Встановлено необхідні та достатні умови, за яких гармонійну в кулі функцію n-вимірного простору, n≥3, можна продовжити до цілої гармонійної. Умови формулюються в термінах найкращого наближення цієї функції гармонійними многочленами. Також отримано вирази для узагальненого та нижнього узагальненого порядків цілої гармонійної в просторі функції через похибку апроксимації функції, яка продовжується гармонійними многочленами

Ключові слова: сферичні гармоніки, ціла гармонійна функція, узагальнений порядок, нижній узагальнений порядок

Установлены необходимые и достаточные условия, при которых гармоническую в шаре функцию n-мерного пространства, n≥3, можно продлить к целой гармонической. Условия формулируются в терминах наилучшего приближения этой функции гармоническими многочленами. Также получены выражения для обобщенного и нижнего обобщенного порядков целой гармонической в пространстве функции через погрешность аппроксимации функции, которая продолжается гармоническими многочленами

Ключевые слова: сферические гармоники, целая гармоническая функция, обобщенный порядок, нижний обобщенный порядок

UDC 517.572

DOI: 10.15587/1729-4061.2017.108387

CRITERION OF THE CONTINUATION OF HARMONIC FUNCTIONS IN THE BALL OF n-DIMENSIONAL SPACE AND REPRESENTATION OF THE GENERALIZED ORDERS OF THE ENTIRE HARMONIC FUNCTIONS IN $\mathbb{R}^n$ IN TERMS OF APPROXIMATION ERROR

O. Veselovska
PhD, Associate Professor
Department of Mathematics
Lviv Polytechnic National University
S. Bandery str., 12, Lviv, Ukraine, 79013
E-mail: veselovskaov@gmail.com

K. Drohomyretska
PhD, Associate Professor
Department of Mathematics
Lviv Polytechnic National University
S. Bandery str., 12, Lviv, Ukraine, 79013
E-mail: christdr@mail.lviv.ua

L. Kolyasa
PhD
Department of Mathematics
Lviv Polytechnic National University
S. Bandery str., 12, Lviv, Ukraine, 79013
E-mail: kolyasa.lubov@gmail.com

1. Introduction

In the theory of entire functions of one complex variable, one of the most thoroughly examined issues is the relation between the growth of such functions and behavior of Taylor coefficients. Along with this, in [1, 2], the approximation of entire functions on compact sets was studied and generalized characteristics of such functions in terms of error approximation or interpolation were obtained. Similar studies were conducted for harmonic functions as they expand into series in spherical harmonics in space $\mathbb{R}^n$, $n\geq 3$ and, in addition, into series in the adjoined Legendre polynomials in space $\mathbb{R}^3$. Also the problem on expressing the characteristics of growth of harmonic functions in terms that are not related to the coefficients of their expansion into series was considered. In particular, characteristics of the growth of a harmonic function in an n-dimensional space were expressed in terms of the norm of their gradient at the origin, while in a three-dimensional space – in terms of approximation errors of harmonic functions in the ball by harmonic polynomials.

The relevance of such problems is due to the fact that the harmonic functions play an important role not only in theoretical mathematical research, but are used in physics and mechanics in order to describe different stationary processes. Therefore it is important to study generalized characteristics of the growth of a harmonic function in an n-dimensional space.

2. Literature review and problem statement

Paper [3] expressed generalized characteristics of growth of entire functions of several complex variables by using the best polynomial approximation and interpolation on compact sets, paper [4] – in terms of the best approximation error in L^p norm. Relative generalized and lower generalized orders for the entire functions of two complex variables were introduced and investigated in [5]. Similar problems were considered for analytic functions of several complex variables in [6, 7].

The first studies into harmonic functions of a three-dimensional space were made in [8], in particular, by using an integral Bergman representation of a harmonic function in $\mathbb{R}^3$ by an entire function of a complex variable and a real parameter, they represented the order of a harmonic function and the type of an axisymmetric harmonic function in $\mathbb{R}^3$ in terms of expansion coefficients of these functions by the adjoined Legendre polynomials. This result was refined and generalized in [9, 10]. Paper [11] derived formulae for the generalized order of a harmonic function in space $\mathbb{R}^n$ in terms of its Fourier coefficients, while articles [12, 13] represented characteristics of the growth of a harmonic function in space in terms of the norm of its gradient at the origin. Paper [14] investigated the growth of a *J*-universal harmonic function in space.

Paper [15] addresses the study of uniform approximation by polynomials of the generalized axisymmetric potentials. The order and type of potentials are represented in terms of an approximation error. In [16], authors investigated approximation with the Chebyshev polynomials of the entire solutions to the Helmholtz equation and obtained certain estimates for the growth parameters of these solutions in terms of coefficients and an approximation error in the *sup* norm. Study of the generalized q-type and generalized lower q-type solutions of ordinary elliptic differential equation in partial derivatives is given in article [17]. Approximation by the Chebyshev polynomials of the entire solutions of the Helmholtz equation in Banach spaces $B(p,q,m)$ is considered in [18]. Paper [19] builts on the studies of [16]. Expressions for the order and type of solutions of certain linear differential equations in partial derivatives in terms of error in the axisymmetric harmonic polynomial approximation and the Lagrange interpolation were derived in [20]. In [21], a slow growth and the approximation of pseudoanalytic functions on disk were investigated.

The necessary and sufficient conditions under which a harmonic function in the ball of a three-dimensional space continues to the entire harmonic one were established in [22]. It also represented the order and type of an entire harmonic function in terms of an approximation error of the continued function with harmonic polynomials. A similar issue was examined in [23] for the (p,q) orders and types of harmonic function in $\mathbb{R}^3$.

A review of the scientific literature revealed that a number of papers were devoted to examining a relation between the growth of harmonic functions of a three-dimensional or an n-dimensional spaces and the behavior of expansion coefficients of these functions into series by the adjoined Legendre or Chebyshev polynomials, or in spherical harmonics. Along with this, the approximation of harmonic functions was studied, as well as solutions of some differential equations in partial derivatives using different polynomials with respect to different norms, while expressions for the various characteristics of growth of such functions in terms of approximation or interpolation errors were obtained.

Papers [22, 23] considered uniform approximation of harmonic functions in a three-dimensional space and, in order to characterize a growth of the continued functions, the order, type, the (p,q) orders and types were applied.

Of significant interest, therefore, is the examination of a uniform approximation in an n-dimensional space, and in order to characterize the growth of an entire harmonic function in space $\mathbb{R}^n$, more general characteristics of growth, outlined in [24], are employed.

3. The aim and objectives of the study

The goal of present work is to establish conditions under which a harmonic function in the ball of an n-dimensional space continues to the entire harmonic function, and to derive formulae for the generalized growth characteristics of harmonic function in space.

To accomplish the set goal, the following tasks have been resolved:

– to obtain an estimate for the uniform norm of spherical harmonics in terms of best approximation of harmonic function in the ball by harmonic polynomials;

– to assess approximation error of a harmonic function in the ball in terms of a maximum modulus of a harmonic function in space;

– to assess the maximum modulus of a harmonic function in space in terms of a maximum modulus of some entire function of one complex variable or a maximum term of its power series.

4. Criterion of the continuation of harmonic function in the ball of *n*-dimensional space to the entire harmonic

Let $S^n = \{x \in \mathbb{R}^n : |x| = 1\}$ be a unit sphere in $\mathbb{R}^n$ centered at the origin, while

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

is its surface area, where Γ denotes the Gamma function.

A spherical harmonic or a sperical Laplace function of degree k, $k \in Z_+ = \{0, 1, 2,..\}$ which is denoted by $Y^{(k)}$, is called a restriction of a homogeneous harmonic polynomial of degree k on the unit sphere S^n, $n \geq 2$, [25].

A set of spherical harmonics of degree k can be considered as a sub-space of the space $L^2(S^n)$ of real-valued functions with the scalar product

$$(f,g) = \frac{1}{\omega_n} \int_{S^n} f(x) g(x) \mathrm{d}S,$$

where dS is the element of the surface area on the sphere S^n. If $\left\{Y_1^{(k)},\dots,Y_{\gamma_k}^{(k)}\right\}$ is an orthonormal base in the given subspace, then

$$\bigcup_{k=0}^{\infty} \left\{Y_1^{(k)},\dots,Y_{\gamma_k}^{(k)}\right\}$$

will be an orthonormal base in space $L^2(S^n)$. Here

$$\gamma_k = \frac{(2k+n-2)(k+n-3)!}{k!(n-2)!}$$

is the quantity of linearly independent spherical harmonics of degree k.

Let u be an entire harmonic function in $\mathbb{R}^n$, that is, the harmonic function over the whole space $\mathbb{R}^n$. Then it expands into a Fourier-Laplace series [26]

$$u(rx)=\sum_{k=0}^{\infty}Y^{(k)}(x;u)\,r^k, \qquad (1)$$

where $x\in S^n$,

$$Y^{(k)}(x;u)=a_1^{(k)}Y_1^{(k)}(x)+a_2^{(k)}Y_2^{(k)}(x)+\ldots+a_{\gamma_k}^{(k)}Y_{\gamma_k}^{(k)}(x),$$

$$a_j^{(k)}=\left(u,Y_j^{(k)}\right),\quad j=\overline{1,\gamma_k},$$

$\left(u,Y_j^{(k)}\right)$ is the scalar product in $L^2(S^n)$.

At n=2, spherical harmonics reduce to ordinary trigonometric functions of an angle. At $n\geq 3$, they have a more complicated structure and are connected with some polynomials of special form.

Assume that $d_2=1$, $d_n=n-2$ at $n>2$, while

$$\nu=\frac{n-2}{2}.$$

Then

$$Y^{(k)}(x;u)r^k=\frac{2(k+\nu)}{d_n\omega_n}\int_{S^n}C_k^{\nu}\left[(x,y)\right]u(ry)\mathrm{d}S(y), \qquad (2)$$

where $k\in Z_+$, $x\in S^n$, $(\cdot,\cdot)$ is the sclar product in $\mathbb{R}^n$ and C_k^{ν} are the Gegenbauer polynomials of degree k and order ν [27], which are determined by

$$\frac{1-\tau^2}{\left(1-2\tau t+\tau^2\right)^{\nu+1}}=1+2\sum_{k=1}^{\infty}\frac{k+\nu}{d_n}C_k^{\nu}(t)\,\tau^k,$$

where $|t|\leq 1$, $0\leq\tau<1$.

Let

$$K_R^n=\left\{y\in\mathbb{R}^n:|y|\leq R\right\}$$

be the ball of radius R in space $\mathbb{R}^n$, $n\geq 3$, centered at the origin, and $\overline{K_R^n}$ be the closure of K_R^n. A class of harmonic in K_R^n and continuous on $\overline{K_R^n}$ functions will be denoted by H_R, where $0<R<\infty$.

Let Π_k be a set of harmonic polynomials of degree no higher than k. Approximation error of function $u\in H_R$ by harmonious polynomials $P\in\Pi_k$ will be determined as

$$E_R^k(u)=\inf_{P\in\Pi_k}\left\{\max_{y\in\overline{K_R^n}}\left|u(y)-P(y)\right|\right\}. \qquad (3)$$

Theorem 1. Function $u\in H_R$ continues to the entire harmonic function of an n-dimensional space $\mathbb{R}^n$, $n\geq 3$, if and only if the following equality holds

$$\lim_{k\to\infty}\sqrt[k]{E_R^k(u)}=0, \qquad (4)$$

where $E_R^k(u)$ are determined by expression (3).

To prove the theorem, the following lemmas are required.

Lemma 1. If $u\in H_R$, then for all $k\in N$ inequality

$$\max_{x\in S^n}\left|Y^{(k)}(x;u)\right|R^k\leq\frac{4(k+2\nu)^{2\nu}}{(2\nu)!}E_R^{k-1}(u)$$

holds, where

$$\nu=\frac{n-2}{2},$$

$E_R^k(u)$ are determined by expression (3).

Proof. Since a harmonic polynomial is the sum of homogeneous harmonic polynomials, then on the basis of addition theorem [27] for the Gegenbauer polynomials C_k^{ν}, we obtain

$$\int_{S^n}C_k^{\nu}\left[(x,\xi)\right]P(\tau\xi)dS(\xi)=0,$$

where $P\in\Pi_{k-1}$, $0<\tau<R$, $x\in S^n$. Considering this, we shall re-write (2) as

$$Y^{(k)}(x;u)\tau^k=$$

$$=\frac{k+\nu}{\nu\omega_n}\int_{S^n}C_k^{\nu}\left[(x,\xi)\right]\left\{u(\tau\xi)-P(\tau\xi)\right\}dS(\xi).$$

Hence, taking into account equality

$$\max_{-1\leq t\leq 1}\left|C_k^{\nu}(t)\right|=C_k^{\nu}(1),$$

from [27], where

$$C_k^{\nu}(1)=\frac{(k+2\nu-1)!}{(d_n-1)!k!},$$

we find

$$\left|Y^{(k)}(x;u)\right|\;\tau^k\leq\frac{k+\nu}{\nu\omega_n}\max_{\tau\xi\in\overline{K_R^n}}\left|u(\tau\xi)-P(\tau\xi)\right|C_k^{\nu}(1)\omega_n\leq$$

$$\leq\frac{2(k+2\nu)^{2\nu}}{(2\nu)!}\max_{\tau\xi\in\overline{K_R^n}}\left|u(\tau\xi)-P(\tau\xi)\right|. \qquad (5)$$

Next, it follows from determining an error $E_R^k(u)$ that there exists polynomial $P^*\in\Pi_{k-1}$, for which

$$\max_{\tau\xi\in\overline{K_R^n}}\left|u(\tau\xi)-P^*(\tau\xi)\right|\leq 2E_R^{k-1}(u). \qquad (6)$$

Putting $P=P^*$ into inequalities (5) and taking into account inequality (6), as well as arbitrariness of τ, we obtain the assertion of Lemma 1.

Let

$$B_k=\sqrt{\frac{(2\nu)!}{2}}\frac{1}{(k+2\nu)^{\nu}}\max_{x\in S^n}\left|Y^{(k)}(x;u)\right|$$

and

$$M(r,\mathrm{u})=\max_{x\in S^m}\left|u(rx)\right|,\quad r>0 \qquad (7)$$

be a maximum modulus of function u.

Lemma 2. For an entire harmonic function u in $\mathbb{R}^n$, $n\geq 3$, which is assigned by the series (1), the following inequalities hold

$$B_k\leq M(r,u)r^{-k}$$

for all $k\in Z_+$ and $r>0$.

The proof of this Lemma is given in [10].

Lemma 3. For an entire harmonic function u in $\mathbb{R}^n$, $n\geq 3$, the following estimation holds

$$E_R^k(u)\leq\sqrt{\frac{2}{(2\nu)!}}(2\nu+1)!(k+2\nu)^{2\nu}M(r,u)\left(\frac{R}{r}\right)^k$$

for all $r>eR$ and $k\in Z_+$.

Proof. Let us consider function

$$Q(r\xi)=\sum_{j=0}^{k}Y^{(j)}(\xi;u)r^j,$$

where $r>0$, $\xi\in S^n$. It is a harmonic polynomial of degree no higher than k, that is $Q\in\Pi_k$. Taking into acount the definition of error $E_R^k(u)$ and considering Lemma 2, and that the function $u\in H_R$ expands into a seies (1) for all r, $0<r<R$, we find

$$E_R^k(u)\leq\max_{\tau\xi\in K_R^n}\left|u(\tau\xi)-Q(\tau\xi)\right|\leq\sum_{j=k+1}^{\infty}\max_{\xi\in S^n}\left|Y^{(j)}(\xi;u)\right|R^j\leq$$

$$\leq\sqrt{\frac{2}{(2\nu)!}}M(r,u)\sum_{j=k+1}^{\infty}(j+2\nu)^{\nu}\left(\frac{R}{r}\right)^j=$$

$$=\sqrt{\frac{2}{(2\nu)!}}M(r,u)\left(\frac{R}{r}\right)^k\sum_{j=k+1}^{\infty}(j+2\nu)^{\nu}\left(\frac{R}{r}\right)^{j-k}.$$

Let us estimate the last sum. For $r>eR$, we obtain

$$\sum_{j=k+1}^{\infty}(j+2\nu)^{\nu}\left(\frac{R}{r}\right)^{j-k}\leq$$

$$\leq e^k\sum_{j=k+1}^{\infty}(j+2\nu)^{2\nu}e^{-j}\leq e^k\int_k^{\infty}(t+2\nu)^{2\nu}e^{-t}dt.$$

Selecting $s=2\nu$ and $h_s(t)=(t+s)^s$ and integrating s+1 times in parts, we obtain

$$\int_k^{\infty}h_s(t)e^{-t}dt=\left[-e^{-t}\left(h_s(t)+h_s'(t)+\ldots+h_s^{(s)}(t)\right)\right]_k^{\infty}.$$

Considering that

$$h_s^{(i)}(t)=\frac{s!}{(s-i)!}(t+s)^{s-i}$$

for $i=\overline{1,s}$, we find

$$\int_k^{\infty}h_s(t)e^{-t}dt=e^{-k}\sum_{i=0}^{s}\frac{s!(k+s)^{s-i}}{(s-i)!}=$$

$$=e^{-k}\sum_{i=0}^{2\nu}\frac{(2\nu)!(k+2\nu)^{2\nu-i}}{(2\nu-i)!}.$$

The last sum does not exceed $(2\nu+1)!(k+2\nu)^{2\nu}$, which proves Lemma 3.

Proof of Theorem 1. Assume that function $u\in H_R$ continues to the entire harmonic function in space $\mathbb{R}^n$, $n\geq 3$, which will also be denoted by u. Then equality (4) directly follows from Lemma 3. On the contrary, employing Lemma 1, we obtain

$$\left|\sum_{k=0}^{\infty}Y^{(k)}(\xi;u)r^k\right|\leq\left|Y^{(0)}(\xi;u)\right|+$$

$$+\frac{4}{(2\nu)!}\sum_{k=1}^{\infty}(k+2\nu)^{2\nu}E_R^{k-1}(u)\left(\frac{r}{R}\right)^k, \quad (8)$$

hence, based on (4), a uniform convergence of series in the right side of equality (1) on compact subsets of the $\mathbb{R}^n$ follows. Therefore, setting the function $u\in H_R$ by a series (1), we shall continue it over the whole space $\mathbb{R}^n$.

5. Formulae for the generalized and lower generalized orders of a harmonic function of an *n*-dimensional space

Let the function γ be defined and differentiable on interval $[a;+\infty)$ at some $a\geq 0$, strictly monotonically increasing, and $t\to\infty$ as ∞. According to [24], it belongs to the class L^0, if for any real function ψ, so that $\psi(t)\to 0$ as $t\to\infty$, the following equality holds

$$\lim_{t\to\infty}\frac{\gamma\left[(1+\psi(t))t\right]}{\gamma(t)}=1$$

and belongs to the class Λ if for all c, $0<c<\infty$, we have

$$\lim_{t\to\infty}\frac{\gamma(ct)}{\gamma(t)}=1.$$

Using functions α, β from the classes of L^0, Λ, by analogy with [24], we shall introduce a generalised and a lower generalized order of the entire harmonic function u in $\mathbb{R}^n$ by equalities

$$\rho_{\alpha\beta}(u)=\overline{\lim_{r\to\infty}}\frac{\alpha(\ln M(r,\mathrm{u}))}{\beta(r)},\ \lambda_{\alpha\beta}(\mathrm{u})=\underline{\lim_{r\to\infty}}\frac{\alpha(\ln M(r,\mathrm{u}))}{\beta(r)},$$

where $M(r, u)$ are determined by equality (7).

Put

$$F(t,c)=\beta^{-1}(c\alpha(t)), \quad (9)$$

where β^{-1} is the function, inverse to β.

Theorem 2. Let u be an entire harmonic function of an n-dimensional space, $n\geq 3$. If for all c, $0<c<\infty$, one of the following conditions is satisfied

a) $\alpha,\beta\in\Lambda,\dfrac{d\ln F(t,c)}{d\ln t}=O(1),\quad t\to\infty;$

b) $\alpha,\beta\in L^0,\lim\limits_{t\to\infty}\dfrac{d\ln F(t,c)}{d\ln t}=p,\quad 0<p<\infty,$

where function $F(t,c)$ is determined by (9), then the generalized order $\rho_{\alpha\beta}(u)$ of the entire harmonic function u in $\mathbb{R}^n$ is determined by equality

$$\rho_{\alpha\beta}(u)=\overline{\lim_{k\to\infty}}\frac{\alpha(pk)}{\beta\left(e^pR\left[E_R^k(u)\right]^{-1/k}\right)},$$

In the case, condition a) is satisfied, the number p is considered to be an arbitrary positive one.

Proof of Theorem 2. Consider the entire functions of complex variable z:

$$f_1(z)=\sum_{k=0}^{\infty}\frac{\sqrt{(2\nu)!}}{\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}}E_R^k(u)\left(\frac{z}{R}\right)^k,$$

$$f_2(z)=\sum_{k=1}^{\infty}\frac{4}{(2\nu)!}(k+2\nu)^{2\nu}E_R^{k-1}(u)\left(\frac{z}{R}\right)^k.$$

For $r>eR$, by Lemma 3 and inequality (8), we obtain

$$\mu(r;f_1)\le M(r;u)\le\left|Y^{(0)}(\xi;u)\right|+M(r;f_2),\qquad(10)$$

where $\mu(r;f_1)$ is the maximum term of power series of function $f_1(z)$ on circle $\{z:|z|=r\}$, and

$$M(r;f_2)=\max_{|z|=r}|f_2(z)|$$

is the maximum of the module of function $f_2(z)$. Hence

$$\rho_{\alpha\beta}(f_1)\le\rho_{\alpha\beta}(u)\le\rho_{\alpha\beta}(f_2).\qquad(11)$$

Applying a formula that expresses the generalized order of an entire function of one complex variable in term of coefficients of its power series [24] and using the fact that function β belongs to one of classes Λ or L^0, we obtain the equality

$$\rho_{\alpha\beta}(f_1)=\rho_{\alpha\beta}(f_2)=\overline{\lim_{k\to\infty}}\frac{\alpha(pk)}{\beta\left(e^pR\left[E_R^k(u)\right]^{-1/k}\right)},$$

which, together with (11), completes the theorem proving.

Note that for an entire harmonic function u in $\mathbb{R}^n$ from Theorem 2 it is possible to obtain:

1) at $\alpha(t)=\beta(t)=\ln t$ the formula for the order $\rho(u)$:

$$\rho(u)=\overline{\lim_{k\to\infty}}\frac{k\ln k}{\ln\frac{1}{E_R^k(u)}};$$

2) at $\alpha(t)=t,\ \beta(t)=t^{\rho},\ p=\frac{1}{\rho},$ where ρ is the order of function u, the formula for the type $\sigma(u)$:

$$R(\sigma(u)\rho e)^{1/\rho}=\overline{\lim_{k\to\infty}}k^{1/\rho}\sqrt[k]{E_R^k(u)};$$

3) at $\alpha(t)=t,\ \beta(t)=t^{\rho(t)},$ where $\rho(t)$ is the proximate order of function u, the formula for the type $\sigma^*(u)$ relatively to the proximate order $\rho(t)$:

$$R(\sigma^*(u)\rho e)^{1/\rho}=\overline{\lim_{k\to\infty}}\psi(k)\sqrt[k]{E_R^k(u)},$$

where $t=\psi(\tau)$ is the function, inverse to $\tau=t^{\rho(t)}$.

Theorem 2 is complemented by the following theorem.

Theorem 3. Let u be an entire harmonic function of an n-dimensional space, $n\ge3,\ E_R^k,\ F(t,c)$ are determined by equalities (3) and (9), respectively. If $\beta\in L^0$, and α is such that $\alpha(e^t)\in L^0$, and for all c, $0<c<\infty$, the following condition is satisfied

$$\ln\left(\frac{d\ln F(t,c)}{d\ln t}\right)=o(\ln t),\quad t\to\infty,$$

then the generalized order $\rho_{\alpha\beta}(u)$ for an entire harmonic function u in $\mathbb{R}^n$ is determined by equality

$$\rho_{\alpha\beta}(u)=\overline{\lim_{k\to\infty}}\frac{\alpha(k)}{\beta\left(R\left[E_R^k(u)\right]^{-1/k}\right)}.$$

In the case of entire harmonic functions of zero order, a sharper characteristic of growth is given by the following theorem.

Theorem 4. Let u be an entire harmonic function of an n-dimensional space $n\ge3$, E_R^k are determined by equality (3), $\alpha\in\Lambda$, $\Phi(t,c)=\alpha^{-1}(c\alpha(t))$ and for all c, $0<c<\infty$, at sufficiently large t, the following inequality holds

$$0\le\frac{d\Phi(t,c)}{dt}\le A_1e^{A_2\Phi(t,c)},$$

where $A_1,\ A_2$ – constants, such that

$$0<A_1<\infty,\quad 0<A_2<\infty.$$

Then

$$\overline{\lim_{k\to\infty}}\frac{\alpha(\ln\ln M(r,u))}{\alpha(\ln\ln r)}=$$

$$=\max\left\{1,\overline{\lim_{k\to\infty}}\frac{\alpha(\ln k)}{\alpha\left[\ln\left(\frac{1}{k}\ln\left[E_R^k(u)\right]^{-1}\right)\right]}\right\}.$$

Here it is possible to choose $\alpha(x)=\ln_j x$, where $j\ge1$, and

$$\ln_1 x=\ln x,\quad \ln_j x=\ln(\ln_{j-1}x)$$

is the j-th iteration of the logarithm.

Theorems 3, 4 directly follow from inequality (10) and similar results for entire functions of one complex variable [28].

Theorem 5. Let $u\in H_R$. If condition (4) is satisfied, then function u can be continued to the entire harmonic function in space $\mathbb{R}^n$, $n\ge3$, for which

$$\lambda_{\alpha\beta}(u)\ge\underline{\lim_{k\to\infty}}\frac{\alpha(pk)}{\beta\left(e^pR\left[E_R^k(u)\right]^{-1/k}\right)}.\qquad(12)$$

If, in addition, ratio

$$\frac{E_R^k(u)}{E_R^{k+1}(u)}$$

is a nondecreasing function of k, and one of the a), b) conditions of theorem 2 is satisfied, then inequality (12) transforms into the equality.

The proof of this theorem is analogous to Theorem 2 proving.

Corollary. Let u be an entire harmonic function in $\mathbb{R}^n$, $n\ge3$, Then

$$\lambda(u)\ge\underline{\lim_{k\to\infty}}\frac{k\ln k}{\ln\frac{1}{E_R^k(u)}}.$$

The inequality becomes an equality when ratio $\frac{E_R^k(u)}{E_R^{k+1}(u)}$ is a nondecreasing function of k.

In the course of studies, we obtained a criterion of the continuation of harmonic function in the ball of n-dimensional space to the entire harmonic one. This criterion is formulated in terms of the best approximation of the harmonic function in the ball by harmonic polynomials. The Lemma earlier proved in [10] was used to obtain the formulae for generalized characteristics of growth of a harmonic function in n-dimensional space.

Taking into account that the harmonic functions are used not only in mathematics, but also in physics, mechanics, and other applied sciences, the obtained results are important.

6. Discussion of results of examining a uniform approximation of harmonic functions of several variables

Present study has examined a uniform approximation of harmonic functions of several variables. An approximation error of the harmonic function in the ball K_R^n by harmonic polynomials of degree no higher than k is determined as

$$E_R^k(u)=\inf_{P\in\Pi_k}\left\{\max_{y\in K_R^n}\left|u(y)-P(y)\right|\right\}.$$

We established a rate of decrease of error $E_R^k(u)$, so that a harmonic function in the ball of several variables continued to the entire harmonic function. We obtain estimates on the maximum modulus of harmonic function and approximation error in terms of uniform norm of spherical harmonics in expansion of a harmonic function in series. These results made it possible to obtain the most general characteristics of the growth of a harmonic function in terms of error $E_R^k(u)$, which allows us to estimate the growth of harmonic function by the behavior of this error.

Since harmonic functions describe various stationary processes in mechanics and physics, it is important to estimate their growth by expansion coefficients in series or by other characteristics, for example, approximation error.

Other areas in which further research conducted is using other norms than the uniform one, as well as consideration of harmonic functions not in balls but in other domains of the n-dimensional space and their continuation to the entire harmonic ones.

7. Conclusions

1. We obtain the estimate of uniform norm of spherical harmonics in the expansion of a harmonic function into series in terms of the approximation error, which made it possible to obtain the necessary and sufficient conditions under which a harmonic function in the ball of an n-dimensional space continues to the entire harmonic function.

2. We obtain the estimates for a maximum modulus of an entire harmonic function of several variables in terms of the maximum modulus and the maximum term of some entire functions of a complex variable, which allowed us to represent the generalized and the lower generalized orders of the entire harmonic function in terms of the approximation error by harmonic polynomials of the continued function.

3. The maximum modulus of an entire harmonic function was evaluated. It made it possible to explore a slow growth of harmonic functions and to represent growth characteristics through the approximation error.

References

1. Srivastava, G. S. Uniform approximation of entire functions on compact sets and their generalized growth [Text] / G. S. Srivastava, S. Kumar // New Zealand Journal of Mathematics. – 2009. – Vol. 39 – P. 33–43.
2. Srivastava, G. S. Approximation of entire functions of slow growth on compact sets [Text] / G. S. Srivastava, S. Kumar // Archivum Mathematicum (BRNO). – 2009. – Vol. 45 – P. 137–146.
3. Harfaoui, M. Generalized order and best approximation of entire function in Lp-norm [Text] / M. Harfaoui // International Journal of Mathematics and Mathematical Sciences. – 2010. – Vol. 2010 – P. 1–15. doi: 10.1155/2010/360180
4. Kumar, D. Generalized growth and best approximation of entire functions in Lp-norm in several complex variables [Text] / D. Kumar // Annali dell'universita' di ferrara. – 2011. – Vol. 57, Issue 2. – P. 353–372. doi: 10.1007/s11565-011-0130-8
5. Datta, S. K. Growth analysis of entire functions of two complex variables [Text] / S. K. Datta, T. Biswas // Sahand Communications in Mathematical Analysis. – 2016. – Vol. 3, Issue 2, – P. 13–24.
6. Srivastava, G. S. On approximation and generalized type of analytic functions of several complex variables [Text] / G. S. Srivastava, S. Kumar // Analysis in Theory and Applications. – 2011. – Vol. 27, Issue 2, – P. 101–108. doi: 10.1007/s10496-011-0101-z
7. Harfaoui, M. Best Approximation in Lp-norm and Generalized (α, β)-growth of Analytic Functions [Text] / M. Harfaoui, D. Kumar // Theory and Applications of Mathematics & Computer Science. – 2014. – Vol. 4, Issue 1, – P. 65–80.
8. Temliakov, A. A. K probleme rosta harmonycheskykh funktsyi trekhmernoho prostranstva [Text] / A. A. Temliakov // Matematicheskii sbornik. – 1935. – Vol. 42, Issue 6. – P. 707–718.
9. Kapoor, G. P. On the growth of harmonic functions in $\mathbb{R}^3$ [Text] / G. P. Kapoor, A. Nautiyal // Demonstr. math. – 1983. – Vol. 16, Issue 4. – P. 811–819.
10. Veselovskaia, O. V. O roste tselykh harmonycheskykh v funktsyi [Text] / O. V. Veselovskaia // Yzv. Vuzov. Matem. – 1983. – Vol. 10. – P. 13–17.
11. Srivastava, G. S. Generalized growth of entire harmonic functions [Text] / G. S. Srivastava // Fasciculi Mathematici. – 2008. – Vol. 40. – P. 79–89.

12. Fugard, T. B. Growth of entire harmonic functions in $\mathbb{R}^n$, n≥2 [Text] / T. B. Fugard // Journal of Mathematical Analysis and Applications. – 1980. – Vol. 74, Issue 1. – P. 289–291. doi: 10.1016/0022-247x(80)90127-4
13. Srivastava, G. S. Growth of entire harmonic functions in $\mathbb{R}^n$, n≥2 and generalized orders [Text] / G. S. Srivastava // Bulletin of the Greek Mathematical Society. – 2008. – Vol. 55. – P. 49–58.
14. Kumar, D. Growth of universal entire harmonic functions [Text] / D. Kumar, S. K. Gupta // TJMM. – 2012. – Vol. 3, Issue 2. – P. 111–116.
15. Kumar, D. Growth and approximation properties of generalized axisymmetric potentials [Text] / D. Kumar, K. N. Arora // Demonstratio Mathematica. – 2010. – Vol. 43, Issue 1. – P. 107–116. doi: 10.1515/dema-2013-0215
16. Kumar, D. Growth and Chebyshev Approximation of Entire Function Solutions of Helmholtz Equation in $\mathbb{R}^2$ [Text] / D. Kumar // European Journal of Pure and Applied Mathematics. – 2010. – Vol. 3, Issue 6. – P. 1062–1069.
17. Kumar, D. Growth and Approximation of Generalized Bi-Axially Symmetric Potentials [Text] / D. Kumar, A. Basu // Journal of Mathematical Research with Applications. – 2015. – Vol. 35, Issue 6. – P. 613–624.
18. Khan, H. H. Slow Growth and Approximation of Entire Solution of Generalized Axially Symmetric Helmholtz Equation [Text] / H. H. Khan, R. Ali // Asian Journal of Mathematics & Statistics. – 2012. – Vol. 5, Issue 4. – P. 104–120. doi: 10.3923/ajms.2012.104.120
19. Kumar, D. On the (p, q)-growth of entire function solution of Helmholtz equation [Text] / D. Kumar // J. Nonlinear Sci. Appl. – 2011. – Vol. 4, Issue 2. – P. 92–101.
20. Kumar, D. Growth and approximation of solutions to a class of certain linear partial differential equations in $\mathbb{R}^n$ [Text] / D. Kumar // Mathematica Slovaca. – 2014. – Vol. 64, Issue 1. – P. 139–154. doi: 10.2478/s12175-013-0192-4
21. Kumar, D. Slow Growth and Optimal Approximation of Pseudoanalytic Functions on the Disk [Text] / D. Kumar // International Journal of Analysis and Applications. – 2013. – Vol. 2, Issue 1. – P. 26–37.
22. Kapoor, G. P. Approximation of entire harmonic functions in $\mathbb{R}^3$ [Text] / G. P. Kapoor, A. Nautiyal // Indian Journal of Pure and Applied Mathematics. – 1982. – Vol. 13, Issue 9. – P. 1024–1030.
23. Shaker Abdu-Hussein M. On the generalized type and approximation of entire harmonic functions in $\mathbb{R}^3$ having index pair (p, q) [Text] / M. Shaker Abdu-Hussein, G. S. Srivastava // Istanbul Univ. Fem. Fak. Mat. Der. – 2001. – Vol. 60. – P. 1–17.
24. Šeremeta, M. N. On the connection between the growth of the maximum modulus of an entire function and the modulii of the coefficients of its power series expansion [Text] / M. N. Šeremeta // American Mathematical Society Translations: Series 2. – 1970. – P. 291–301. doi: 10.1090/trans2/088/11
25. Stein, Y. Vvedenye v harmonycheskyi analyz na evklydovykh prostranstvakh [Text] / Y. Stein, H. Veis. – Moscow: Myr, 1974. – 336 p.
26. Tyman, A. F. Vvedenye v teoryiu harmonycheskykh funktsii [Text] / A. F. Tyman, V. N. Trofymov. – Moscow: Nauka, 1968. – 208 p.
27. Beitmen, H. Vysshye transtsendentnye funktsyy. 2nd edition [Text] / H. Beitmen, A. Erdeiy. – Moscow: Nauka, 1974. – 296 p.
28. Sheremeta, M. N. O sviazy mezhdu rostom tselykh ili analytycheskykh v kruhe funktsyi nulevoho poriadka i koeffytsyentamy ikh stepennykh razlozhenyi [Text] / M. N. Sheremeta // Yzv. vuzov. Matem. – 1968. – Vol. 6. – P. 115–121.