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A STUDY OF SYNCHRONIZATION PROCESSES OF NONLINEAR SYSTEMS IN THE DIFFERENCE SPACE OF PHASE VARIABLES

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Приведений аналіз розв'язків звичайних диференційних рівнянь з класифікацією фазових траєкторій. З використанням матриці синхронізації проведений аналіз процесів синхронізації систем Реслера. Встановленні комбінації елементів матриці, при яких реалізується повна, фазова та топологічна синхронізації систем. Показано, що для систем з нелінійною динамікою може мати місце топологічна синхронізація навіть у випадку відсутності зв'язку між ними

Ключові слова: система Реслера, повна та фазова синхронізація, матриця синхронізації, топологічна синхронізація

Приведен анализ решений обыкновенных дифференциальных уравнений с классификацией фазовых траекторий. С использованием матрицы синхронизации проведен анализ процессов синхронизации систем Реслера. Установлены комбинации элементов матрицы, при которых реализуется полная, фазовая и топологическая синхронизации систем. Показано, что для систем с нелинейной динамикой может иметь место топологическая синхронизация даже в случае отсутствия связи между ними

Ключевые слова: система Ресслера, полная и фазовая синхронизация, матрица синхронизации, топологическая синхронизация

1. Introduction

Due to the prospects of using systems with nonlinear dynamics in information and communication networks, the study of synchronous work of systems with nonlinear dynamics is important and promising in various fields of modern science [1], despite the fact that the methods of solving differential equations with nonlinear functions are already known. It should be noted that the known methods for solving differential equations by linearization technique do not exclude the possibility of having incorrect solutions in the process of system buckling analysis. Lyapunov and Sylvester stability criterion is one of the best known criteria.

In addition, the use of linearization techniques provides the study of only complete synchronization of the two systems, the essence of which is that over time solutions of the main and controlled systems are identical. Herewith, the phase space will have a fixed point.

Phase synchronization will occur in case of solutions to differential equations that describe the behaviour of the main and controlled system and form closed trajectories in the phase space.

The case when the solutions to the systems of differential equations with nonlinear right-hand side form an attractor in the phase space is feasible for systems with nonlinear dynamics. Synchronization is not possible in case of an unlimited increase of the distance between phase trajectories.

2. Literature review and problem statement

The study of synchronization of systems with nonlinear dynamics is in the scope of interest of experts in various scientific and technical fields.

In [1, 2] there is given the study of synchronization of magnetically coupled electronic circuits, the dynamics of which is subject to Colpitt's and Van der Pol's models. The authors consider only complete or phase synchronization. All possible variants of interaction between magnetically coupled electronic circuits may appear to be important in the analysis of unauthorized information leakage by means of interaction through an electromagnetic field.

In [3, 4] the method of checking the controlled system is considered, and the processes of complete synchronization are investigated. The authors emphasize the importance of controlling the process of synchronization in the process of designing powerful data processing systems. At the same time, they were using classical methods of controlling synchronization that are based on constructing Lyapunov functions. Herewith, more complicated cases of synchronization, namely the topological nature of the phenomenon, are not taken into account.

In [5] there is given a generalized approach to description of coherent motion of the two systems, and comparison of phase and complete synchronization is made. In accordance with the elaborated concept, there exists a weaker type of coherent motion – topological synchronization.

The authors of [6] investigated the synchronization of Rössler and Rikitaki systems only in the case of complete synchronization by means of the method of Lyapunov function construction. Other types of synchronization were not considered.

In [7] there is given an analysis of the so-called "projective synchronization" on the example of Liu system. Herewith, there is introduced a special scale factor that helps analyze the possible synchronization error, caused by a linear combination of variables of the main and controlled systems, the behaviour of which can be nonlinear.

The literature review shows that a significant number of problems in the theory of nonlinear systems can be solved in the framework of the theory of linear approximation. Herewith, more complicated types of synchronization, unlike phase and complete one, can not be considered with the help of methods of linearization of differential equations.

3. The aim and objectives of the study

The aim of the paper is to study the behaviour of systems with nonlinear dynamics in the phase space and establish complete, phase and topological synchronization between them.

In achieving this aim, the following objectives have been addressed:

– to classify the solutions of ordinary differential equations that describe the behaviour of systems with nonlinear dynamics;

– to improve the model of synchronized systems by means of applying matrix synchronization;

– to study the so-called topological synchronization of systems with nonlinear dynamics according to the obtained attractors in the phase space.

4. Modelling synchronized systems with the help of synchronization matrix

Research of synchronous oscillations in systems with nonlinear dynamics is a fairly complex mathematical problem.

We shall consider autonomous systems for which in differential equations there are no terms that are functions of time:

$$\begin{cases} \dot{x}_1 = F_1(x_1, x_2, \ldots, x_{n-1}, x_n), \\ \ldots \\ \dot{x}_n = F_n(x_1, x_2, \ldots, x_{n-1}, x_n), \end{cases} \quad (1)$$

where $(x_1, x_2, ..., x_n)$ – variables of the system; $(F_1, F_2, ..., F_n)$ – certain functions from the variables of the system.

Further, we shall consider differential equations with three variables. We shall analyze the properties of solutions to the system by the given classification (Fig. 1), which can be used to study the processes of synchronization of two dynamic systems with nonlinear dynamics.

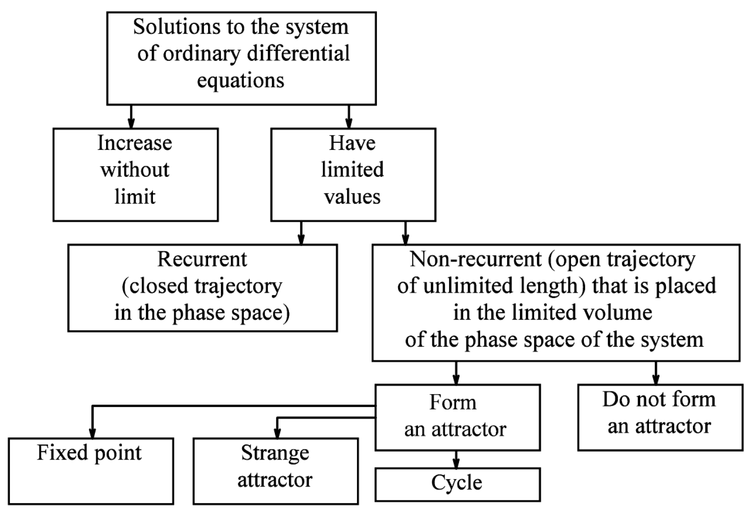


Fig. 1. Classification of solutions to the systems of ordinary differential equations

The solutions to systems of linear differential equations can be found by the roots of the characteristic equation.

Under such conditions, solutions to the system (1) can be classified according to the following properties (Fig. 1):

– unlimited exponential growth in time with positive real part of the exponential index;

– frequency for the case of existence of complex conjugated roots of the characteristic equation, the real part of which equals zero;

– exponential time dependency with negative real part of the exponential index with an attractor formed as a fixed point;

– form a cyclic attractor – if solutions are a sum of exponentially time-dependent solutions and trigonometric functions.

If the right-hand side of differential equations contains nonlinear functions, the following two types of solutions can occur:

– limited non-recurrent solutions that do not form an attractor;

– limited non-recurrent solutions that form a strange attractor.

The latter two types of solutions, which, by their nature, are complex nonlinear oscillations, are the most promising for practical application.

Let us consider one of the known methods of studying the synchronization process, the essence of which is to replace the variables in the right-hand side of the differential equations that describe the controlled system, with the corresponding variable values of the main system [2]. This method will be further called a replacement method.

The generalized mathematical model for synchronizing two identical systems with nonlinear dynamics, which is realized by replacing variables, can be set by a system (2) of $2\times n$ equations, where n is the order of synchronized systems described by equations (1).

$$\begin{cases} \dot{x}_{01} = F_1\left(x_{01}, x_{02}, \ldots, x_{0(n-1)}, x_{0n}\right), \\ \ldots \\ \dot{x}_{0n} = F_n\left(x_{01}, x_{02}, \ldots, x_{0(n-1)}, x_{0n}\right), \\ \dot{x}_{11} = F_1\left(x_{01}\cdot\delta_{11} + x_{11}\cdot(1-\delta_{11}), x_{02}\cdot\delta_{12} + x_{12}\cdot(1-\delta_{12}), \ldots, x_{0n}\cdot\delta_{1n} + x_{1n}\cdot(1-\delta_{1n})\right), \\ \ldots \\ \dot{x}_{1n} = F_n\left(x_{01}\cdot\delta_{n1} + x_{11}\cdot(1-\delta_{n1}), x_{02}\cdot\delta_{n2} + x_{12}\cdot(1-\delta_{n2}), \ldots, x_{0n}\cdot\delta_{nn} + x_{1n}\cdot(1-\delta_{nn})\right). \end{cases} \quad (2)$$

In differential equations (2) variables x_{0j} are variables of the main system, and variables x_{1j} are variables of the controlled system. Synchronization matrix $\|\delta_{ij}\|$ with $n\times n$ dimension determines the choice of variables, according to which the main and the controlled systems are synchronized. The elements of the matrix are Kronecker symbols that can take on a value of "1" – if there occurs replacement of variables of the controlled system, otherwise – "0" – if no replacement occurs. The first index of the Kronecker symbol in the synchronization matrix means the number of the equation, and the second – the number of the variable in this equation.

Research of synchronization of the two systems with nonlinear dynamics is convenient to be carried out in the phase space, the coordinates of which are presented by the difference of values of the corresponding random oscillations of the main and controlled systems:

$$\begin{cases} \dot{e}_{11} = F_1\left(x_{01}\cdot\delta_{11} + x_{11}\cdot(1-\delta_{11}), x_{02}\cdot\delta_{12} + x_{12}\cdot(1-\delta_{12}), \ldots, x_{0n}\cdot\delta_{1n} + x_{1n}\cdot(1-\delta_{1n})\right) - F_1\left(x_{01}, x_{02}, \ldots, x_{0(n-1)}, x_{0n}\right), \\ \ldots \\ \dot{e}_{1n} = F_n\left(x_{01}\cdot\delta_{n1} + x_{11}\cdot(1-\delta_{n1}), x_{02}\cdot\delta_{n2} + x_{12}\cdot(1-\delta_{n2}), \ldots, x_{0n}\cdot\delta_{nn} + x_{1n}\cdot(1-\delta_{nn})\right) - F_n\left(x_{01}, x_{02}, \ldots, x_{0(n-1)}, x_{0n}\right), \end{cases} \quad (3)$$

where $e_i = x_{1i} - x_{0i}$.

Herewith, the study of the synchronization process can be reduced to the solution of the problem of stability by Lyapunov by the technique of a linearized system [1]. Lyapunov stability theory postulates the necessary and sufficient conditions for the stability of zero solution to the system of differential equations by real parts of the eigenvalues of the linearized system. Provided that the real parts of the eigenvalues are negative, the solution to the system is stable for small disturbances. This corresponds to the case of the so-called complete synchronization at which, over time, the systems will have similar solutions.

To study the systems with nonlinear dynamics, it is necessary to consider the options of processes of interaction between the two systems, in which not only complete, but also more complicated types of synchronization occur.

5. The study of synchronization in two Rössler systems with similar parameters

Establishing of synchronous oscillations and their stability were studied on the example of Rössler system, which is one of the simplest systems capable of generating complex oscillations and, under certain conditions, forming a strange attractor. The mentioned system is a third-order system with one nonlinearity (4):

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + a\cdot y, \\ \dot{z} = b + (x - r)\cdot z. \end{cases} \quad (4)$$

Nonlinear oscillations occurred when the values of the parameters of the system (4) were a=b=0.2 and r=5.7. Herewith, solutions to the system (4) form a band attractor of the Rössler system (Fig. 2).

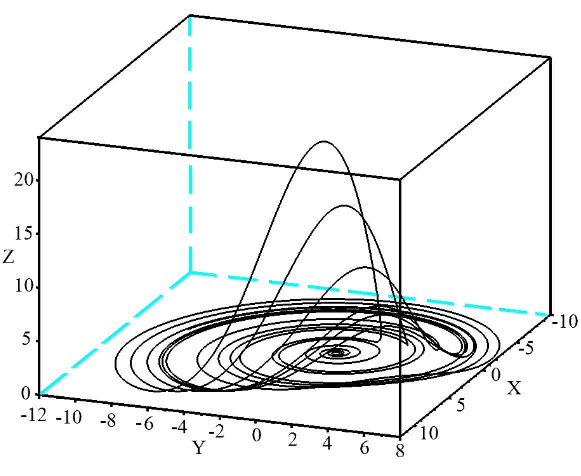
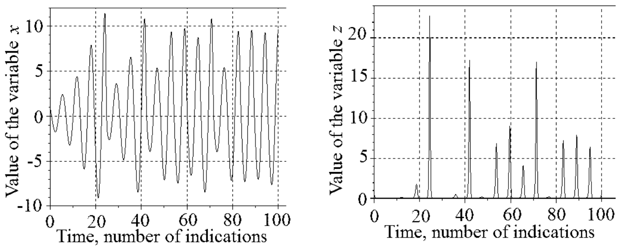


Fig. 2. Band attractor of the Rössler system

The research was conducted in the following sets of solutions to the system of equations

$$x\in[-9.1;11.43];\quad y\in[-10.8;7.8];\quad z\in[0.01;22.8].$$

Time diagrams of variables x and z of the Rössler systems are shown in Fig. 3.



a *b*

Fig. 3. Time diagrams: *a* — variables x of the Rössler system; *b* — variables *z* of the Rössler system

Generalized differential equations that describe the coupled main and controlled systems with nonlinear dynamics can be written as follows:

$$\begin{cases}\dot{x}_0=-y_0-z_0,\\ \dot{y}_0=x_0+0.2\cdot y_0,\\ \dot{z}_0=0.2+(x_0-5.7)\cdot z_0,\\ \dot{x}_1=-\delta_{12}\cdot y_0-(1-\delta_{12})\cdot y_1-\delta_{13}\cdot z_0-(1-\delta_{13})\cdot z_1,\\ \dot{y}_1=\delta_{21}\cdot x_0+(1-\delta_{21})\cdot x_1+0.2\cdot\delta_{22}\cdot y_0+0.2\cdot(1-\delta_{22})\cdot y_1,\\ \dot{z}_1=0.2+(\delta_{31}\cdot x_0+(1-\delta_{31})\cdot x_1)\cdot(\delta_{33}\cdot z_0+(1-\delta_{33})\cdot z_1)-5.7\cdot\delta_{33}\cdot z_0-5.7\cdot(1-\delta_{33})\cdot z_1,\end{cases}\quad (5)$$

where (x_0, y_0, z_0) and (x_1, y_1, z_1) – values of chaotic oscillations generated by the main and controlled systems, δ_{ij} – Kronecker symbols

The order of the synchronization matrix of the system (5) will be as follows:

$$\left\|\delta_{ij}\right\|=\begin{Vmatrix} * & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & *\\ \delta_{31} & * & \delta_{33}\end{Vmatrix},\quad (6)$$

where insignificant elements of the matrix are marked with *.

Let us consider the variables that are equal to the difference of values of oscillations, generated by the main and controlled systems. Then the system (5) will be as follows:

$$\begin{cases}\dot{e}_x=-(1-\delta_{12})\cdot e_y-(1-\delta_{13})\cdot e_z,\\ \dot{e}_y=(1-\delta_{21})\cdot e_x+0.2\cdot(1-\delta_{22})\cdot e_y,\\ \dot{e}_z=(\delta_{31}\cdot x_0+(1-\delta_{31})\cdot x_1)\cdot(\delta_{33}\cdot z_0+(1-\delta_{33})\cdot z_1)-\\ -5.7\cdot(1-\delta_{33})\cdot e_z-x_0\cdot z_0.\end{cases}\quad (7)$$

In the linear approximation, the phase space that corresponds to the above-listed variables will be called the difference space. Neglecting the term $(1-\delta_{13})\cdot(1-\delta_{31})\cdot e_x\cdot e_z$, we will obtain a linearized system (8), which has the following matrix form (9):

$$\begin{cases}\dot{e}_x=-(1-\delta_{12})\cdot e_y-(1-\delta_{13})\cdot e_z,\\ \dot{e}_y=(1-\delta_{21})\cdot e_x+0.2\cdot(1-\delta_{22})\cdot e_y,\\ \dot{e}_z=(1-\delta_{31})\cdot z_0\cdot e_x+(1-\delta_{33})\cdot(x_0-5.7)\cdot e_z,\end{cases}\quad (8)$$

$$\begin{Vmatrix}\dot{e}_x\\ \dot{e}_y\\ \dot{e}_z\end{Vmatrix}=\begin{Vmatrix}0 & -(1-\delta_{12}) & -(1-\delta_{13})\\ (1-\delta_{21}) & 0.2\cdot(1-\delta_{22}) & 0\\ (1-\delta_{31})\cdot z_0 & 0 & (1-\delta_{33})\cdot(x_0-5.7)\end{Vmatrix}\times\begin{Vmatrix}e_x\\ e_y\\ e_z\end{Vmatrix}.\quad (9)$$

Let us consider the process of synchronization of the main and controlled oscillators in the case of replacing all variables of the controlled oscillator $\delta_{12}=\delta_{13}=\delta_{21}=\delta_{22}=\delta_{31}=\delta_{33}=1$.

In this case, the system (9) has a sustainable over time trivial solution:

$$e_x=C_x;\ e_y=C_y;\ e_z=C_z,\quad (10)$$

that depends only on the difference of the initial conditions of the main and controlled systems y.

$$C_x=x_1(0)-x_0(0);\ C_y=y_1(0)-y_0(0);\ C_z=z_1(0)-z_0(0).\quad (11)$$

Dependencies between variables of the main and controlled systems for this case of synchronization are shown in Fig. 4.

When the values of the matrix elements are $\delta_{12}=0$; $\delta_{13}=\delta_{21}=\delta_{22}=\delta_{31}=$ $=\delta_{33}=1$, the solutions to the system (9) will be as follows:

$$e_x=C_x-C_y\cdot t;\ e_y=C_y;\ e_z=C_z.\quad (13)$$

From the above-stated solution, it follows that synchronization will be stable only if $C_y=0$. When $C_y\neq 0$ the difference of oscillations x_0 and x_1 of the main and controlled systems will have a linear growth in time.

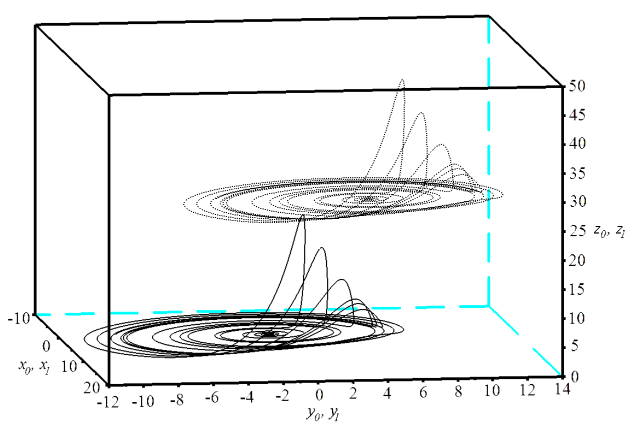


Fig. 4. Phase trajectories of the main (solid curve) and controlled (dotted curve) systems at replacement of all variables of the controlled system

The trajectories of systems in the phase space of variables are shown in Fig. 5.

Table 1 shows the solutions to the system of equations (9) at such values of the synchronization matrix elements, which provide the unlimited growth of the difference of coordinates in time.

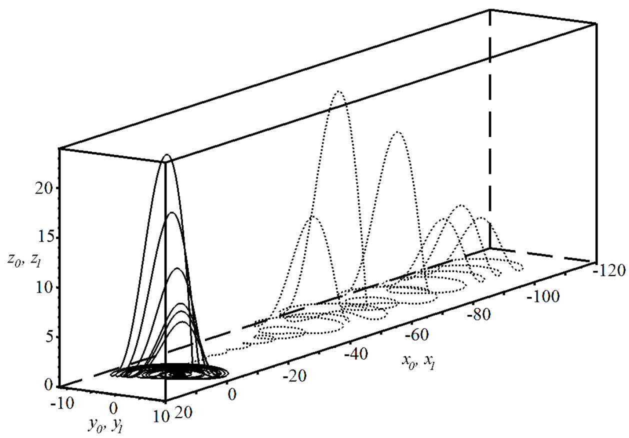


Fig. 5. The trajectories of the main (solid curve) and controlled (dotted curve) systems in the phase space of values of generated random oscillations

Table 1

Solutions to the system (9) at the values of the matrix coefficients with no synchronization

Values of coefficients	Solutions to the system
$\delta_{12}=0$; $\delta_{13}=\delta_{21}=\delta_{22}=\delta_{31}=\delta_{33}=1$	$e_x=C_x-C_y\cdot t$; $e_y=C_y$; $e_z=C_z$
$\delta_{13}=0$; $\delta_{12}=\delta_{21}=\delta_{22}=\delta_{31}=\delta_{33}=1$	$e_x=C_x-C_z\cdot t$; $e_y=C_y$; $e_z=C_z$
$\delta_{12}=\delta_{13}=0$; $\delta_{21}=\delta_{22}=\delta_{31}=\delta_{33}=1$	$e_x=C_x-(C_y+C_z)\cdot t$; $e_y=C_y$; $e_z=C_z$
$\delta_{22}=0$; $\delta_{12}=\delta_{13}=\delta_{21}=\delta_{31}=\delta_{33}=1$	$e_x=C_x$; $e_y=C_y\cdot\exp(0.2\cdot t)$; $e_z=C_z$
$\delta_{21}=0$; $\delta_{12}=\delta_{13}=\delta_{22}=\delta_{31}=\delta_{33}=1$	$e_x=C_x$; $e_y=C_y+C_x\cdot t$; $e_z=C_z$
$\delta_{21}=\delta_{22}=0$; $\delta_{12}=\delta_{13}=\delta_{31}=\delta_{33}=1$	$e_x=C_x$; $e_y=(5\cdot C_x+C_y)\cdot\exp(0.2\cdot t)-5\cdot C_x$; $e_z=C_z$
$\delta_{12}=\delta_{21}=\delta_{22}=0$; $\delta_{13}=\delta_{31}=\delta_{33}=1$	$e_x(t)=A\cdot\exp(0.1\cdot t)\cdot\sin(\sqrt{99}\ t+\alpha)$; $e_y(t)=A\cdot\exp(0.1\cdot t)\cdot\sin(\sqrt{99}\ t+\alpha)$

Fig. 7 shows the phase trajectories, obtained for the values of the synchronization matrix elements given in the last line of Table 1.

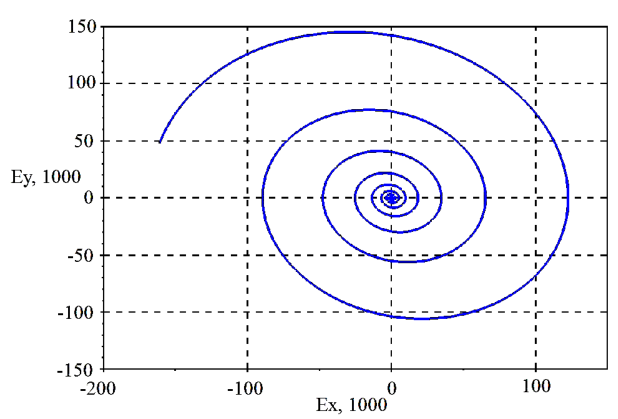


Fig. 6. The projection of phase trajectories onto a plane e_xe_y in the difference space at the values of the parameters of the synchronization matrix $\delta_{12}=\delta_{21}=\delta_{22}=0$, $\delta_{13}=\delta_{31}=\delta_{33}=1$

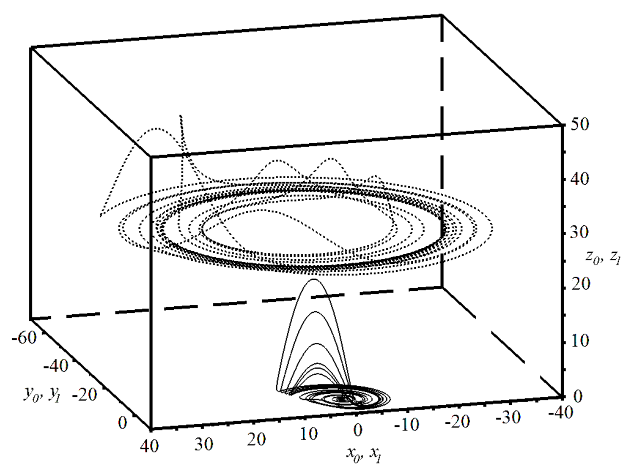
When parameter values are $\delta_{12}=\delta_{21}=0$, $\delta_{13}=\delta_{33}=\delta_{22}=1$ and $\delta_{13}=0$ (Table 2), in the difference space we will have closed phase trajectories that correspond to phase synchronization of the main and controlled systems. The systems will have synchronous oscillations. Phase synchronization is set between them.

Table 2

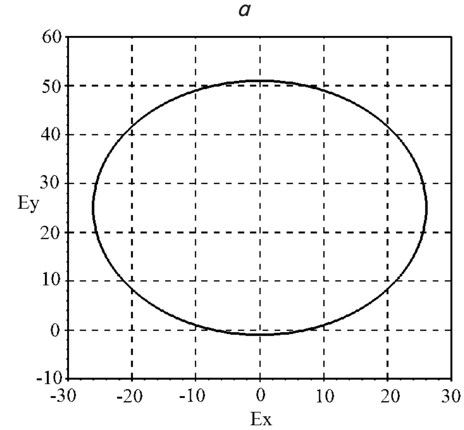
The solutions to the system of differential equations at the values of the synchronization matrix coefficients at which phase synchronization is observed

Values of coefficients	Solutions to the system
$\delta_{12}=\delta_{21}=0$, $\delta_{13}=1$, $\delta_{22}=\delta_{31}=\delta_{31}=1$	$e_x=\sqrt{(C_x)^2+(C_y)^2}\cdot\sin\left[t+\text{arctg}\left(\frac{C_y}{C_x}\right)\right]$; $e_z=C_z$
$\delta_{12}=\delta_{21}=0$, $\delta_{13}=0$, $\delta_{22}=\delta_{31}=\delta_{31}=1$	$e_x=\sqrt{(C_x)^2+(C_y)^2}\cdot\sin\left[t+\text{arctg}\left(\frac{C_y}{C_x}\right)\right]$; $e_y=\sqrt{(C_x)^2+(C_y)^2}\cdot\cos\left[t+\text{arctg}\left(\frac{C_y}{C_x}\right)\right]$; $e_z=C_z$

Attractors of the main and controlled system and the projection of phase trajectories onto the plane xy for this case are shown in Fig. 7.



a



b

Fig. 7. Phase synchronization of the main and controlled systems in the space of phase variables at the values of the synchronization matrix elements given in Table 2: *a* – attractors of the main (dotted line) and controlled system (solid line) in the space of phase variables; *b* – phase trajectory projection onto a plane e_xe_y in the difference space

More complicated types of synchronization occur if the coefficients in the third equation of the system are equal to zero. In this case, there will be a complex dependency of coefficients of differential equations on the time, determined by time dependencies of variables x_0 and z_0. Analytical solutions to the system of differential equations as a function of time are impossible.

It is obvious that the presence of coefficients with complex time dependency is caused by nonlinearity of the primary system. The greater the number of nonlinear terms and equations, the more complicated attractors of the systems in the difference space are.

Table 3 shows solutions to the system of differential equations that describe the interaction between the main and controlled systems when exposed to the process of synchronization of nonlinear terms in the right–hand side of the differential equations.

Table 3

Influence of nonlinearities on the process of synchronization of the main and controlled systems

Synchronization matrix coefficient values	Solutions to the system
$\delta_{31}=0$; $\delta_{12}=\delta_{13}=\delta_{21}=\delta_{22}=\delta_{33}=1$	$e_x=C_x$; $e_y=C_y$; $e_z(t)=C_x\cdot\int_0^t z_0(\xi)d\xi+C_z$
$\delta_{33}=0$; $\delta_{12}=\delta_{13}=\delta_{21}=\delta_{22}=\delta_{31}=1$	$e_x=C_x$; $e_y=C_y$; $e_z(t)=C_z\cdot e^{-5.7\cdot t}\cdot e^{\int_0^t x_0(\xi)d\xi}$
$\delta_{31}=\delta_{33}=0$; $\delta_{12}=\delta_{13}=\delta_{21}=\delta_{22}=1$	$e_x=C_x$; $e_y=C_y$; $e_z(t)=C_x\cdot\int_0^t z_0(\xi)d\xi+C_z\cdot e^{-5.7\cdot t}\cdot e^{\int_0^t x_0(\xi)d\xi}$

We shall analyze the influence of the first term $z_0\cdot e_x$ on the process of synchronization when the values of the synchronization matrix elements are $\delta_{31}=0$; $\delta_{12}=\delta_{13}=\delta_{21}=\delta_{22}=\delta_{33}=1$. For these values of the elements, the variable e_z in the difference space is determined by the timing diagram of the coordinate z, which has an impulse nature and is always positive (Fig. 3, *b*). Thus, the growth intervals of the variable e_z in time will alternate with the intervals, where its values will remain constant (Fig. 8).

It is obvious that the influence of the second term $x_0\cdot e_z$ will be determined by the time dependency of the integral $\int_0^t x_0(t)dt$ values (Fig. 9). From the time dependency of x (Fig. 3, *a*) it follows that the value of this integral is a priori less than 5.7. This indicates the stability of solutions to the system by Lyapunov, as evidenced by the diagram of the time dependency e_z (Fig. 10).

Thus, when the values of the parameters are $\delta_{13}=0$; $\delta_{12}=\delta_{21}=\delta_{22}=\delta_{31}=\delta_{33}=1$, there occurs complete synchronization, since variables e_x, e_y in the difference space (e_x, e_y) are constant, and the third variable e_z asymptotically tends to zero.

Let us consider the conditions under which the systems with nonlinear dynamics realize a strange attractor. Herewith, there occurs phase synchronization ($\delta_{12}=\delta_{21}=0$) on the variables x and y and consistent disturbance cycle ($\delta_{33}=0$). The values of parameters δ_{22} and δ_{33} are equal to one, since otherwise, the trajectories will indefinitely grow in time.

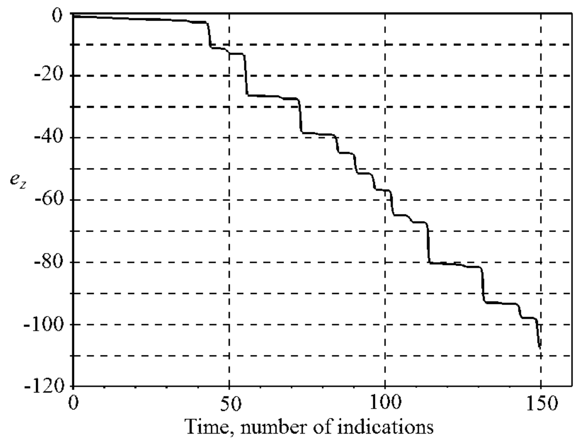


Fig. 8. Time dependency e_z at the values of the synchronization matrix elements $\delta_{31}=0$; $\delta_{12}=\delta_{13}=\delta_{21}=\delta_{22}=\delta_{33}=1$

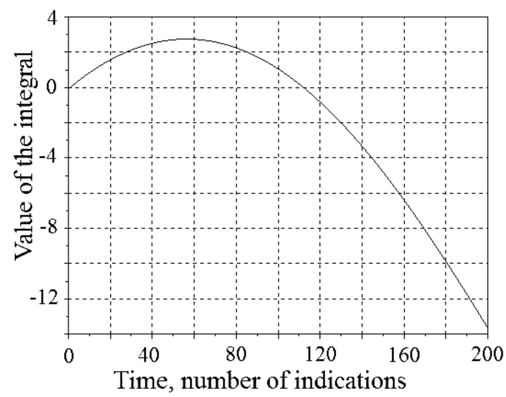


Fig. 9. Time dependency of the integral $\int_0^t x_0(\xi)d\xi$ value

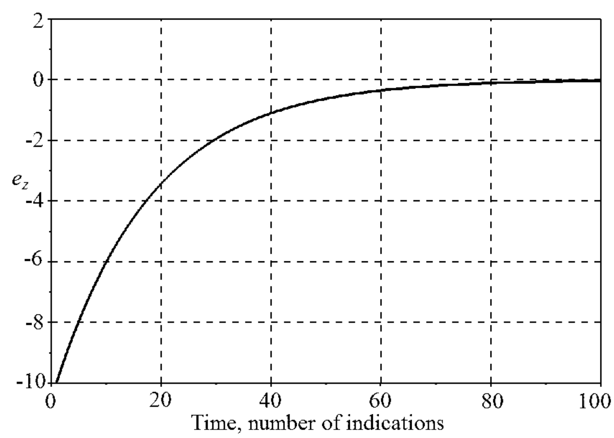


Fig. 10. Time dependency e_z at the values of the synchronization matrix elements $\delta_{13}=0$; $\delta_{12}=\delta_{21}=\delta_{22}=\delta_{31}=\delta_{33}=1$

Phase trajectories at the values of the synchronization matrix coefficients (the first and second lines in Table 4) are shown in Fig. 11, 12, respectively.

Such type of synchronization is called topological synchronization [8].

It should be noted that in the absence of communication between the systems ($\delta_{12}=\delta_{13}=\delta_{21}=\delta_{22}=\delta_{31}=\delta_{33}=0$), a strange attractor will also occur (Fig. 13).

Table 4

Conditions of realization of a strange attractor (limited non-recurrent trajectories) in the difference space

Values of coefficients	Solutions to the system
$\delta_{12}=\delta_{21}=$ $=\delta_{31}=0;$ $\delta_{13}=\delta_{22}=$ $=\delta_{33}=1$	$e_x=\sqrt{(C_x)^2+(C_y)^2}\cdot\sin\left[t+\operatorname{arctg}\left(\frac{C_y}{C_x}\right)\right];$ $e_y=\sqrt{(C_x)^2+(C_y)^2}\cdot\cos\left[t+\operatorname{arctg}\left(\frac{C_y}{C_x}\right)\right];$ $e_z(t)=C_z\cdot e^{-5.7\cdot t}\cdot e^{\int_0^t x_0(\xi)d\xi}$
$\delta_{12}=\delta_{21}=$ $=\delta_{13}=\delta_{31}=0;$ $\delta_{22}=\delta_{33}=1$	$e_x=\sqrt{(C_x)^2+(C_y)^2}\cdot\sin\left[t+\operatorname{arctg}\left(\frac{C_y}{C_x}\right)\right];$ $e_y=\sqrt{(C_x)^2+(C_y)^2}\cdot\cos\left[t+\operatorname{arctg}\left(\frac{C_y}{C_x}\right)\right]-C_y+C_z;$ $e_z(t)=C_z\cdot e^{-5.7\cdot t}\cdot e^{\int_0^t x_0(\xi)d\xi}$

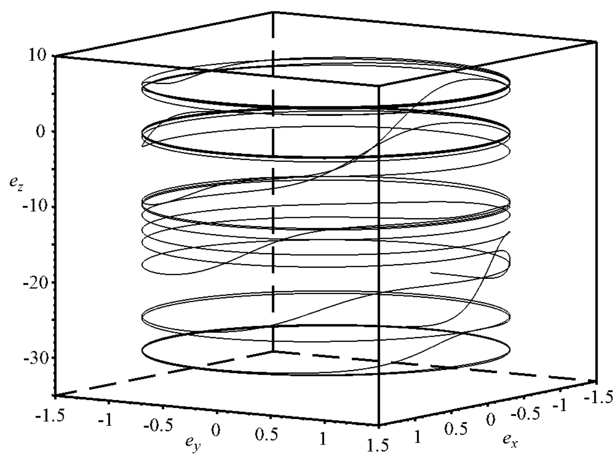


Fig. 11. Strange attractor of the coupled systems with nonlinear dynamics at the values of the synchronization matrix elements $\delta_{12}=\delta_{21}=\delta_{31}=0$, $\delta_{13}=\delta_{22}=\delta_{33}=1$

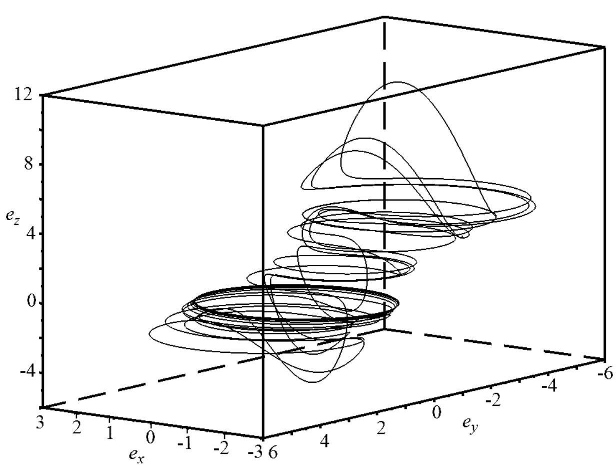


Fig. 12. Strange attractor of the coupled systems with nonlinear dynamics at the values of the synchronization matrix elements $\delta_{12}=\delta_{21}=\delta_{13}=\delta_{31}=0$, $\delta_{22}=\delta_{33}=1$

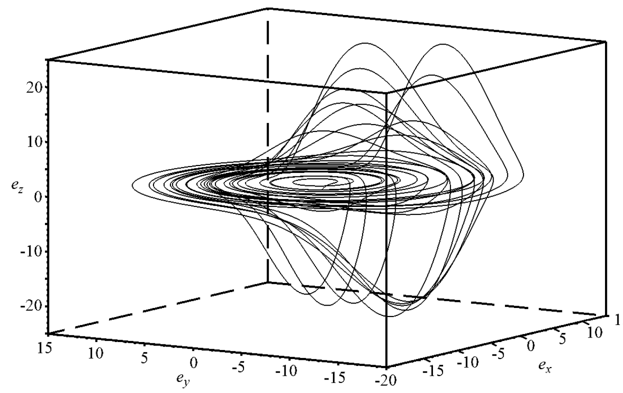


Fig. 13. Strange attractor in the difference space of Rössler uncoupled systems

This means that the Rössler systems with different initial conditions can be synchronized even in the absence of communication between them.

6. Discussing the results of research of synchronization processes in Rössler systems

Based on the analysis of the behaviour of systems with nonlinear dynamics, there was made a classification of solutions of ordinary differential equations and there were improved models of synchronization processes of the systems under analysis by means of matrix synchronization. With the help of the obtained models, the trajectories in the phase space of the above-mentioned equation systems were analyzed and classified.

The processes of synchronization of oscillations in the main and controlled nonlinear systems were studied by means of replacement and transition to a linearized system of variables equal to the difference of phase variables of the main and controlled systems.

As a result of the analysis, there were determined the values of the synchronization matrix elements in which there are different types of synchronization: complete, phase and topological. It was discovered that topological synchronization occurs even in the absence of communication between Rössler systems in the difference space of phase variables of the main and controlled systems with nonlinear dynamics. Herewith, in the phase space, there is formed an attractor with low spatial complexity that is an uncoupled trajectory with limited values. The criterion for the absence of synchronization of nonlinear systems is the unlimited growth of the phase variables difference.

The disadvantage of the proposed research method is the lack of quantitative analysis of complex synchronization methods and the conditions under which they are possible.

The obtained results can be used in modelling nonlinear electronic circuits, coding and cryptographic protection of information flows in telecommunication systems.

The work is a continuation of thematic studies of nonlinear processes [9–11], done at the Department of Radio Engineering and Information Security under the theme adopted by the MES of Ukraine in the line of further improvement of the proposed method.

7. Conclusions

1. By means of synchronization matrix, it was discovered that in the difference space of Rössler systems, there are observed trajectories corresponding to complete, phase and topological synchronization. Even a relatively simple system with one nonlinearity has almost all possible solutions of ordinary differential equations in accordance with the exact classification (except those that do not form an attractor because of being constrained).

2. Taking for the basis the synchronization matrix, there were analysed, by means of replacement, the synchronization processes of Rössler systems. The analysis can serve as a prototype for the development and implementation of software algorithms in the study of differential equations with nonlinear functions. It was discovered that nonlinear links in the right-hand side can cause a variety of solutions to equations and the absence of a unique algorithm for classification and search for solutions in a given system.

3. It is shown that in the difference space of phase variables in Rössler uncoupled systems, there exists an attractor with low spatial complexity that enables the establishment of coherent oscillations even in the absence of communication between the systems with nonlinear dynamics. In particular, this means that analysis of synchronization processes will require other methods that are not related to the analysis of the spectrum of Lyapunov exponents and construction of Lyapunov functions. This is especially true when chaos is used to protect information in television and information and communication systems, as hereat, little coherence could allow unauthorized access to information.

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