jh)}\right)\Bigg(\Big(ch\xi\left(y-y_k^{(j)^*}\right)S_-\left(y-y_k^{(j)^*}\right)-$$
$$-ch\xi\left(y-y_j\right)S_+\left(y-y_j\right)-N\left(y,y_k^{(j)^*},y_j\right)\Big)-$$
$$-\frac{ch\xi y}{sh\xi y_n}\left(sh\xi\left(y_n-y_j\right)-sh\xi\left(y_n-y_k^{(j)^*}\right)\right)\Bigg)\Bigg]+\frac{q_0}{\xi}\frac{ch\xi\left(y-y_n\right)}{sh\xi y_n}\Bigg\}\Bigg\rangle d\xi. \qquad (18)$$

The unknown approximated values of excess temperature $\theta_k^{(jh)}$ $(k=\overline{1,m})$ and $\theta_l^{(j)}$ $(l=\overline{1,p+t})$, $(j=\overline{1,n})$ will be found by solving the system of n(m+p+t) linear algebraic equations obtained from expression (18). Therefore, the desired temperature field in a layered plate with a through-inclusion is expressed by formula (18). From this formula we obtain temperature values in arbitrary point of the structure "layered plate – inclusion".

4. 3. Isotropic thermosensitive plate with a through-inclusion

4. 3. 1. The object of study and its mathematical model

Let us consider a thermosensitive (thermal parameters depend on temperature) plate, isotropic relative to the thermal parameters, which contains a through-inclusion (Fig. 1). With regard to thermal sensitivity of the system at the surfaces of inclusion $K_{\pm}=\{(\pm h,y,z): -l\le y\le l, |z|\le\delta\}$, conditions for the perfect thermal contact will be written in the form:

$$t_0=t_1,\ \lambda_0(t)\frac{\partial t_0}{\partial x}=\lambda_1(t)\frac{\partial t_1}{\partial x}\ \text{ for } |x|=h.$$

The distribution of temperature t(x,y) by spatial coordinates, taking into account thermal sensitivity, will be obtained upon solving the nonlinear equation of thermal conductivity [10, 11]:

$$\frac{\partial}{\partial x}[\lambda(x,t)\frac{\partial t}{\partial x}]+\frac{\partial}{\partial y}\left[\lambda(x,t)\frac{\partial t}{\partial y}\right]=0 \tag{19}$$

with boundary conditions:

$$t\Big|_{|x|\to\infty}=0,\quad \frac{\partial t}{\partial x}\Big|_{|x|\to\infty}=0,\quad \frac{\partial t}{\partial y}\Big|_{y=l}=0,$$

$$\lambda_0(t)\frac{\partial t}{\partial y}\Big|_{y=-l}=-q_0S_-(h-|x|), \tag{20}$$

where $\lambda(x,t)=\lambda_1(t)+[\lambda_0(t)-\lambda_1(t)]S_-(h-|x|)$ is the coefficient of thermal conductivity of a non-uniform thermosensitive plate; $\lambda_0(t)$, $\lambda_1(t)$ are the coefficients of thermal conductivity of materials of the inclusion and the plate, respectively.

Introduce a linearizing function [15, 16]:

$$\vartheta(x,y)=\int_0^{t(x,y)}\lambda_1(\zeta)d\zeta+S_-(x+h)\times$$
$$\times\int_{t(-h,y)}^{t(x,y)}\left[\lambda_0(\zeta)-\lambda_1(\zeta)\right]d\zeta+$$
$$+S_+(x-h)\int_{t(h,y)}^{t(x,y)}\left[\lambda_0(\zeta)-\lambda_1(\zeta)\right]d\zeta, \tag{21}$$

upon differentiating which by variables x and y, we shall obtain:

$$\lambda(t,x)\frac{\partial t}{\partial x}=\frac{\partial\vartheta}{\partial x},$$

$$\lambda(t,x)\frac{\partial t}{\partial y}=\frac{\partial\vartheta}{\partial y}+$$
$$+\left\{\left[\lambda_0(t)-\lambda_1(t)\right]\frac{\partial t}{\partial y}\right\}\Big|_{x=-h}S_-(x+h)-$$
$$-\left\{\left[\lambda_0(t)-\lambda_1(t)\right]\frac{\partial t}{\partial y}\right\}\Big|_{x=h}S_+(x-h). \tag{22}$$

Considering expressions (22), the original equation (19) will take the following form:

$$\Delta\vartheta+\frac{\partial}{\partial y}\left\{\left[\lambda_0(t)-\lambda_1(t)\right]\frac{\partial t}{\partial y}\right\}\Big|_{x=-h}S_-(x+h)-$$
$$-\frac{\partial}{\partial y}\left\{\left[\lambda_0(t)-\lambda_1(t)\right]\frac{\partial t}{\partial y}\right\}\Big|_{x=h}S_+(x-h)=0. \tag{23}$$

Boundary conditions with the use of relation (21) will be written as:

$$\frac{\partial\vartheta}{\partial y}\Big|_{y=l}=0,\quad \vartheta\Big|_{|x|\to\infty}=0,\quad \frac{\partial\vartheta}{\partial x}\Big|_{|x|\to\infty}=0, \tag{24}$$

$$\frac{\partial\vartheta}{\partial y}\Big|_{y=-l}=$$
$$=-\left\{q_0S_-(h-|x|)+\left[\left(\lambda_0(t)-\lambda_1(t)\right)\frac{\partial t}{\partial y}\right]\Big|_{\substack{x=-h\\y=-l}}S_-(x+h)-\right.$$
$$\left.-\left[\left(\lambda_0(t)-\lambda_1(t)\right)\frac{\partial t}{\partial y}\right]\Big|_{\substack{x=h\\y=-l}}S_+(x-h)\right\}. \tag{25}$$

Linearizing function (21) allowed us to reduce a non-linear boundary problem (19), (20) to the partially linearized equation (23) with discontinuous coefficients with boundary conditions (24), (25).

4. 3. 2. Analytical-numerical solution

Let us approximate function $t(\pm h,y)$ by variable y (Fig. 2) with expression (7) and substitute it in relations (23), (25). As a result, we shall receive a linear differential equation with partial derivatives relative to the linearizing function:

$$\Delta\vartheta=$$
$$=-\sum_{j=1}^{n-1}\left(t_{j+1}-t_j\right)\left[\lambda_0(t_{j+1})-\lambda_1(t_{j+1})\right]S_-(h-|x|)\delta_-'\left(y-y_j\right) \tag{26}$$

with boundary condition:

$$\frac{\partial\vartheta}{\partial y}\Big|_{y=-l}=-q_0S_-(h-|x|). \tag{27}$$

Let us apply the integral Fourier transform by the x coordinate to equation (26) and boundary conditions (27). Upon solving the obtained boundary problem relative to representation

$$\bar{\vartheta}(\xi,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vartheta(x,y)e^{i\xi x}dx$$

of function $\vartheta(x,y)$, and then passing over to the original, we shall receive the solution to problem (26), (27) in the form:

$$\vartheta=\frac{2}{\pi}\int_0^{\infty}\frac{1}{\xi}\sin h\xi\cos x\xi\times$$
$$\times\left\{\sum_{j=1}^{n-1}(t_{j+1}-t_j)\left[\lambda_0(t_{j+1})-\lambda_1(t_{j+1})\right]\times\right.$$
$$\times\left[\frac{ch\xi(y+l)}{sh2\xi l}sh\xi\left(l-y_j\right)-\right.$$
$$\left.\left.-ch\xi\left(y-y_j\right)S_-\left(y-y_j\right)\right]+\frac{q_0}{\xi}\frac{ch\xi(y-l)}{sh2\xi l}\right\}d\xi. \tag{28}$$

A system of nonlinear equations will be obtained by using relations (21), (28) to determine the unknown approximated values of temperature t_j $(j=\overline{1,n})$.

The desired temperature field for the specified system will be defined by using the resulting nonlinear equation employing relations (21), (28).

4. 3. 3. A partial example

In order to solve many practical problems, the following dependence of thermal conductivity coefficient on the temperature is applied [17, 18]:

$$\lambda=\lambda_m^0\left(1-k_m t\right), \quad (29)$$

where λ_m^0, k_m is the reference and temperature coefficients of thermal conductivity of materials for an inclusion (m=0) and a plate (m=1).

Considering ratios (29), from expressions (21), (28) we shall obtain formulas for determining the temperature t(x,y):

– in region Ω_0 of the inclusion:

$$t=\frac{1}{k_0}\left(1-\sqrt{1-k_0\left(\frac{2\vartheta}{\lambda_0^0}+\vartheta_1\right)}\right), \quad (30)$$

– in region $\Omega_1=\{(x,y,z):|x|>h,|y|\le l,|z|\le\delta\}$ of the plate (beyond the inclusion):

$$t=\frac{1}{k_1}\left(1-\sqrt{1-\frac{2k_1\vartheta}{\lambda_1^0}}\right). \quad (31)$$

Here

$$\vartheta_1=\left\{t\left[2-k_0 t-\frac{\lambda_1^0}{\lambda_0^0}(2-k_1 t)\right]\right\}\Big|_{x=h};$$

$$t\Big|_{x=h}=\frac{1}{k_1}\left(1-\sqrt{1-\frac{2k_1}{\lambda_1^0}\vartheta\Big|_{x=h}}\right).$$

Formulas (30), (31) completely describe a temperature field in the thermosensitive structure "plate – inclusion".

4. 4. Isotropic thermosensitive multi-layered plate with a through-inclusion

4. 4. 1. The object of study and its mathematical model

Let us consider an isotropic thermosensitive layered infinite plate with a through-inclusion (Fig. 3). With regard to thermal sensitivity of the system at the surfaces of layers $K_j=\{(x, y_j, z)$: $|x|<\infty$, $|z|\le\delta\}$ $(j=\overline{1,n-1})$ and inclusion $K_\pm=\{(\pm h, y, z)$: $0\le y\le y_n$, $|z|\le\delta\}$ of the plate, conditions for the ideal thermal contact will be written in the form:

$$t_j=t_{j+1},\ \ \lambda_j(t)\frac{\partial t_j}{\partial y}=\lambda_{j+1}(t)\frac{\partial t_{j+1}}{\partial y},$$

where $y=y_j$ $(j=\overline{1,n-1})$;

$$t_0=t_j,\ \ \lambda_0(t)\frac{\partial t_0}{\partial x}=\lambda_j(t)\frac{\partial t_j}{\partial x}\ \ (j=\overline{1,n})\ \text{ for } |x|=h.$$

In the specified structure, it is necessary to determine the distribution of temperature t(x, y) by spatial coordinates, which we shall obtain upon solving the nonlinear equation of thermal conductivity (19) with boundary conditions:

$$t\Big|_{|x|\to\infty}=0,\ \frac{\partial t}{\partial x}\Big|_{|x|\to\infty}=\frac{\partial t}{\partial y}\Big|_{y=y_n}=0,$$

$$\lambda_0(t)\frac{\partial t}{\partial y}\Big|_{y=0}=-q_0 S_-(h-|x|), \quad (32)$$

where

$$\lambda(x,y,t)=\sum_{j=1}^{n}\{\lambda_j(t)+[\lambda_0(t)-\lambda_j(t)]S_-(h-|x|)\}N(y,y_{j-1})$$

is the coefficient of thermal conductivity of a non-uniform plate; $\lambda_j(t)$, $\lambda_0(t)$ are the coefficients of thermal conductivity of materials of the j-th layer of the plate and the inclusion, respectively; $y_0=0$; $N(y,y_{j-1})=S_+(y-y_{j-1})-S_+(y-y_j)$.

Introduce a linearizing function [13]:

$$\vartheta(x,y)=\sum_{j=1}^{n}\{N(y,y_{j-1})\int_0^{t(x,y)}\lambda_j(\zeta)d\zeta\ +S_-(x+h)\,[N(y,y_{j-1})\times$$
$$\times\int_{t(-h,y)}^{t(x,y)}(\lambda_0(\zeta)-\lambda_j(\zeta))d\zeta-S_+(y-y_{j-1})\times$$
$$\times\int_{t(-h,y_{j-1})}^{t(x,y_{j-1})}(\lambda_0(\zeta)-\lambda_j(\zeta))d\zeta\ S_+(y-y_{j-1})+$$
$$+S_+(y-y_j)\int_{t(-h,y_j)}^{t(x,y_j)}(\lambda_0(\zeta)-\lambda_j(\zeta))d\zeta\ ]-S_+(x-h)\,[N(y,y_{j-1})\times$$
$$\times\int_{t(h,y)}^{t(x,y)}(\lambda_0(\zeta)-\lambda_j(\zeta))d\zeta-S_+(y-y_{j-1})\int_{t(h,y_{j-1})}^{t(x,y_{j-1})}(\lambda_0(\zeta)-\lambda_j(\zeta))d\zeta\ +$$
$$+S_+(y-y_j)\int_{t(h,y_j)}^{t(x,y_j)}(\lambda_0(\zeta)-\lambda_j(\zeta))d\zeta\ ]-S_+(y-y_{j-1})\times$$
$$\times\int_0^{t(x,y_{j-1})}\lambda_j(\zeta)d\zeta\ +S_+(y-y_j)\int_0^{t(x,y_j)}\lambda_j(\zeta)d\zeta\ \}. \quad (33)$$

By differentiating expression (33) by variables x and y, we shall obtain:

$$\lambda(x,y,t)\frac{\partial t}{\partial x}=\frac{\partial\vartheta}{\partial x}-F_1(x,y),$$

$$\lambda(x,y,t)\frac{\partial t}{\partial y}=\frac{\partial\vartheta}{\partial y}+F_2(x,y), \quad (34)$$

where

$$F_1(x,y)=S_+(|x|-h)\times$$
$$\times\sum_{j=1}^{n}\left\{[(\lambda_0(t)-\lambda_j(t))\frac{\partial t}{\partial x}]\Big|_{y=y_{j-1}}S_+(y-y_{j-1})-\right.$$
$$\left.-[(\lambda_0(t)-\lambda_j(t))\frac{\partial t}{\partial x}]\Big|_{y=y_j}S_+(y-y_j)\right\},$$

$$F_2(x,y)=S_-(h-|x|)\sum_{j=1}^{n}[(\lambda_0(t)-\lambda_j(t))\frac{\partial t}{\partial y}]\Big|_{|x|=h}N(y,y_{j-1}).$$

Considering expressions (34), the original equation (19) takes the following form:

$$\frac{\partial^2\vartheta}{\partial y^2}-\frac{\partial}{\partial x}[F_1(x,y)]+\frac{\partial}{\partial y}[F_2(x,y)]=0. \quad (35)$$

Boundary conditions (32) using ratio (33) will be written as:

$$\vartheta\Big|_{|x|\to\infty}=0,\ \frac{\partial\vartheta}{\partial x}\Big|_{|x|\to\infty}=\frac{\partial\vartheta}{\partial y}\Big|_{y=y_n}=0,\ \frac{\partial\vartheta}{\partial y}\Big|_{y=0}=-q_0 S_-(h-|x|). \quad (36)$$

Linearizing function (33) allowed us to reduce a nonlinear equation of thermal conductivity (19) to the partially linearized equation (35) with discontinuous coefficients and completely linearized boundary conditions (36).

4. 4. 2. Analytical-numerical solution

Let us approximate functions $t(\pm h, y)$, $t(x, y_j)$ by expressions:

$$t(\pm h,y)=t_1^{(jh)}+\sum_{k=1}^{m-1}(t_{k+1}^{(jh)}-t_k^{(jh)})S_-(y-y_k^{(j)*}),$$

$$t(x,y_j)=t_1^{(j)}+\sum_{l=1}^{p-1}(t_{l+1}^{(j)}-t_l^{(j)})S_-(x-x_l), \qquad (37)$$

where $y_k^{(j)*}\in]y_{j-1};\ y_j[$; $y_1^{(j)*}\le y_2^{(j)*}\le\ldots\le y_{m-1}^{(j)*}$; $x_l\in]h;\ x_*[$; $x_1\le x_2\le\ldots\le x_{p-1}$; m, p is the number of partitions of intervals $]y_{j-1};y_j[$ and $]h;x_*[$, respectively; $t_k^{(jh)}$ $(k=1,2,\ldots,m)$, $t_l^{(j)}$ $(l=\overline{1,p})$ are the unknown approximated values of temperature; x_* is the value of the x coordinate, in which temperature is almost equal to zero (to be found from the corresponding linear boundary problem).

By substituting expressions (37) in relation (35), we shall obtain a linear differential equation with partial derivatives relative to linearizing function $\vartheta(x,y)$:

$$\Delta\vartheta=\sum_{j=1}^{n}[\sum_{l=1}^{p-1}F_j^{(l)}(y)\delta_-'(x-x_l)-S_-(h-|x|)\sum_{k=1}^{m-1}F_j^{(k)}(y)]. \qquad (38)$$

Here

$$F_j^{(k)}(y)=(t_{k+1}^{(jh)}-t_k^{(jh)})[\lambda_0(t_{k+1}^{(jh)})-\lambda_j(t_{k+1}^{(jh)})]\delta_-'(y-y_k^{(j)*});$$

$$F_j^{(l)}(y)=(t_{l+1}^{(j-1)}-t_l^{(j-1)})[\lambda_0(t_{l+1}^{(j-1)})-\lambda_j(t_{l+1}^{(j-1)})]S_+(y-y_{j-1})-$$
$$-(t_{l+1}^{(j)}-t_l^{(j)})[\lambda_0(t_{l+1}^{(j)})-\lambda_j(t_{l+1}^{(j)})]S_+(y-y_j).$$

Let us apply the integral Fourier transform by the x coordinate to equation (38) and boundary conditions (36). Upon solving the obtained boundary problem relative to representation $\bar{\vartheta}(\xi,y)$ of function $\vartheta(x,y)$, and then passing over to the original, we shall receive the solution to problem (38), (36) in the form:

$$\vartheta=\frac{1}{\pi}\int_0^\infty\frac{1}{\xi}\Bigg\{\sum_{j=1}^{n}\Bigg[\sum_{l=1}^{p-1}\sin\xi(x-x_l)\Bigg(\Bigg(\Big(\big(1-\mathrm{ch}\xi(y-y_{j-1})\big)S_+(y-y_{j-1})+$$
$$+\frac{\mathrm{ch}\xi y}{\mathrm{sh}\xi y_n}\mathrm{sh}\xi(y_n-y_{j-1})\Big)\Big(t_{l+1}^{(j-1)}-t_l^{(j-1)}\Big)\Big(\lambda_0(t_{l+1}^{(j-1)})-\lambda_j(t_{l+1}^{(j-1)})\Big)-$$
$$-\big(1-\mathrm{ch}\xi(y-y_j)\big)S_+(y-y_j)+$$
$$+\frac{\mathrm{ch}\xi y}{\mathrm{sh}\xi y_n}\mathrm{sh}\xi(y_n-y_j)\Bigg)\Big(t_{l+1}^{(j)}-t_l^{(j)}\Big)\Big(\lambda_0(t_{l+1}^{(j)})-\lambda_j(t_{l+1}^{(j)})\Big)\Bigg)-$$
$$-2\sin\xi h\cos\xi x\sum_{k=1}^{m-1}\Big(t_{k+1}^{(jh)}-t_k^{(jh)}\Big)\Big(\lambda_0(t_{k+1}^{(jh)})-\lambda_j(t_{k+1}^{(jh)})\Big)\times$$
$$\times\Big(\mathrm{ch}\,\xi\big(y-y_k^{(j)^*}\big)S_-\big(y-y_k^{(j)^*}\big)-\Big(t_{l+1}^{(j)}-t_l^{(j)}\Big)\Big(\lambda_0(t_{k+1}^{(jh)})-\lambda_j(t_{k+1}^{(jh)})\Big)$$
$$-\frac{\mathrm{ch}\xi y}{\mathrm{sh}\xi y_n}\mathrm{sh}\xi(y_n-y_k^{(j)^*})\Big)\Bigg]+\frac{2q_0}{\xi}\sin\xi h\cos\xi x\frac{\mathrm{ch}\xi(y-y_n)}{\mathrm{sh}\xi y_n}\Bigg\}d\xi. \qquad (39)$$

A system of nonlinear equations will be obtained using relations (33), (39) for determining the unknown approximated values of temperature $t_k^{(jh)}$ $(k=\overline{1,m})$ and $t_l^{(j)}$ $(l=\overline{1,p})$.

The desired temperature field for the specified structure will be defined by using the resulting nonlinear equation with ratios (33), (39).

4. 4. 3. A partial example

Let us apply a dependence of thermal conductivity coefficient on the temperature in the form:

$$\lambda_s=\lambda_s^0(1-k_s t), \qquad (40)$$

where λ_s^0, k_s are the reference and temperature coefficients of thermal conductivity of materials for the inclusion (s=0) and the j-th layer of the plate (s=j), $j=\overline{1,n}$. From expressions (33), (39) we shall obtain formulas for determining the temperature t(x,y) for a two-layer plate (n=2) in region $\Omega_1=\{(x,y): |x|>h,\ 0\le y<y_1\}$ of the first layer beyond the inclusion:

$$t=\frac{1-\sqrt{1-2\frac{k_1}{\lambda_1^0}(\vartheta+\vartheta_1)}}{k_1}, \qquad (41)$$

in region $\Omega_2=\{(x,y): |x|>h,\ y_1\le y<y_2\}$ of the second layer beyond the inclusion:

$$t=\frac{1-\sqrt{1-2\frac{k_2}{\lambda_2^0}(\vartheta+\vartheta_2)}}{k_2}, \qquad (42)$$

in region $\Omega_3=\{(x,y): |x|\le h,\ 0\le y<y_1\}$ of the inclusion of the first layer:

$$t=\frac{1-\sqrt{1-2\frac{k_0}{\lambda_0^0}(\vartheta+\vartheta_3)}}{k_0}, \qquad (43)$$

in region $\Omega_4=\{(x,y): |x|\le h,\ y_1\le y\le y_2\}$ of the inclusion of the second layer:

$$t=\frac{1-\sqrt{1-2\frac{k_0}{\lambda_0^0}(\vartheta+\vartheta_4)}}{k_0}. \qquad (44)$$

Here

$$\vartheta_1=\lambda_1^0\left[(1-\frac{k_1}{2}t)t\right]\Bigg|_{y=0};$$

$$\vartheta_2=\vartheta_m+\vartheta_1;\quad \vartheta_m=\left[(\lambda_2^0-\lambda_1^0+\frac{\lambda_1^0k_1-\lambda_2^0k_2}{2}t)t\right]\Bigg|_{y=y_1};$$

$$\vartheta_3=\vartheta_v^{(1)}-\vartheta_v^{(1)}\Big|_{y=0}+\vartheta_0;\ \vartheta_0=\lambda_0^0\left[(1-\frac{k_0}{2}t)t\right]\Bigg|_{y=0};$$

$$\vartheta_v^{(i)}=\left[(\lambda_0^0-\lambda_i^0+\frac{\lambda_i^0k_i-\lambda_0^0k_0}{2}t)t\right]\Bigg|_{|x|=h} \quad i=1,2;$$

$$\vartheta_4=\vartheta_v^{(2)}-\vartheta_v^{(1)}\Big|_{y=0}+\vartheta_m\Big|_{|x|=h}+\vartheta_0;$$

the values of temperature $t(x,0)$ will be found by formula (31); $t(\pm h,y)$, $t(x,y_1)$ – by formula (41).

Formulas (41), (44) fully describe a temperature field in the thermosensitive two-layered infinite plate with a foreign through inclusion.

5. Analysis of the obtained numerical results

Let us consider a two-layered plate with uniformly distributed heat sources at the surfaces of layer interface (Fig. 4). Suppose that at the boundary surfaces of plate $y=-y_1$, $y=y_2$, temperature is $t_1=0$ °C, $t_2=700$ °C, respectively. Material of the plate's layers is steel U12 and 08.

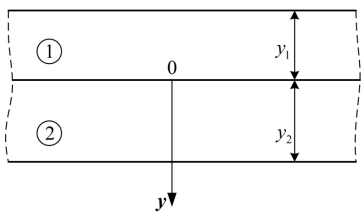

Fig. 4. Two-layered plate

In the temperature range of [0 °C; 700 °C] these materials are described by the following dependences of thermal conductivity coefficient on the temperature:

$$\lambda_1(t)=47{,}5\frac{W}{Km}\left(1-0{,}00037\frac{1}{K}t\right),$$

$$\lambda_2(t)=64{,}5\frac{W}{Km}\left(1-0{,}00049\frac{1}{K}t\right). \qquad (45)$$

We performed numerical calculations of temperature field for a linear model (constant thermal conductivity coefficient of materials of layers of the plate; λ_1=38.7W/(Km), λ_2= =48.7W/(Km)) (Fig. 5, curve 1). The distribution of temperature for a nonlinear model (linearly variable thermal conductivity coefficient of materials of layers of the plate, expressed by ratios (45)) is shown in Fig. 5 (curve 2); $y_1=y_2=1$ m.

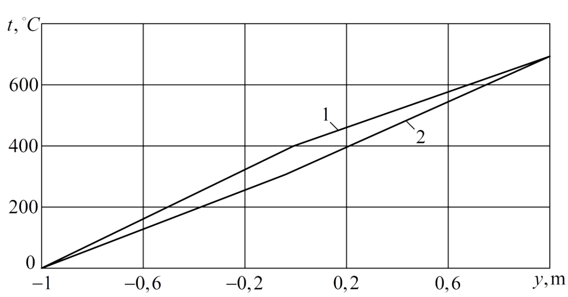

Fig. 5. Dependence of temperature t on the y coordinate for a stable (curve 1) and linearly variable (curve 2) coefficient of thermal conductivity of materials of layers of the plate

Behavior of the curves indicates conformity of the mathematical model with a real physical process because at the surfaces of layer interface of the plates (x=0) we observe how conditions for an ideal thermal contact are satisfied (temperature jump is missing). The results obtained for the chosen materials by a linear dependence of thermal conductivity coefficient on the temperature differ from the results obtained for a stable coefficient of thermal conductivity by 15 %.

6. Discussion of results of examining the mathematical models for the thermal conductivity process

In the process of developing and examining the linear and nonlinear mathematical models of the thermal conductivity process for designs that are geometrically described by the presented piecewise uniform structures, we established that the numerical results of temperature field for the examined materials in the case of a stable thermal conductivity coefficient and a linearly variable one differ by 15 %. This indicates that taking into account the dependence of thermal-physical parameters on the temperature of materials of design elements in complex systems is important, as the results obtained with the use of nonlinear models are more accurate. Important in the studies presented is also the consideration of piecewise-uniform structure of the elements in designs, which considerably complicates the solution of the appropriate linear and nonlinear boundary problems, but the solutions to these problems describe the distribution of temperature more adequately in terms of real process.

7. Conclusions

1. We developed a mathematical model for calculating the temperature field in an isotropic plate with a through-inclusion. The analytical-numerical solution constructed for the entire system as a single entity allows us to analyze the distribution of temperature in the inclusion and in the plate using spatial coordinates.

2. A mathematical model for the calculation of temperature field in an isotropic layered plate with a through-inclusion is built. The analytical-numerical solution constructed for the entire system as a single entity makes it possible to analyze the distribution of temperature in the inclusion and in the layers of plate using spatial coordinates.

3. We devised a non-linear mathematical model for the calculation of temperature field in a thermosensitive isotropic plate with a through-inclusion. A linearizing function is introduced, which allowed us to linearize the original non-linear boundary problem on thermal conductivity and obtain, for a linearly variable thermal conductivity coefficient, calculation formulas for determining the temperature field in an inclusion and in a plate.

4. A non-linear mathematical model for the calculation of temperature field in a thermosensitive isotropic layered plate with a through-inclusion is developed. We introduced a linearizing function, which made it possible to linearize the original non-linear boundary problem on thermal conductivity and receive, for a linearly variable thermal conductivity coefficient, calculation formulas for determining the temperature field in an inclusion and in the layers of plate.

References

1. Golicyna, E. V. Kvazistacionarnaja trehmernaja zadacha teploprovodnosti vo vrashhajushhemsja sploshnom cilindre iz kompozicionnogo materiala s nelinejnymi granichnymi uslovijami [Text] / E. V. Golicyna, Ju. R. Osipov // Konstrukcii iz kompozicionnyh materialov. – 2007. – Issue 4. – P. 47–58.

2. Kudryashov, N. A. Priblizhennye resheniya odnomernyh zadach nelinejnoj teploprovodnosti pri zadannom potoke [Text] / N. A. Kudryashov, V. A. Chmyhov // Zhurnal vychislitel'noj matematiki i matematicheskoj fiziki. – 2007. – Vol. 47, Issue 1. – P. 110–120.
3. Kudinov, V. A. Analiz nelinejnoj teploprovodnosti na osnove opredelenija fronta temperaturnogo vozmushhenija [Text] / V. A. Kudinov, B. V. Averin, E. V. Stefanjuk, S. A. Nazarenko // Teplofizika vysokih temperatur. – 2006. – Vol. 44, Issue 4. – P. 577–585.
4. Carpinteri, A. Thermo-elastic mismatch in nonhomogeneous beams [Text] / A. Carpinteri, M. Paggi // Journal of Engineering Mathematics. – 2008. – Vol. 61, Issue 2-4. – P. 371–384. doi: 10.1007/s10665-008-9212-8
5. Xu, Y. Analysis of Steady Thermal Stress in a ZrO2/FGM/Ti-6Al-4V Composite ECBF Plate with Temperature-Dependent Material Properties by NFEM [Text] / Y. Xu, D. Tu // 2009 WASE International Conference on Information Engineering. – 2009. doi: 10.1109/icie.2009.73
6. Ghannad, M. A thermoelasticity solution for thick cylinders subjected to thermo-mechanical loads under various boundary conditions [Text] / M. Ghannad, M. P. Yaghoobi // International Journal of Advanced Design & Manufacturing Technology. – 2015. – Vol. 8, Issue 4. – P. 1–12.
7. Jabbari, M. Radially Symmetric Steady State Thermal and Mechanical Stresses of a Poro FGM Hollow Sphere [Text] / M. Jabbari, S. Karampour, M. R. Eslami // ISRN Mechanical Engineering. – 2011. – Vol. 2011. – P. 1–7. doi: 10.5402/2011/305402
8. Bayat, A. Thermo-mechanical analysis of functionally graded thick spheres with linearly time-dependent temperature [Text] / A. Bayat, H. Moosavi, Y. Bayat // Scientia Iranica. – 2015. – Vol. 22, Issue 5. – P. 1801–1812.
9. Mohazzab, A. H. Two-Dimensional Stresses in a Hollow FG Sphere with Heat Source [Text] / A. H. Mohazzab, M. Jabbari // Advanced Materials Research. – 2011. – Vol. 264-265. – P. 700–705. doi: 10.4028/www.scientific.net/amr.264-265.700
10. Podstrigach, Ja. S. Termouprugost' tel neodnorodnoj struktury [Text] / Ja. S. Podstrigach, V. A. Lomakin, Ju. M. Koljano. – Moscow: Nauka, 1984. – 368 p.
11. Koljano, Ju. M. Metody teploprovodnosti i termouprugosti neodnorodnogo tela [Text] / Ju. M. Koljano. – Kyiv: Naukova dumka, 1992. – 280 p.
12. Korn, G. Spravochnik po matematike dlja nauchnyh rabotnikov i inzhenerov [Text] / G. Korn, T. Korn. – Moscow: Nauka, 1977. – 720 p.
13. Gavrysh, V. I. Modeljuvannja temperaturnyh rezhymiv u kuskovo – odnoridnyh strukturah [Text] / V. I. Gavrysh, D. V. Fedasjuk. – Lviv: V-vo Nac. un-tu «L'vivs'ka politehnika», 2012. – 176 p.
14. Gavrysh, V. I. Chislenno – analiticheskoe reshenie nelinejnoj stacionarnoj zadachi teploprovodnosti dlja beskonechnoj termochuvstvitel'noj mnogoslojnoj plastiny [Text] / V. I. Gavrysh // Elektronnoe modelirovanie. – 2014. – Vol. 36, Issue 3. – P. 59–70.
15. Gavrysh, V. I. Doslidzhennja temperaturnyh rezhymiv u termochutlyvij plastyni z chuzhoridnym naskriznym vkljuchennjam [Text] / V. I. Gavrysh // Fizyko-matematychne modeljuvannja ta informacijni tehnologii'. – 2013. – Issue 18. – P. 43–50.
16. Gavrysh, V. I. Nelinijna krajova zadacha teploprovidnosti dlja sharuvatoi' plastyni z vkljuchennjam [Text] / V. I. Gavrysh // Fizyko- himichna mehanika materialiv. – 2015. – Vol. 51, Issue 3. – P. 32–38.
17. Lomakin, V. A. Teorija uprugosti neodnorodnyh tel [Text] / V. A. Lomakin. – Moscow: Izd- vo Mosk. un- ta, 1976. – 376 p.
18. Berman, R. Teploprovodnost' tverdyh tel [Text] / R. Berman. – Moscow: Mir, 1979. – 288 p.