Вирішені динамічні задачі теорії пружності. Їх особливістю є використання поєднання плоских функцій, час-координата. Визначено умови існування рішень, яким відповідають аргументи обумовлених функцій. Рівняння, яким функції повинні задовольняти, можуть відноситися до рівнянь в часткових похідних гіперболічного типу. Отримане загальне рішення корелюється з відомими рішеннями лінійного хвильового рівняння. У прикладної задачі з'являється можливість обліку взаємодії, через смугу, суміжних клітей безперервного прокатного стану

Ключові слова: динамічна задача, хвильове рівняння, плоскі функції, суміжні кліті, аргумент-функції, умови існування рішень

Решены динамические задачи теории упругости. Их особенностью является использование сочетания плоских функций, время-координата. Определены условия существования решений, которым отвечают аргументы определяемых функций. Уравнения, которым функции должны удовлетворять, могут относиться к уравнениям в частных производных гиперболического типа. Полученное общее решение коррелируется с известными решениями линейного волнового уравнения. В прикладной задаче появляется возможность учета взаимодействия, через полосу, смежных клетей непрерывного прокатного стана

Ключевые слова: динамическая задача, волновое уравнение, плоские функции, смежные клети, аргумент-функции, условия существования решений

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DEVELOPMENT OF A DYNAMIC MODEL OF TRANSIENTS IN MECHANICAL SYSTEMS USING ARGUMENT-FUNCTIONS

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1. Introduction

There are a number of applied problems in which it is necessary to take into account the dynamic component of the process or phenomenon including the fact that the load is applied not instantaneously but in time. For example, during continuous rolling, there are such combinations of mechanical systems in which action transfer from one rolling stand to another via the strip occurs with some delay. This is reflected in the transient processes and the strip gripping capacity in adjacent continuous mill stands. The strip between the mill stands is in an elastic state. When the rolls start acting on it during gripping in the subsequent stand, they transmit disturbance to the strip in a form of oscillations or in a form of a stationary action. In this period, strip gage variation appears reducing dimensional accuracy of the rolled product, i. e. the product quality worsens.

2. Literature review and problem statement

It is of practical and theoretical interest to consider the wave problem as the process of propagation of the initial deviation and the initial velocity. At the same time, a need of defining general schemes (both linear and spatial ones) of solving dynamic problems arises.

Reference book [2] outlines general approaches to solution of the simplest dynamical problems. In one of the classic papers [3], loads that vary in time are considered. In solving the dynamical problem, unknown scalar functions ϕ and ψ were introduced for consideration. Their choice is determined by solving differential equations of the form:

$$c_1^2 \cdot \nabla^2 \cdot \phi - \frac{\partial^2 \phi}{\partial t^2} = 0; \quad c_2^2 \cdot \nabla^2 \cdot \psi - \frac{\partial^2 \psi}{\partial t^2} = -\psi.$$

Each differential equation corresponds to a certain type of waves. In seismology, such waves are called primary and secondary waves: wave P and wave S (shear wave), respectively. It should be emphasized that the vector solution takes place when the selected functions satisfy the reduced differential equations and actually are argument-functions. However, analysis shows that this is not enough for a number of applied problems. It is necessary to establish a differential relationship between these scalar argument-functions.

Solution of the wave equation also assumes dependences on coordinates and time, which ensure a smaller Rayleigh wave amplitude according to the exponential law [3]. Such a structure of solution can be useful in considering dynamic problems in the field of equipment for continuous rolling mills during the roll bite when the pulse action of the rolls applied to the elastic strip in the inter-stand space is represented as a variable in time and space. It should be added that the solution presented does not imply its use in other applied examples. There are no general laws governing determination of unknown functions under new boundary conditions associated with new processes and equipment operation.

It is noted in monograph [4] that the problem in the dynamics of the elasticity theory consists in formulation of the boundary problem variants and assessment of their application field. Homogeneous and inhomogeneous solutions of dynamic problems for hollow bodies were proposed. Although the work repeatedly emphasized the diversity of dynamic problems in the elasticity theory and the variety of boundary conditions to which the new solutions correspond, the solutions were restricted by the dynamic problem for hollow bodies.

Another approach to the determination of solutions of the dynamic problem for boundary elements with distributed loads was presented in publication [5]. Influence of impact loading on the indices of the stress-strain state of an elastic medium was shown. The method of boundary integral equations was used. It is indicative that the author limited himself by solution of an applied problem, found necessary theoretical basis and obtained the result acceptable for practice. In this case, there are no recommendations for using the proposed solution in applied problems with other or similar boundary conditions.

Quadratures of solutions for the third dynamic problem with mixed boundary conditions were constructed in work [6]. Displacements are specified on one part of the surface, and forces on the other part. Construction of the solution extends the possibilities of its use in applied problems with mixed boundary conditions but such generalizations are not enough for their use in the first and second dynamics problems for boundary conditions associated with a variable damped effect on the elastic medium.

Work [7] describes algorithms of the R-function method for solving dynamic problems in the elasticity theory for bodies of finite dimensions deformation of which proceeds in an elastic region. Using the theory of R-functions, the problem of constructing coordinate (trial) functions was solved constructively which made it possible to open the possibilities for practical application of the variational and projection solution methods. Variational and difference methods are used to search for new structures introduced into consideration. As the authors suppose, a universal toolkit is presented that allows one not only solve mechanics of the deformed solid and the problems of mathematical physics but also the problems having relation to the development of new technological processes. Very attractive is the fact that a powerful mathematical apparatus for result generalization was proposed and that it can be applied in finding new solutions for dynamic problems of the elasticity theory. It is necessary to clarify some details of the proposed approach. Coordinate test functions were introduced. They can play role of the proposed argument-functions. However, their definition by variational or other methods may appear to be not the exclusive option. Not so fundamental but more intuitive and practical options are possible.

Spatial self-oscillations of orthotropic plates were considered in the presence of an internal viscous resistance, in proportion to the velocity of the medium points [8]. By applying the asymptotic method, equations for longitudinal and shear oscillation frequencies were obtained. The use of new boundary conditions determines a new result and equations of longitudinal and shear oscillation frequencies are obtained. The discussed example is a partial result, which does not apply to solutions for other boundary conditions.

The problem of determining stresses on the boundary of an elastic half-space from the given displacements was shown in [9]. The solution was found by the method of Laplace and Fourier integral transforms. The initial data were reduced to a system of three integral Fredholm equations. Numerical solutions were obtained. Introduction to the consideration of transforms allows one to approach the problem solution by a numerical method as one of the variants of the problem under consideration. Variation of boundary and obvious conditions change the solution approaches and the result obtained. However, there are no generalizing relations that are superimposed on the closing equations for obtaining a series of partial solutions and a possibility of a broad analysis of the boundary conditions of new applied problems.

In work [10], abilities of another method are considered. It is the method of potentials and the theory of multidimensional singular integral equations for solving three-dimensional stationary and non-stationary boundary problems in the theory of elasticity.

Classical solutions of a linear dynamical problem are presented in work [11]. The linear wave equation has the form:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Write an equation of characteristics for (1):

$$dx^2 - a^2 dt^2 = 0.$$

There are two equations in new variables:

$$\xi = x - at, \ \eta = x + at.$$

The oscillation equation (1) is converted to a simpler expression:

$$u_{\xi\eta} = 0. \tag{2}$$

The common integral of equation (2) is:

$$u(x,t) = f_1(x+at) + f_2(x-at). \tag{3}$$

Taking into account the boundary conditions, expression (3) is transformed to the form:

$$u(x,t) = \frac{\phi(x+at)+\phi(x-at)}{2} + \frac{1}{2a} \cdot \int_{x-at}^{x+at} \psi(\alpha) d\alpha. \tag{4}$$

Expression (4) is called the d'Alembert formula. It should be emphasized that the fraction on the right side of (4) is a function of both the coordinate and the time. In this case, the function ϕ is not defined which makes it impossible to instantiate the boundary conditions of the applied problem.

A method of separation of variables or the Fourier method is known. The solution is represented as:

$$u(x,t) = X(x) \cdot T(t).$$

Substitution of the proposed form of solution in (1) gives the following:

$$X'' \cdot T = \frac{1}{a^2} \cdot X \cdot T'' \ \text{ or } \ \frac{X''}{X} = \frac{1}{a^2} \cdot \frac{T''}{T} = -\lambda.$$

Ordinary differential equations for determination of X and T functions:

$$X'' + \lambda X = 0, \quad T'' + a^2\lambda T = 0.$$

A partial solution is known:

$$u_n(x,t) = X_n \cdot T_n = \left(A_n \cos\frac{\pi n}{l} \cdot at + B_n \sin\frac{\pi n}{l} \cdot at \right) \cdot \sin\frac{\pi n}{l} \cdot x. \quad (5)$$

Expression (5) satisfies the boundary and edge conditions and equation (1). In the general case, by virtue of linearity and homogeneity, the sum of the partial solutions of (5) is:

$$u(x,t) = \sum_1^\infty u_n(x,t) =$$

$$= \sum_1^\infty \left(A_n \cos\frac{\pi n}{l} \cdot at + B_n \sin\frac{\pi n}{l} \cdot at \right) \cdot \sin\frac{\pi n}{l} \cdot x. \quad (6)$$

The constants of integration in (6) are determined by the boundary conditions of the problem. The class of functions that define boundary and edge conditions for the expression (6) is limited. There are difficulties in using solution (6) in practical problems. It is necessary to develop approaches enabling determination of conditions for existence of several solutions corresponding to the specified boundary and edge conditions of various applied problems.

Work [12] expands the scope of solutions but it does not show convincing general schemes for determining required dependencies taking into account practical diversity of the initial and boundary conditions.

Monograph [13] does not show the abilities of using complex initial and boundary conditions determining the non-stationary action on the strip such as ones occurring, e. g. during rolling in adjacent continuous mill stands.

In applied work [14], non-stationary problems are considered with reference to the metal forming equipment. However, the rolling features in the transient processes associated with loading during the period of strip gripping are not disclosed.

It should be mentioned that one of the first works where the method of solving applied problems using argument-functions was applied was paper [15]. The solution is presented in the theory of plasticity with no consideration of the loading dynamic component.

The method of argument-functions with examples from the applied theory of plasticity and elasticity was given and generalized in [16], but the possibility of its use in solving dynamic problems of the theory of elasticity was not demonstrated.

Complication of the problem with the use of argument-functions [17] indicates potentials of the method but the dynamic problem was not considered.

The first generalizations of the dynamics results were given in [18] but their further use for various boundary conditions and obtaining a new result was not stated.

As analysis of the papers presented shows, there is a wide use of various approaches and methods for solving dynamic problems in the elasticity theory. These include the method of potentials, the method of integral transformants, variational and asymptotic methods. Besides, the theory of R-functions, the method of boundary integral equations, d'Alembert method, Fourier method, the method of argument-functions, etc. can be mentioned. Most of them are used to solve specific problems while having no mathematical generalizations for their further use. In work [4], one of the most important problems of the dynamic theory of elasticity was emphasized: it is formulation of variants of boundary problems and estimation of the field of their application. Practical implementation of such approaches broadens the possibilities of using the resulting solutions by linking and selecting for them boundary conditions of various technological processes and equipment operating conditions. In the course of their development, works appeared which implement such generalizations and algorithms and not just dynamic tasks, e. g. papers [3, 6, 7, 9, 15]. Scalar functions [3], R-functions (tested) [7], integral transformants [9], argument-functions [15] are used, which determine not the functional dependencies themselves but the conditions for their existence.

In the latter case, the stages of the further, closing solution are considered. But this is just another problem. In the first approximation, one can restrict his attention to solving the simplest invariant differential relations. They are of interest since in many respects they, as a special case, coincide with the known classical solutions.

As it follows from works [3–18], application of argument-functions involves introduction into consideration of the main functional dependences, which can include exponential, trigonometric, hyperbolic, logarithmic, complex, etc. functions. Their arguments are also functions depending on coordinates and time. These closing unknown coordinate-time relationships are determined in the process of solving the problem, more precisely, conditions for their existence are found. Ultimately, the conditions for the existence of a number of basic functions of the problem being solved are determined.

With this statement of a question, it becomes possible to define a whole class of unknown functions the implementation of which expands abilities of the method under consideration, which has been repeatedly emphasized in the course of analysis. Examples of successful use of such approaches are listed in the list of literature in question.

3. The study objective and tasks

This work objective was to determine common approaches or conditions of existence of various solutions, which are determined by differential equations of the dynamic problem and boundary conditions.

To achieve this objective, the following tasks were set:

– development of general approaches to solution of differential wave equations using argument-functions, various boundary conditions of problems in the theory of elasticity;

– definition of conditions for existence of trailing partial solutions using invariant differential relations and equations for argument-functions;

– development of a dynamic model of the transient processes taking place during rolling in adjacent continuous mill stands.

4. Approaches and solution of dynamical problems of the elasticity theory

4. 1. Approaches of analytical solution of dynamical problems of the elasticity theory with the use of argument-functions

The totality of solutions of concrete differential equations is a practical necessity with the purpose of choosing the mathematical model that satisfies to the right degree desired boundary conditions of the problem. Considering their diversity, which is associated with the variety of applied problems, problems arise in obtaining the necessary solution for the initial data. In this case, it is expedient to obtain not the concrete result of solution but the conditions for its existence, i.e. determine those restrictions that are imposed on the functions from the side of differential equations and boundary conditions. Thus, the final result is not the functions themselves but the conditions for their existence, in other words, the invariants of the differential relations of the argument-functions introduced into consideration. This approach is partially described in literature. However, the possibilities of transition from one boundary condition to another are not indicated. Some examples of using argument-functions are given thereinafter.

Solution of a differential equation of hyperbolic type:

$$\frac{\partial^2\tau}{\partial x^2}-\frac{\partial^2\tau}{\partial y^2}=2\frac{\partial^2}{\partial x\partial y}\sqrt{k^2-\tau^2},$$

was proposed to perform with the help of unknown θ, AF argument-functions introduced into consideration [9–12]:

$$\tau=C\cdot\exp\theta\cdot\sin A\Phi,$$

where θ and AF are the argument-functions of the deformation-zone coordinates. In this case, differential constraints on the argument functions introduced into consideration are shown:

$$\theta_x=\mp A\Phi_y,\ \ \theta_y=\pm A\Phi_x,\ \ \theta_{xx}+\theta_{yy}=0,\ \ A\Phi_{xx}+A\Phi_{yy}=0.$$

The conditions themselves determine the type of equations that must be used to find the unknown coordinate dependences θ, AF. The simplest option of solution of the Laplace equation for the AF function is:

$$A\Phi_{xx}=A\Phi_{yy}=0,\ \ A\Phi=AA_6\cdot x\cdot y,\ \ \theta=\frac{1}{2}AA_6\left(x^2-y^2\right).$$

It should be emphasized that the argument-functions are determined not only by Laplace's equations but to a greater extent by differential relations between adjacent dependencies as well. The last expression θ is also a harmonic function satisfying the Laplace equation.

The same approach was used to solve the wave equations of the elasticity theory. Restrictions are imposed on the side of the differential equations themselves and boundary

4. 2. Solution of a dynamic problem using argument-functions

Solution of the dynamic problem in analytical form was presented in [18]. With the help of argument-functions, solution of a linear dynamic problem of a limited application was presented.

Use the approaches formulated in these papers, write a fairly simple relationship and introduce argument-functions θ, AF into consideration:

$$u=C\cdot\sin\theta\cdot\sin A\Phi, \tag{7}$$

where C and A are constants characterizing the process; Θ, Φ are unknown argument-functions of time and coordinate, continuous, having second derivatives in time and a corresponding coordinate.

Substitution of expression (7) in (1) gives a differential equation of the form:

$$\begin{aligned}&\left(\theta_{tt}-a^2\cdot\theta_{xx}\right)\cdot\cos\theta\cdot\sin A\Phi-\\&-\left[\left(\theta_t+a\cdot A\Phi_x\right)\cdot\left(\theta_t-a\cdot A\Phi_x\right)+\left(A\Phi_t+a\theta_x\right)\cdot\left(A\Phi_t-a\theta_x\right)\right]\times\\&\times\sin\theta\cdot\sin A\Phi+2\left(\theta_t A\Phi_t-a^2\theta_x A\Phi_x\right)\cos\theta\cdot\cos A\Phi+\\&+\left(A\Phi_{tt}-a^2A\Phi_{xx}\right)\Big]\cdot\sin\theta\cdot\cos A\Phi=0,\end{aligned} \tag{8}$$

where following notations were in parentheses $\theta_t=\frac{\partial\theta}{\partial t}$, $A\Phi_x=\frac{\partial A\Phi}{\partial x}$, etc. Further analysis shows that equation (8) will be substantially simplified if nonlinearity is eliminated and the brackets taken equal to zero, i.e:

$$\theta_t=\mp aA\Phi_x,\ \ A\Phi_t=\mp a\theta_x. \tag{9}$$

In this case, all the summed operators on the left-hand side will be zero. On the basis of the result obtained, taking into account (9), solution can be presented in a more general form:

$$u=C_0\left(C_1\sin\theta+C_2\cos\theta\right)\left(C_3\sin A\Phi+C_4\cos A\Phi\right), \tag{10}$$

provided that the following relations exist for the argument-functions:

$$\begin{aligned}&\theta_t=\mp aA\Phi_x,\ \ A\Phi_t=\mp a\theta_x,\\&\theta_{tt}-a^2\cdot\theta_{xx}=0,\ \ A\Phi_{tt}-a^2A\Phi_{xx}=0.\end{aligned} \tag{11}$$

The differential relations (9), (11) differ by signs from the above Cauchy-Riemann relations. Hence, differential constraints of the functions introduced for consideration change with the change in the form of the differential equations. The trailing constraints (11) define the basic solution (10). Besides, the unknown argument-functions become known for the shown differential equations. The final result can be represented as a superposition of solutions:

$$\begin{aligned}&u=\sum_{i=1}^{i=n}C_{i,0}\left(C_{i,1}\sin\theta_i+C_{i,2}\cos\theta_i\right)\left(C_{i,3}\sin A_i\Phi_i+C_{i,4}\cos A_i\Phi_i\right),\\&\theta_{i,t}=\mp aA_i\Phi_{i,x},\ \ A_i\Phi_{i,t}=\mp a\theta_{i,x},\\&\theta_{i,tt}-a^2\cdot\theta_{i,xx}=0,\ \ A\Phi_{i,tt}-a^2A\Phi_{i,xx}=0.\end{aligned} \tag{12}$$

Solution (12) resembles Fourier solution since a product of trigonometric functions takes place. However, there are a number of fundamental differences. The arguments of trigonometric functions are functions as well not of one variable as in the method of separation but of two variables. In addition, the differential dependencies of solution (12) show the variants of new solutions without being tied to a specific result but to the boundary conditions of the process.

The next version of solution of the linear wave equation (1) can be applied to other boundary conditions, e. g. to damped, periodical influences on the elastic medium.

Consider the following variant of partial solution using argument-functions. In this case, the θ argument-function is not in the trigonometric, but in the exponential dependence for the displacement u.

$$u = C \cdot \exp\theta \cdot \cos A\Phi, \qquad (13)$$

where the argument-functions θ, AF are to be determined by solution of the problem. Substitute (13) into (1) taking into account the fact that the indicated functions admit second derivatives to obtain:

$$a^2 u_{xx} = Ca^2 \exp\theta \times$$
$$\times\left[\left(\theta_{xx} + \theta_x^2 - A\Phi_x^2\right)\cos A\Phi - \left(2\theta_x A\Phi_x + A\Phi_{xx}\right)\sin A\Phi\right],$$

$$u_{tt} = C \cdot \exp\theta \times$$
$$\times\left[\left(\theta_{tt} + \theta_t^2 - A\Phi_t^2\right)\cdot\cos A\Phi - \left(2\theta_t \cdot A\Phi_t + A\Phi_{tt}\right)\cdot\sin A\Phi\right].$$

Substitute the last differential relations into equation (1) to obtain:

$$\left[-\theta_{tt} + a^2\theta_{xx} + a^2\theta_x^2 - a^2A\Phi_x^2 - \theta_t^2 - A\Phi_t^2\right]\cos A\Phi -$$
$$-\left[2a^2\theta_x \cdot A\Phi_x - 2\theta_t \cdot A\Phi_t + a^2A\Phi_{xx} - A\Phi_{tt}\right]\sin A\Phi = 0.$$

There are operators in square brackets:

$$\left[\left(a^2\theta_{xx} - \theta_{tt}\right) - \left(a^2A\Phi_x^2 - A\Phi_t^2\right) + \left(a^2\theta_x^2 - \theta_t^2\right)\right]\cos A\Phi -$$
$$-\left[\left(a^2A\Phi_{xx} - A\Phi_{tt}\right) + 2\left(a^2\theta_x \cdot A\Phi_x - \theta_t \cdot A\Phi_t\right)\right]\sin A\Phi = 0. \qquad (14)$$

Consider the case when the parentheses of the first operator are zero, that is,

$$a^2\theta_x^2 - \theta_t^2 = \left(a\theta_x + \theta_t\right)\left(a\theta_x - \theta_t\right) = 0,$$

$$a^2A\Phi_x^2 - A\Phi_t^2 = \left(aA\Phi_x + A\Phi_t\right)\left(aA\Phi_x - A\Phi_t\right) = 0.$$

Consider some variants.

Variant 1:

$$a\theta_x = \theta_t, \quad aA\Phi_x = A\Phi_t. \qquad (15)$$

Define the second derivatives from the relationships (15):

$$a^2\theta_{xx} = a\theta_{tx}, \quad \theta_{tt} = a\theta_{xt},$$

$$a^2A\Phi_{xx} = aA\Phi_{tx}, \quad A\Phi_{tt} = aA\Phi_{xt},$$

substitute all relationships into the square brackets of the first and second operators (14) to obtain the equality to zero of the remaining parentheses:

$$a^2\theta_{xx} - \theta_{tt} = a\theta_{tx} - a\theta_{xt} = 0,$$

$$a^2A\Phi_{xx} - A\Phi_{tt} = aA\Phi_{tx} - aA\Phi_{xt} = 0,$$

$$a^2\theta_x \cdot A\Phi_x - \theta_t \cdot A\Phi_t = a^2\theta_x \cdot A\Phi_x - a^2\theta_x \cdot A\Phi_x = 0.$$

Thus, under the conditions of (15), equation (14) becomes an identity.

Variant 2:

$$a\theta_x = -\theta_t, \quad aA\Phi_x = -A\Phi_t. \qquad (16)$$

Define the second derivatives:

$$a^2\theta_{xx} = -a\theta_{tx}, \quad \theta_{tt} = -a\theta_{xt},$$

$$a^2A\Phi_{xx} = -aA\Phi_{tx}, \quad A\Phi_{tt} = -aA\Phi_{xt},$$

substitute in square brackets to get:

$$a^2\theta_{xx} - \theta_{tt} = -a\theta_{tx} + a\theta_{xt} = 0,$$

$$a^2A\Phi_{xx} - A\Phi_{tt} = -aA\Phi_{tx} + aA\Phi_{xt} = 0,$$

$$a^2\theta_x \cdot A\Phi_x - \theta_t \cdot A\Phi_t = a^2\theta_x \cdot A\Phi_x - a^2\theta_x \cdot A\Phi_x = 0.$$

Consequently, equation (14) also becomes an identity when conditions of (16) are satisfied.

Variant 3:

$$a\theta_x = \theta_t, \quad aA\Phi_x = -A\Phi_t. \qquad (17)$$

In this case, signs between the differential relations are opposite. For the second derivatives in the defining differential equations, the identity is maintained but the identity does not hold in the third one. Really:

$$a^2\theta_x \cdot A\Phi_x - \theta_t \cdot A\Phi_t =$$
$$= a^2\theta_x \cdot A\Phi_x - \left(a\theta_x\right)\cdot\left(-aA\Phi_x\right) = 2a^2\theta_x \cdot A\Phi_x \neq 0.$$

The relationships (17) do not satisfy equation (14). Finally, taking into account the first two variants, the following can be written:

$$u = C_1\exp\theta_1\left(C_1' \cdot \sin A_1\Phi_1 + C_1'' \cdot \cos A_1\Phi_1\right) +$$
$$+ C_2\exp\theta_2\left(C_2' \cdot \sin A_2\Phi_2 + C_2'' \cdot \cos A_2\Phi_2\right). \qquad (18)$$

provided there are solutions for the argument-functions:

$$a\theta_{1x} = \pm\theta_{1t}, \quad aA_1\Phi_{1x} = \pm A_1\Phi_{1t},$$

$$a\theta_{2x} = \pm\theta_{2t}, \quad aA_2\Phi_{2x} = \pm A_2\Phi_{2t},$$

$$a^2\theta_{1xx} - \theta_{1tt} = 0, \quad a^2A_1\Phi_{1xx} - A_1\Phi_{1tt} = 0,$$

$$a^2\theta_{2xx} - \theta_{2tt} = 0, \quad a^2A_2\Phi_{2xx} - A_2\Phi_{2tt} = 0.$$

Proceed from partial solutions (13), (18) to a general solution in the form:

$$u = \sum_1^n C_i\exp\theta_i \cdot\left(C_i' \cdot \sin A_i\Phi_i + C_i'' \cdot \cos A_i\Phi_i\right) +$$
$$+ \sum_2^m C_j\exp\theta_j \cdot\left(C_j' \cdot \sin A_j\Phi_j + C_j'' \cdot \cos A_j\Phi_j\right) \qquad (19)$$

under condition that:

$$a_i\theta_{ix} = \pm\theta_{it}, \quad a_iA_i\Phi_{ix} = \pm A_i\Phi_{it},$$

$$a_i\theta_{jx} = \pm\theta_{jt}, \quad a_iA_j\Phi_{jx} = \pm A_j\Phi_{jt},$$

$$a_i^2\theta_{ixx} - \theta_{itt} = 0, \quad a_i^2A_i\Phi_{ixx} - A_i\Phi_{itt} = 0,$$

$$a_i^2\theta_{jxx} - \theta_{jtt} = 0, \quad a_i^2A_j\Phi_{jxx} - A_j\Phi_{jtt} = 0.$$

Comparison of solutions (7) and (13) shows that the argument-functions satisfy the same type of differential equations (12) and (19). However, the differential relations between the adjacent ones in solution are different. In many cases, this feature is defining and basically distinguishes the solutions shown.

Following the trends in developing methods for solving dynamical problems as applied to changeable boundary conditions, consider solution of a more complex problem, the spatial problem [19]. Wave equation for the spatial problem has the form:

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right). \quad (20)$$

A unified approach to the solution with the use of argument-functions was formulated above. Represent the general solution u in a form of a superposition of partial solutions u_i that simultaneously depend on one of the coordinates and time. The following is obtained:

$$u=u_1+u_2+u_3$$

wherein

$$u_1=C_1\cos A_1\Phi_1\cos\theta_1,$$

$$u_2=C_2\cos A_2\Phi_2\cos\theta_2,$$

$$u_3=C_3\cos A_3\Phi_3\cos\theta_3. \quad (21)$$

Function:

$$A_1\Phi_1=f_1(x,t),\quad \theta_1=f_2(x,t),$$

$$A_2\Phi_2=f_3(y,t),\quad \theta_2=f_4(y,t),$$

$$A_3\Phi_3=f_5(z,t),\quad \theta_3=f_6(z,t).$$

Take the derivatives of (21) taking into account the functional dependences:

$$\frac{\partial^2(u_1+u_2+u_3)}{\partial x^2}=-C_1(A_1\Phi_1)_{xx}\sin A_1\Phi_1\cos\theta_1-$$
$$-C_1(A_1\Phi_1)_x^2\cos A_1\Phi_1\cos\theta_1+$$
$$+C_1(A_1\Phi_1)_x(\theta_1)_x\sin A_1\Phi_1\sin\theta_1-$$
$$-C_1(\theta_1)_{xx}\cos A_1\Phi_1\sin\theta_1+$$
$$+C_1(\theta_1)_x(A_1\Phi_1)_x\sin A_1\Phi_1\sin\theta_1-$$
$$-C_1(\theta_1)_x^2\cos A_1\Phi_1\cos\theta_1; \quad (22)$$

$$\frac{\partial^2(u_1+u_2+u_3)}{\partial y^2}=-C_2(A_2\Phi_2)_{yy}\sin A_2\Phi_1\cos\theta_2-$$
$$-C_2(A_2\Phi_2)_y^2\cos A_2\Phi_2\cos\theta_2+$$
$$+C_2(A_2\Phi_2)_y(\theta_2)_y\sin A_2\Phi_2\sin\theta_2-$$
$$-C_2(\theta_2)_{yy}\cos A_2\Phi_2\sin\theta_2+$$
$$+C_2(\theta_2)_y(A_2\Phi_2)_y\sin A_2\Phi_2\sin\theta_2-$$
$$-C_2(\theta_2)_y^2\cos A_2\Phi_2\cos\theta_2; \quad (23)$$

$$\frac{\partial^2(u_1+u_2+u_3)}{\partial z^2}=-C_3(\theta_3)_{zz}\cos A_3\Phi_3\sin\theta_3+$$
$$+C_3(\theta_3)_z(A_3\Phi_3)_z\sin A_3\Phi_3\sin\theta_3+$$
$$+C_3(A_3\Phi_3)_z(\theta_3)_z\sin A_3\Phi_3\sin\theta_3-$$
$$-C_3(\theta_3)_{zz}\cos A_3\Phi_3\sin\theta_3+$$
$$+C_3(\theta_3)_z(A_3\Phi_3)_z\sin A_3\Phi_3\sin\theta_3-$$
$$-C_3(\theta_3)_z^2\cos A_3\Phi_3\cos\theta_3. \quad (24)$$

Further:

$$\frac{\partial^2(u_1+u_2+u_3)}{\partial t^2}=$$
$$=-C_1(A_1\Phi_1)_{tt}\sin A_1\Phi_1\cos\theta_1-C_1(A_1\Phi_1)_t^2\cos A_1\Phi_1\cos\theta_1+$$
$$+C_1(\theta_1)_{tt}\cos A_1\Phi_1\sin\theta+C_1(\theta_1)_t(A_1\Phi_1)_t\sin A_1\Phi_1\sin\theta-$$
$$-C_2(A_2\Phi_2)_{tt}\sin A_2\Phi_2\cos\theta_2-C_2(A_2\Phi_2)_t^2\cos A_2\Phi_2\cos\theta_2+$$
$$+C_2(\theta_2)_{tt}\cos A_2\Phi_2\sin\theta_2+C_2(\theta_2)_t(A_2\Phi_2)_t\sin A_2\Phi_2\sin\theta_2-$$
$$-C_3(A_3\Phi_3)_{tt}\sin A_3\Phi_3\cos\theta_3-C_3(A_3\Phi_3)_t^2\cos A_3\Phi_3\cos\theta_3+$$
$$+C_3(A_3\Phi_3)_t(\theta_3)_t\sin A_3\Phi_3\sin\theta_3-C_3(\theta_3)_t^2\cos A_3\Phi_3\cos\theta_3. \quad (25)$$

Substituting (22)–(25) into equation (20) and making simple transformations by grouping beside the trigonometric functions, obtain a nonlinear second-order partial differential equation with a possible repeated simplification scheme:

$$-C_1\Big\{\Big[(A_1\Phi_1)_{tt}-c^2(A_1\Phi_1)_{xx}\Big]\cdot\sin A_1\Phi_1\cdot\cos\theta_1-$$
$$-\Big[\Big((A_1\Phi_1)_t^2-c^2(\theta_1)_x^2\Big)+(\theta_1)_t^2-c^2(A_1\Phi_1)_x^2\Big]\cdot\cos A_1\Phi_1\cdot\cos\theta_1+$$
$$+2\Big[(A_1\Phi_1)_t(\theta_1)_t-c^2(A_1\Phi_1)_x(\theta_1)_x\Big]\cdot\sin A_1\Phi_1\cdot\sin\theta_1-$$
$$-\Big[(\theta_1)_{tt}-c^2(\theta_1)_{xx}\Big]\cdot\cos A_1\Phi_1\sin\theta_1\Big\}-$$
$$-C_2\Big\{\Big[(A_2\Phi_2)_{tt}-c^2(A_2\Phi_2)_{yy}\Big]\sin A_2\Phi_2\cos\theta_2-$$
$$-\Big[\Big((A_2\Phi_2)_t^2-c^2(\theta_{21})_y^2\Big)+(\theta_2)_t^2-c^2(A_2\Phi_2)_y^2\Big]\cos A_2\Phi_2\cos\theta_2+$$
$$+2\Big[(A_2\Phi_2)_t(\theta_2)_t-c^2(A_2\Phi_2)_y(\theta_2)_y\Big]\sin A_2\Phi_2\sin\theta_2-$$
$$-\Big[(\theta_2)_{tt}-c^2(\theta_2)_{yy}\Big]\cos A_2\Phi_2\sin\theta_2\Big\}-$$
$$-C_3\Big\{\Big[(A_3\Phi_3)_{tt}-c^2(A_3\Phi_3)_{zz}\Big]\sin A_3\Phi_3\cos\theta_3-$$
$$-\Big[\Big((A_1\Phi_1)_t^2-c^2(\theta_1)_x^2\Big)+(\theta_1)_t^2-c^2(A_1\Phi_1)_x^2\Big]\cos A_1\Phi_1\cos\theta_1+$$
$$+2\Big[(A_1\Phi_1)_t(\theta_1)_t-c^2(A_1\Phi_1)_x(\theta_1)_x\Big]\sin A_1\Phi_1\sin\theta_1-$$
$$-\Big[(\theta_1)_{tt}-c^2(\theta_1)_{xx}\Big]\cos A_1\Phi_1\sin\theta_1\Big\}=0. \quad (26)$$

Eliminating nonlinearity in (26), obtain variants of simplifications with different signs:

$$1.\ (A_1\Phi_1)_t=c(\theta_1)_x,\quad (\theta_1)_t=c(A_1\Phi_1)_x,$$

$$(A_2\Phi_2)_t=c(\theta_2)_y,\quad (\theta_2)_t=c(A_2\Phi_2)_y, \quad (27)$$

$$(A_3\Phi_3)_t=c(\theta_{31})_z,\quad (\theta_3)_t=c(A_3\Phi_3)_z.$$

$$2.\ (A_1\Phi_1)_t = -c(\theta_1)_x,\ \ (\theta_1)_t = -c(A_1\Phi_1)_x,$$
$$(A_2\Phi_2)_t = -c(\theta_2)_y,\ \ (\theta_2)_t = -c(A_2\Phi_2)_y, \quad (28)$$
$$(A_3\Phi_3)_t = -c(\theta_{31})_z,\ \ (\theta_3)_t = -c(A_3\Phi_3)_z.$$

With brackets eliminated, expression (26) will be written as:

$$-C_1\left\{\left[(A_1\Phi_1)_{tt} - c^2(A_1\Phi_1)_{xx}\right]\sin A_1\Phi_1\cos\theta_1 - \right.$$
$$\left.-\left[(\theta_1)_{tt} - c^2(\theta_1)_{xx}\right]\cos A_1\Phi_1\sin\theta_1\right\} -$$
$$-C_2\left\{\left[(A_2\Phi_2)_{tt} - c^2(A_2\Phi_2)_{yy}\right]\sin A_2\Phi_2\cos\theta_2 - \right.$$
$$\left.-\left[(\theta_2)_{tt} - c^2(\theta_2)_{yy}\right]\cos A_2\Phi_2\sin\theta_2\right\} -$$
$$-C_3\left\{\left[(A_3\Phi_3)_{tt} - c^2(A_3\Phi_3)_{zz}\right]\sin A_3\Phi_3\cos\theta_3 - \right.$$
$$\left.-\left[(\theta_1)_{tt} - c^2(\theta_1)_{xx}\right]\cos A_1\Phi_1\sin\theta_1\right\} = 0. \quad (29)$$

On the one hand, simplification of equation (26) has taken shape, on the other hand, constraints on unknown argument-functions appeared in the form of (27), (28).

Second derivatives of equation (26) are determined from (27) and (28) in the variants:

$$1.\ (A_1\Phi_1)_{tt} = c(\theta_1)_{xt},\ \ (\theta_1)_{tt} = c(A_1\Phi_1)_{xt},$$
$$(A_1\Phi_1)_{tx} = c(\theta_1)_{xx},\ \ (\theta_1)_{tx} = c(A_1\Phi_1)_{xx},$$
$$(A_2\Phi_2)_{tt} = c(\theta_2)_{yt},\ \ (\theta_2)_{tt} = c(A_2\Phi_2)_{yt},$$
$$(A_2\Phi_2)_{ty} = c(\theta_2)_{yy},\ \ (\theta_2)_{ty} = c(A_2\Phi_2)_{yy}, \quad (30)$$
$$(A_3\Phi_3)_{tt} = c(\theta_3)_{zt},\ \ (\theta_3)_{tt} = c(A_3\Phi_3)_{zt},$$
$$(A_3\Phi_3)_{tz} = c(\theta_3)_{zz},\ \ (\theta_3)_{tz} = c(A_3\Phi_3)_{zz};$$

$$2.\ (A_1\Phi_1)_{tt} = -c(\theta_1)_{xt},\ \ (\theta_1)_{tt} = -c(A_1\Phi_1)_{xt},$$
$$(A_1\Phi_1)_{tx} = -c(\theta_1)_{xx},\ \ (\theta_1)_{tx} = -c(A_1\Phi_1)_{xx},$$
$$(A_2\Phi_2)_{tt} = -c(\theta_2)_{yt},\ \ (\theta_2)_{tt} = -c(A_2\Phi_2)_{yt},$$
$$(A_2\Phi_2)_{ty} = -c(\theta_2)_{yy},\ \ (\theta_2)_{ty} = -c(A_2\Phi_2)_{yy}, \quad (31)$$
$$(A_3\Phi_3)_{tt} = -c(\theta_3)_{zt},\ \ (\theta_3)_{tt} = -c(A_3\Phi_3)_{zt},$$
$$(A_3\Phi_3)_{tz} = -c(\theta_3)_{zz},\ \ (\theta_3)_{tz} = -c(A_3\Phi_3)_{zz}.$$

Using (30) and (31), define square brackets in equation (29). It can be shown that they are zero. Indeed, subtraction of the second derivatives gives defining differential relationships of the form:

$$(A_1\Phi_1)_{tt} - c^2(A_1\Phi_1)_{xx} = 0,\ \ (\theta_1)_{tt} - c^2(\theta_1)_{xx} = 0.$$
$$(A_2\Phi_2)_{tt} - c^2(A_2\Phi_2)_{yy} = 0,\ \ (\theta_2)_{tt} - c^2(\theta_2)_{yy} = 0. \quad (32)$$
$$(A_3\Phi_3)_{tt} - c^2(A_3\Phi_3)_{zz} = 0,\ \ (\theta_3)_{tt} - c^2(\theta_3)_{zz} = 0.$$

Substitution of (32) into (29) results in a further simplification of the problem. Solution of the partial differential equation (20) is represented as:

$$u = C_1\cos A_1\Phi_1\cos\theta_1 +$$
$$C_2\cos A_2\Phi_2\cos\theta_2 + C_3\cos A_3\Phi_3\cos\theta_3. \quad (33)$$

Expression (33) will also be valid for various combinations of trigonometric functions taking into account (27), (28), (30), (31):

$$u = C_1'\cdot(C_1\sin\theta_1 + C_2\cos\theta_1)\cdot(C_3\sin A_1\Phi_1 + C_4\cos A_1\Phi_1) +$$
$$+C_2'\cdot(C_5\sin\theta_2 + C_6\cos\theta_2)\cdot(C_7\sin A_2\Phi_2 + C_8\cos A_2\Phi_2) +$$
$$+C_3'\cdot(C_9\sin\theta_3 + C_{10}\cos\theta_3)\cdot(C_{11}\sin A_3\Phi_3 + C_{12}\cos A_3\Phi_3), \quad (34)$$

under conditions:

$$(A_1\Phi_1)_t = \pm c(\theta_1)_x,\ (\theta_1)_t = \pm c(A_1\Phi_1)_x,$$
$$(A_1\Phi_{1i})_{tt} - c^2(A_1\Phi_1)_{xx} = 0,\ (\theta_1)_{tt} - c^2(\theta_1)_{xx} = 0,$$
$$(A_2\Phi_2)_t = \pm c(\theta_2)_y,\ \ (\theta_2)_t = \pm c(A_2\Phi_2)_y,$$
$$(A_2\Phi_2)_{tt} - c^2(A_2\Phi_2)_{yy} = 0,\ (\theta_2)_{tt} - c^2(\theta_2)_{yy} = 0,$$
$$(A_3\Phi_3)_t = \pm c(\theta_3)_z,\ \ (\theta_3)_t = \pm c(A_3\Phi_3)_z,$$
$$(A_3\Phi_3)_{tt} - c^2(A_3\Phi_3)_{zz} = 0,\ \ (\theta_3)_{tt} - c^2(\theta_3)_{zz} = 0.$$

Conditions for existence of various solutions in the form of defining differential equations of hyperbolic type for the arguments of trigonometric functions were shown. In general form, solution (34):

$$u = \sum_{i=1}^{n} C_i\cdot(C_i'\sin\theta_i + C_i''\cos\theta_i)\cdot(C_i'''\sin A_i\Phi_i + C_i''''\cos A_i\Phi_i) +$$
$$+\sum_{j=2}^{m} C_j\cdot(C_j'\sin\theta_j + C_j''\cos\theta_2)\cdot(C_j'''\sin A_j\Phi_j + C_j''''\cos A_j\Phi_j) +$$
$$+\sum_{k=3}^{g} C_k\cdot(C_k'\sin\theta_k + C_k''\cos\theta_k)\cdot(C_k'''\sin A_k\Phi_k + C_k''''\cos A_k\Phi_k) \quad (35)$$

under conditions:

$$(A_i\Phi_i)_t = \pm c(\theta_i)_x,\ \ (\theta_i)_t = \pm c(A_i\Phi_i)_x,$$
$$(A_i\Phi_i)_{tt} - c^2(A_i\Phi_i)_{xx} = 0,\ (\theta_i)_{tt} - c^2(\theta_i)_{xx} = 0,$$
$$(A_j\Phi_j)_t = \pm c(\theta_j)_y,\ \ (\theta_j)_t = \pm c(A_j\Phi_j)_y,$$
$$(A_j\Phi_j)_{tt} - c^2(A_j\Phi_j)_{yy} = 0,\ (\theta_j)_{tt} - c^2(\theta_j)_{yy} = 0,$$
$$(A_k\Phi_k)_t = \pm c(\theta_k)_z,\ \ (\theta_k)_t = \pm c(A_k\Phi_k)_z,$$
$$(A_k\Phi_k)_{tt} - c^2(A_k\Phi_k)_{zz} = 0,\ (\theta_k)_{tt} - c^2(\theta_k)_{zz} = 0.$$

Thus, the result (35) which was obtained is a superposition of the flat coordinate-time solutions. In this case, each pair is determined by its differential constraints on the argument-function. In this case, complication of the problem is a kind of a generalizing factor of the proposed approach.

5. Comparison of the study results

Validity of the presented result is determined not only by finding a solution that satisfies the conditions of the problem but also by comparing it with the solutions already known in literature, assessed and tested. Consider a list of solutions with the use of argument-functions for a flat problem of the plasticity theory, namely (7) and (13).

Plasticity theory

$$\frac{\partial^2\tau}{\partial x^2}-\frac{\partial^2\tau}{\partial y^2}=2\frac{\partial^2}{\partial x\partial y}\sqrt{k^2-\tau^2},\quad \tau=C\cdot\exp\theta\cdot\sin A\Phi. \tag{36}$$

Restrictions on expressions (36):

$$\theta_x=\mp A\Phi_y,\quad \theta_y=\pm A\Phi_x,\ \theta_{xx}+\theta_{yy}=0,\ A\Phi_{xx}+A\Phi_{yy}=0.$$

Linear dynamic problem with trigonometric solution and constraints (11):

$$\frac{\partial^2 u}{\partial t^2}=a^2\cdot\frac{\partial^2 u}{\partial x^2}.$$

Solution has the form:

$$u=C_0\left(C_1\sin\theta+C_2\cos\theta\right)\left(C_3\sin A\Phi+C_4\cos A\Phi\right). \tag{37}$$

Restrictions on the expression (37):

$$\theta_t=\mp aA\Phi_x,\quad A\Phi_t=\mp a\theta_x,$$

$$\theta_{tt}-a^2\cdot\theta_{xx}=0,\ A\Phi_{tt}-a^2A\Phi_{xx}=0.$$

Linear dynamical problem with a trigonometric and exponential form of solution (18):

$$u=C_1\exp\theta_1\cdot\left(C_1'\cdot\sin A_1\Phi_1+C_1''\cdot\cos A_1\Phi_1\right)+$$
$$+C_2\exp\theta_2\cdot\left(C_2'\cdot\sin A_2\Phi_2+C_2''\cdot\cos A_2\Phi_2\right). \tag{38}$$

Restrictions on the expression (38):

$$a\theta_{1x}=\pm\theta_{1t},\quad aA_1\Phi_{1x}=\pm A_1\Phi_{1t},$$

$$a\theta_{2x}=\pm\theta_{2t},\quad aA_2\Phi_{2x}=\pm A_2\Phi_{2t},$$

$$a^2\theta_{1xx}-\theta_{1tt}=0,\quad a^2A_1\Phi_{1xx}-A_1\Phi_{1tt}=0,$$

$$a^2\theta_{2xx}-\theta_{2tt}=0,\quad a^2A_2\Phi_{2xx}-A_2\Phi_{2tt}=0.$$

The following results of solutions are obtained using differential relationships between adjacent argument-functions in all above variants (36)–(38). For each variant, the simplest schemes of solutions of invariant differential equations were considered.

Theory of plasticity

The simplest variant of solution (36) is taken:

$$A\Phi_{xx}=A\Phi_{yy}=0,$$

when $\theta_x=\mp A\Phi_y,\ \theta_y=\pm A\Phi_x,\ \theta_{xx}+\theta_{yy}=0,\ A\Phi_{xx}+A\Phi_{yy}=0.$

Argument-functions: $A\Phi=AA_6\cdot x\cdot y,\ \theta=\frac{1}{2}\cdot\left(x^2-y^2\right).$

It can be shown that the θ argument-function satisfies the Laplace equation. Really:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=\frac{\partial^2\left[\frac{1}{2}\cdot AA_6\cdot\left(x^2-y^2\right)\right]}{\partial x^2}+$$
$$+\frac{\partial^2\left[\frac{1}{2}\cdot AA_6\cdot\left(x^2-y^2\right)\right]}{\partial y^2}=AA_6-AA_6=0.$$

There can be another, more complicated solution for the AF function:

$$A\Phi=AA_{13}\cdot x\cdot y\cdot\left(x^2-y^2\right),$$

through the Cauchy-Riemann relationships, the argument-function has the form:

$$\theta=-AA_{13}\cdot\left[\frac{1}{4}\left(x^4+y^4\right)-\frac{3}{2}\cdot x^2\cdot y^2\right].$$

Substitute argument-functions into Laplace's equations to see that they are identically satisfied:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=\left\{\left[-AA_{13}\cdot 3\cdot\left(x^2-y^2\right)\right]+\left[AA_{13}\cdot 3\cdot\left(x^2-y^2\right)\right]\right\}=0,$$

$$\frac{\partial^2 A\Phi}{\partial x^2}+\frac{\partial^2 A\Phi}{\partial y^2}=6\cdot AA_{13}\cdot x\cdot y-6\cdot AA_{13}\cdot x\cdot y.$$

It is possible to obtain a more complex coordinate dependence for the AF function:

$A\Phi=AA_6\cdot x\cdot y\pm AA_{13}\cdot x\cdot y\cdot\left(x^2-y^2\right)$ and so on.

Multiple calculations of the stressed state of metal in the processes of plastic metal working showed qualitative and quantitative convergence of the presented results with experimental data and data of other authors [16, 17].

It can be seen from the last examples that there can be many solutions of the same differential equations and for each of them there are certain boundary or initial conditions that must also be ensured. Taking into account different boundary conditions for different applied problems, it becomes possible to choose the required solution provided with the method of argument-functions.

As noted above, the same problem is encountered in solving dynamic problems of the theory of elasticity.

Linear dynamic problem with basic trigonometric functions

Expression (37) can be simplified and reduced to the form:

$$u=\left(C_1\cdot\sin\theta+C_2\cdot\cos\theta\right)\cdot\sin A\Phi.$$

Argument-functions are defined by relationships (9), (11), i. e:

$$\theta_t=\mp aA\Phi_x,\quad A\Phi_t=\mp a\theta_x,$$

$$\theta_{tt}-a^2\cdot\theta_{xx}=0,\ A\Phi_{tt}-a^2A\Phi_{xx}=0.$$

To satisfy differential equations of hyperbolic type for argument functions, use the simplest case of equation solution in the following form:

$$\theta_{xx}=\theta_{tt}=A\Phi_{xx}=A\Phi_{tt}=0,$$

then one of the variants of the solution for the AF argument-function will be written as:

$$A\Phi=AA_1\cdot x+AA_2\cdot t.$$

Taking into account differential relationships, pass to θ function:

$$a\cdot\theta=AA_2\cdot x+f(t).\text{ or }\ \theta=a\cdot AA_1\cdot t+f(x),$$

$$A\Phi_t=a\theta_x=AA_2\ \text{ or }\ a\cdot\theta=AA_2\cdot x+f(t).$$

Eventually:

$$\theta=a\cdot AA_1\cdot t+AA_2\cdot x,\ \theta=a\cdot AA_1\cdot t+\frac{AA_2}{a}\cdot x. \qquad (39)$$

Expressions for argument-functions satisfy differential equations of hyperbolic type (18), (19) and relationships between adjacent functions. In further simplification, take $AA_2=0$ to obtain:

$$A\Phi=AA_1\cdot x,\ \ \theta=a\cdot AA_1\cdot t. \qquad (40)$$

Demonstrate now that the expressions (40) are trailing solutions for the functions obtained by the method of separation of variables or by Fourier method. Taking the boundary conditions same as in [11] and relationships (18), (19) presented above, the following is obtained:

$$AA_1=\frac{\pi\cdot n}{l},\ \ C_1=A_n,\ \ C_2=B_n.$$

In this case, the differential relationships (18), (19) are also sustained, that is,

$$\theta_t=aA\Phi_x=\frac{\pi n}{l}\cdot a,\ \ A\Phi_t=a\theta_x=0.$$

The relationships are identically satisfied. Taking the boundary and boundary conditions same as in [2] and using (39), (40), the following is obtained:

$$AA_1=\frac{\pi\cdot n}{l},\ \ C_1=A_n,\ \ C_2=B_n.$$

Finally:

$$u=\left[A_n\cos\left(a\cdot\frac{\pi\cdot n}{l}t\right)+B_n\sin\left(a\cdot\frac{\pi\cdot n}{l}t\right)\right]\cdot\sin\left(\frac{\pi\cdot n}{l}x\right),$$

which corresponds to the solution (5) shown in work [11]. It follows that the solution obtained by the Fourier method is a particular case of the solution obtained with the help of argument-functions. In this case, simplifications in the compared solution from work [11] are elementary and multiple with respect to the argument-functions.

If the solution is not simplified, then the following variant can be considered:

$$A\Phi=AA_1\cdot x+AA_2\cdot t,\ \ \theta=a\cdot AA_1\cdot t+\frac{AA_2}{a}\cdot x.$$

Factoring out the AA_1 gives the following:

$$A\Phi=AA_1\left(x+\frac{AA_2}{AA_1}\cdot t\right),\ \ \theta=AA_1\left(a\cdot t+\frac{AA_2}{a\cdot AA_1}\cdot x\right).$$

Taking

$$AA_1=\frac{AA_2}{a},$$

to simplify the result, obtain the following:

$$A\Phi=AA_1(x+a\cdot t),\ \theta=AA_1(a\cdot t+x). \qquad (41)$$

The argument-functions have the same coordinate-time dependence (41).

Taking into account the last expressions for the argument-functions, write expression (37) as:

$$u=(C_1\cdot\sin\theta+C_2\cdot\cos\theta)\cdot\sin A\Phi=$$
$$=\left\{C_1\cdot\sin\left[AA_1(a\cdot t+\cdot x)\right]+C_2\cdot\cos\left[AA_1(a\cdot t+\cdot x)\right]\right\}\times$$
$$\times\sin\left[AA_1(x+a\cdot t)\right].$$

The solution can be reduced to other coordinate-time dependence if negative sign is used in relationships (18), (19). Indeed, what is obtained is:

$$\theta_t=-aA\Phi_x,\ \ A\Phi_t=-a\theta_x,$$

from where:

$$\theta=-a\cdot AA_1\cdot t+f_1(x),\ \ a\cdot\theta=-AA_2\cdot x+f_2(t)$$

or

$$A\Phi=AA_1\cdot x+AA_2\cdot t,\ \ \theta=-a\cdot AA_1\cdot t-\frac{AA_2}{a}\cdot x.$$

After simplifications, the following can be written by analogy with (41):

$$A\Phi=AA_1(x+a\cdot t),\ \ \theta=-AA_1(a\cdot t+x). \qquad (42)$$

Solution (37) for (42) becomes:

$$u=\left\{-C_1\cdot\sin\left[AA_1(a\cdot t+\cdot x)\right]+C_2\cdot\cos\left[AA_1(a\cdot t+\cdot x)\right]\right\}\times$$
$$\times\sin\left[AA_1(x+a\cdot t)\right].$$

The last variants of the argument functions (41) and (42) represent a fragment of the d'Alembert formula (4). Thus, a rather unexpected result was obtained. Maximum simplifications of the argument-functions result in a Fourier solution and the simplifications associated with smaller assumptions lead to the d'Alembert solution. Moreover, in the d'Alembert solution itself there are differences in (41) and (42). It is easy to see that the solutions obtained correspond to different boundary conditions.

Consider a variant of the analytic definition of the d'Alembert formula using solution (38) and corresponding differential constraints of the argument-functions.

Linear dynamic problem with basic trigonometric and exponential functions

This version of the problem is of particular interest since its solution can successfully represent the damped or increasing effect of the rolling tool on the elastic strip during the unsteady roll bite process.

Consider (38). There are elementary differential relationships that make it possible to simplify solution of hyperbolic equations of the form (38). Simplified equations are as follows:

$$\theta_{xx} = \theta_{tt} = A\Phi_{xx} = A\Phi_{tt} = 0.$$

The obtained solutions are in a form of (38)

$$u = C_1 \exp\theta_1 \cdot \left(C_1^{'} \cdot \sin A_1\Phi_1 + C_1^{''} \cdot \cos A_1\Phi_1\right) +$$
$$+C_2 \exp\theta_2 \cdot \left(C_2^{'} \cdot \sin A_2\Phi_2 + C_2^{''} \cdot \cos A_2\Phi_2\right).$$

Differential relationships of argument-functions are as follows:

$$a\theta_{1x} = \pm\theta_{1t}, \quad aA_1\Phi_{1x} = \pm A_1\Phi_{1t},$$

$$a\theta_{2x} = \pm\theta_{2t}, \quad aA_2\Phi_{2x} = \pm A_2\Phi_{2t},$$

$$a^2\theta_{1xx} - \theta_{1tt} = 0, \quad a^2 A_1\Phi_{1xx} - A_1\Phi_{1tt} = 0,$$

$$a^2\theta_{2xx} - \theta_{2tt} = 0, \quad a^2 A_2\Phi_{2xx} - A_2\Phi_{2tt} = 0.$$

Solutions (38) and (37) feature a different differential constraint which defines argument-functions. In (38), there are relationships not with adjacent dependences of the form as for (37) but between the partial derivatives belonging to the same argument-function, i. e.:

$$a\theta_{1x} = \pm\theta_{1t}, \; aA_1\Phi_{1x} = \pm A_1\Phi_{1t},$$

$$a\theta_{2x} = \pm\theta_{2t}, \quad aA_2\Phi_{2x} = \pm A_2\Phi_{2t}.$$

Solving the differential equations presented at the beginning of this variant, one can obtain the following dependences:

$$A_1\Phi_1 = A_1A_3 \cdot x + A_1A_4 \cdot t,$$

$$\theta_1 = A_1A_5 \cdot t + A_1A_6 \cdot x,$$

$$A_2\Phi_2 = A_2A_7 \cdot x + A_2A_8 \cdot t,$$

$$\theta_2 = A_2A_9 \cdot t + A_2A_{10} \cdot x.$$

The form of the functions for adjacent argument-functions is the same. Substitute the last expressions in the differential relationships for the given solution:

$$aA_1\Phi_{1x} = A_1\Phi_{1t} = a \cdot A_1A_3 = A_1A_4,$$

$$a\theta_{1x} = \theta_{1t} = aA_1A_6 = A_1A_5,$$

$$aA_2\Phi_{2x} = -A_2\Phi_{2t} = aA_2A_7 = -A_2A_8,$$

$$a\theta_{2x} = -\theta_{2t} = aA_2A_{10} = -A_2A_9.$$

Taking into account the transformations, write the argument-functions:

$$A_1\Phi_1 = A_1A_3 \cdot x + aA_1 \cdot A_3 \cdot t = A_1A_3 \cdot (x + a \cdot t),$$

$$\theta_1 = aA_1A_6 \cdot t + A_1A_6 \cdot x = A_1A_6 \cdot (x + a \cdot t). \quad (43)$$

$$A_2\Phi_2 = A_2A_7 \cdot x - aA_2A_7 \cdot t = A_2A_7 \cdot (x - a \cdot t),$$

$$\theta_2 = -aA_2A_{10} \cdot t + A_2A_{10} \cdot x = A_2A_{10} \cdot (x - a \cdot t). \quad (44)$$

The argument-functions (43), (44) have the form of (3), (4) from d'Alembert's formulas.

Substitute the argument-function (43), (44) into expression (38) and obtain the final solution:

$$u = C_1\left(\exp A_1A_6 (x + a \cdot t)\right) \times$$
$$\times\left(C_1^{'}\sin A_1A_3 (x + a \cdot t) + C_1^{''}\cos A_1A_3 (x + a \cdot t)\right) +$$
$$+C_2\left(\exp A_2A_{10} (x - a \cdot t)\right) \times$$
$$\times\left(C_2^{'}\sin A_2A_7 (x - a \cdot t) + C_2^{''}\cos A_2A_7 (x - a \cdot t)\right). \quad (45)$$

The argument-functions for exponentials and trigonometric dependencies in (45) are the same. An analogy is observed for the expressions (37) as well when taking into account (41) and (42). The result obtained is of theoretical and practical interest. Indeed, (45) can be written in a general form as:

$$u = \phi_1\left((x + a \cdot t)\right) + \phi_2 (x - a \cdot t).$$

However, this general result is represented in expression (4) as the D'Alembert formula [11]. In this case, the φ function is represented by a concrete coordinate-time expression (45). Besides, the initial conditions of the form as in [20] fit well into solution (45):

$$u^* = C^o \cdot \exp(\pm bt) \cdot (C_1\sin kt + C_2\cos kt).$$

This dependence is suitable for characterizing non-stationary impulse action on the elastic strip in the rolling mill. Indeed, expression (45) can be simplified for:

$$x = 0, a \cdot A_1A_6 = b, \; k = A_1A_3a,$$

$$x = 0, A_2A_{10} = b, \; k = A_2A_7a,$$

$$u^* = C_1\left(\exp(bt)\right) \cdot \left(C_1^{'}\sin(kt) + C_1^{''}\cos(kt)\right) +$$
$$+C_2\left(\exp(-bt)\right) \cdot \left(C_2^{'}\sin(-k \cdot t) + C_2^{''}\cos(-ka \cdot t)\right). \quad (46)$$

Variants (46) of increasing, decreasing functions or their joint action appear. The latter solution is representative when using argument-functions.

In the conditions of transient processes taking place during rolling in adjacent stands, rolls as a system are the source of a damped effect on the strip. The impact is transmitted via the strip to the adjacent stand where a stationary rolling process is realized. A dynamic splash in the last stand appears. This leads to oscillations of the gap between the rolls producing longitudinal thickness variation. Solution (45) makes it possible to evaluate this impact and intervene in the rolling process in the mill stream to eliminate defect formation.

6. Discussion of results: generalization using argument-functions

Comparison of the results obtained in solution with the use of argument-functions with known solutions shows that the presented approach is quite acceptable for calculating pulsed stressing of an elastic medium. In this case, it was not the coordinate-time dependences of the argument-functions that were defined but the conditions for existence of various solutions of the problem that can fit any boundary condition.

It can be seen from the analysis that the result presented in [11–14] was the simplest partial solution of differential relationships for argument-functions. There is a prospect of defining new dependencies for new tasks about which, perhaps, nothing is known yet.

The initial differential equations and boundary conditions determine the type of differential equations for the argument-functions that close solution. On the one hand, argument-functions can be bounded by the Cauchy-Riemann relations and the corresponding differential invariants and on the other hand, by differential relations which lead to the fact that the argument-functions are the same for adjacent coordinate-time dependencies. Besides, analytic dependences on the parameters entering into the d'Alembert formula were obtained.

7. Conclusions

The paper presents development of general approaches to solving differential wave equations using argument-functions. The known solutions of the dynamic problem are in accordance with the proposed approaches and are their partial solutions.

The result obtained is a superposition of flat coordinate-time solutions. Besides, each pair is determined by its differential constraints on the argument-function. In this case complication of the problem is a kind of generalizing factor of the proposed approach.

Conditions for existence of new solutions of the wave problem that are restricted by boundary conditions of different processes were determined using known solutions: plasticity theory, linear dynamic problem with trigonometric solution and constraints, and a linear dynamical problem with basic trigonometric and exponential functions. Invariant differential relationships for argument-functions are the closing element of the solution.

A mathematical model of a dynamic problem with an increasing or damped wave action upon an elastic medium was developed which makes it possible to evaluate this effect and intervene in the rolling process in the mill workflow and consequently eliminate defect formation.

References

1. Putnoki, A. Yu. Mathematical model of rolling dynamics when filling finishing train of wide-strip mill with strip [Text] / A. Yu. Putnoki // Metallurgical and Mining Industry. – 2015. – Issue 11. – P. 218–222.
2. Bronshteyn, I. M. Spravochnik po matematike [Text] / I. M. Bronshteyn, K. L. Semendyayev. – Moscow: Nauka, 1964. – 608 p.
3. Sneddon, I. N. Klassicheskaya teoriya uprugosti [Text] / I. N. Sneddon, D. S. Berri; E. I. Grigolyuk (Ed.). – Moscow, 1961. – 219 p.
4. Mehtiev, M. F. Asimptoticheskiy analiz nekotoryih prostranstvennyih zadach teorii uprugosti dlya polyih tel [Text] / M. F. Mehtiev. – Baku, 2008. – 320 p.
5. Krupoderov, A. V. Reshenie nekotoryih dinamicheskih zadach teorii uprugosti metodom granichnyih elementov [Text] / A. V. Krupoderov, S. S. Scherbakov // Teoreticheskaya i prikladnaya mehanika. – 2013. – Issue 28. – P. 294–300.
6. Ermolenko, G. Yu. Reshenie dinamicheskoy zadachi anizotropnoy teorii uprugosti so smeshannyimi kraevyimi usloviyami [Text] / G. Yu. Ermolenko // Vestn. Sam. gos. tehn. un-ta. Ser. Fiz.-mat. nauki. – 2003. – P. 86–88.
7. Sinekop, N. S. Metod R-funktsiy v dinamicheskih zadachah teorii uprugosti [Text] / N. S. Sinekop, L. S. Lobanova, L. A. Parhomenko. – Kharkiv, 2015. – 95 p.
8. Zakaryan, T. V. O sobstvennyih kolebaniyah ortotropnyih plastin v pervoy kraevoy zadachi teorii uprugosti pri nalichie vyazkogo soprotivleniya [Text] / T. V. Zakaryan // Izv. NAN Armenii. Mehanika. – 2013. – Vol. 66, Issue 3. – P. 38–48.
9. Bagdaev, A. G. Reshenie nestatsionarnoy smeshannoy granichnoy zadachi teorii uprugosti dlya poluprostranstva [Text] / A. G. Bagdaev, A. V. Vardanyan, S. V. Vardanyan // Izv. NAN Armenii. Mehanika. – 2007. – Vol. 60, Issue 4. – P. 23–37.
10. Kupradze, V. D. Dinamicheskie zadachi teorii uprugosti i termouprugosti [Text] / V. D. Kupradze, T. V. Burchuladze // Itogi nauki i tehn. Seriya «Sovremennyie problemyi matematiki». – 1975. – Vol. 7. – P. 163–294.
11. Tikhonov, A. N. Uravneniya matematicheskoy fiziki [Text] / A. N. Tikhonov, A. A. Samarskiy. – Moscow: Nauka, 1966. – 724 p.
12. Panovko, Ya. G. Osnovy prikladnoy teorii uprugikh kolebaniy i udara [Text] / Ya. G. Panovko. – Leningrad: Mashinostroyeniye, 1976. – 320 p.
13. Babanov, I. M. Teoriya kolebaniy [Text] / I. M. Babanov. – Moscow: Nauka, 1968. – 560 p.
14. Noritsin, I. A. Proyektirovaniye kuznechnykh i kholodnoshtampovochnykh tsekhov i zavodov [Text] / I. A. Noritsin. – Moscow: Vysshaya shkola, 1977. – 422 p.
15. Chigirinskiy, V. V. Metod resheniya zadach teorii plastichnosti s ispolzovaniyem garmonicheskikh funktsiy [Text] / V. V. Chigirinskiy // Izv. vuzov. Chernaya metallurgiya. – 2009. – Issue 5. – P. 11–16.

16. Chigirinskiy, V. V. Proizvodstvo tonkostennogo prokata spetsialnogo naznacheniya [Text] / V. V. Chigirinskiy, Yu. S. Kresanov, A. Ya. Kachan, A. V. Boguslaev, G. I. Legotkin, A. Ya. Slepyinin et. al. – Zaporozhe, 2014. – 285 p.
17. Chigirinsky, V. V. Determination of integral characteristics of stress state of the point during plastic deformation in conditions of volume loading [Text] / V. V. Chigirinsky, A. A. Lenok, S. M. Echin // Metallurgical and Mining Industry. – 2015. – Issue 11. – P. 153–163.
18. Chigirinskiy, V. V. Novyye podkhody v reshenii dinamicheskikh zadach obrabotki metallov davleniyem [Text] / V. V. Chigirinskiy, S. P. Sheyko, V. V. Plakhotnik // Visnik SevNTU. – 2013. – Issue 137. – P. 99–102.
19. Chigirinskiy, V. V. Vliyaniye dinamicheskogo nagruzheniya v smezhnykh kletyakh prokatnogo stana [Text] / V. V. Chigirinskiy, A. Yu. Putnoki // Fundamentalnyye i prikladnyye problem tekhniki i tekhnologii. – 2015. – Issue 4 (312). – P. 21–26.
20. Targ, S. M. Kratkiy kurs teoreticheskoy mehaniki [Text] / S. M. Targ. – Moscow: Vyisshaya shkola, 1998. – 411 p.

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METHODS OF BALANCING OF AN AXISYMMETRIC FLEXIBLE ROTOR BY PASSIVE AUTO-BALANCERS

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Застосовано емпіричний критерій настання автобалансування для гнучкого осесиметричного ротора, що балансується n пасивними автобалансирами будь-якого типу. Встановлено, що автобалансування може відбуватися тільки на швидкостях, що перевищують n-ю критичну швидкість обертання ротора. Знайдено діапазони кутових швидкостей обертання ротора, на яких наступатиме автобалансування. Запропоновано способи оптимального балансування гнучкого ротора

Ключові слова: гнучкий ротор, пасивний автобалансир, автобалансування, критерій настання автобалансування, критичні швидкості гнучкого ротора

Применен эмпирический критерий наступления автобалансировки для гибкого осесимметричного ротора, балансируемого n пассивными автобалансирами любого типа. Установлено, что автобалансировка может происходить только на скоростях, превышающих n-ю критическую скорость вращения ротора. Найдены диапазоны угловых скоростей вращения ротора, на которых будет наступать автобалансировка. Предложены способы оптимальной балансировки ротора

Ключевые слова: гибкий ротор, пассивный автобалансир, автобалансировка, критерий наступления автобалансировки, критические скорости гибкого ротора

1. Introduction

Many rotors of aircraft engines, gas turbine engines of power plants, agricultural machines, etc. work at speeds above the first critical one, and therefore behave as flexible [1, 2]. The form and unbalance of the flexible rotor depend on the current speed. In addition, during the operation of such rotors, their unbalance can change due to temperature, wear, dirt sticking, etc. Therefore, it is expedient to constantly balance flexible rotors in motion, in the process of exploitation, by passive auto-balancers [3]. For application of passive auto-balancers, it is necessary to know whether it is possible in principle and on what rotation speeds to balance the flexible rotor installed on the certain supports by them in motion.

Dynamics of rotors without auto-balancers is described by rather difficult differential equations of motion [1–5]. Introduction of auto-balancers (masses movable relative to the rotor) to the system makes the equations even more complicated [3, 6–16]. Therefore, an analytical determination of the conditions for the occurrence of auto-balancing is a complex mathematical problem.

Analytically, the conditions for the occurrence of auto-balancing are determined in [3–16]. At the same time, the most general conditions applicable for auto-balancers of any type and with any number of corrective weights, are received using the empirical criteria [3–5].

Thus, it is actual to find the conditions for the occurrence of auto-balancing in the case of balancing of the flexible massive rotor by any number of auto-balancers of any type.