Запропоновано метод видалення розмірності математичної моделі, що дає кількість змінних менше, ніж пропонується π-теоремою. Це спростило вибір початкових рішень, використовуваних в груповому методі побудови апроксимаційних залежностей для відображення рішення нелінійних рівнянь, при відсутності його реалізації в термінах стандартних функцій. Працездатність методу проілюстрована на прикладі дослідження руху маятника, що є в теорії автоматичного управління аналогом інерційної ланки другого порядку

Ключові слова: видалення розмірності математичної моделі, групові методи вирішення, інерційну ланку другого порядку

Предложен метод обезразмеривания математической модели, дающий количество переменных меньше, чем предписывается π-теоремой. Это упростило выбор начальных решений, используемых в групповом методе построения аппроксимационных зависимостей для отображения решения нелинейных уравнений, при отсутствии его реализации в терминах стандартных функций. Работоспособность метода проиллюстрирована на примере исследования движения маятника, являющегося в теории автоматического управления аналогом инерционного звена второго порядка

Ключевые слова: обезразмеривание математической модели, групповые методы решения, инерционное звено второго порядка

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DEVELOPMENT OF A METHOD FOR APPROXIMATE SOLUTION OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS USING PENDULUM MOTION AS AN EXAMPLE

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1. Introduction

When studying behavior of control systems, dynamic conditions require special consideration. Apparatus of ordinary differential equations is most often used as a mathematical tool for studying such systems. General methods of solution have only their linear forms. For this reason, linear (linearized) approaches are used to construct models of object and controller. However more exact models are represented by nonlinear differential equations. This situation leads to the fact that the results obtained can only be used within the range of small variation of parameters of the controlled processes nearby the linearization point.

The use of fuel of a variable composition [1, 2] in power equipment instead of certified fuel with a constant composition brings about a continuous variation of the fuel calorific value, amount of combustion products, their temperature, thermophysical properties, etc. In turn, this causes variation in a wide range of the process parameters subject to control. In such a situation, application of the results obtained by analysis of linear models is very problematic and even unfeasible in most cases.

The search for new approaches to analysis of nonlinear models is an important element in solution of the problem of optimal control of power equipment using non-certified fuel of a variable composition.

2. Literature review and problem statement

An effective way to simplify the study of complex models is to make them dimensionless. Reduction of dimensionality of the modeling space makes it possible both to reduce number of experimental (full-scale and numerical) studies by orders of magnitude and facilitate derivation of analytical dependencies. In most cases, when a standard application of this approach takes place, the number of variables and parameters is reduced by an amount determined by the -theorem. As practice shows, this is not the limit. There are works, for example [3, 4], in which the possibility of a deeper transformation of models is substantiated with an even greater reduction of the number of variables. It was observed that the greatest possible reduction of the number of dimensionless parameters was achieved with the help of the proposed method but without their complete exclusion from consideration.

The smallest possible number of dimensionless quantities is determined by the number of independent variables. In the standard approach to deletion of dimensions, additional parameters appear besides them. Similarity criteria appear in their quality. Consequently, there is a need to consider elimination of all or at least most of the criteria. In [5], this feasibility is illustrated by the simplest example and considered only as a favorable, desirable but a stochastic possibility leading to self-similarity.

Self-similarity of the model and its solution are achievable if the set of parameters determining the system state does not contain characteristic scales of independent variables. The standard approach to deletion of dimensions of mathematical models is based primarily on introduction of such scales. This contradiction is the cause of a rare achievement of self-similarity. The ways of excluding characteristic scales of independent variables from the process of deletion of dimensions have not been found in literature.

As another method of generalized analysis and solution of differential equations, the Lee group theory can be considered. Initially, the group theory was created (the end of the nineteenth century) as a demonstration of universality of approach to already known disconnected solutions of ordinary differential equations and did not make any new solutions in this sense. However, the incorporated idea turned out to be productive in the study of models in the newly emerging fields of research when new types of equations were used. As a rule, these are partial differential equations [6]. Based on the group theory, both analytical and numerical solutions can be obtained [7]. In addition to the exact solutions, there is a trend of using approximate symmetries in the group theory [8] and, correspondingly, obtaining approximate solutions.

A special feature of application of the group methods when trying to solve an equation is the use of arbitrary, convenient, or even random initial solutions for the case under consideration. One of the most known mathematical applications of continuous groups is found in the control theory [9]. In the reviewed literature, it has not been possible to find examples of use of linearized models as the initial most developed solutions for investigating the processes described by nonlinear equations.

Unidirectionality of the methods for deletion of dimensions and the group theory to simplification of solution of differential equations has led to an attempt of their union [10]. In a narrower application in [11], deletion of dimensions of the original differential equations was used to simplify analysis of the algebraic equations derived from them by means of the Laplace transform. This operation is a tool for identifying a group of uniform stretches. Within the framework of this approach, the question is considered in [12]: "What is the goal: deletion of dimensions of variables or reducing the number of model parameters? An unbiased view of this question suggests that the goal is precisely to reduce the number of parameters and the deletion of dimensions is just a means enabling achievement of exactly this result in a number of cases". Sometimes this is called "the problem of reducing to a minimally parametric form" [13]. But as noted above, this attitude to deletion of dimensions is determined by the standard approach used. In the literature [13], it was not possible to find examples of combination of special methods for deletion of dimensions that ensure achievement of self-similarity in terms of parameters (similarity criteria) and the ideas of the group methods for solution.

3. The aim and objectives of the study

This work objective was to develop a method for approximate solution of nonlinear differential equations using results of deletion of dimensions resulting in self-similarity, and group methods of solution.

To achieve this goal, the following tasks were set:

– to show feasibility of deletion of dimensions with no use of characteristic scales of independent variables, which will reduce the number of similarity criteria to the values less than those prescribed by the π-theorem;

– to choose the initial solutions convenient for the case under consideration, obtained from the initial equations with dimensions deleted by the proposed method;

– to show efficiency of the developed method and estimate feasibility of its application in engineering calculations.

4. Deletion of dimensions of the mathematical models with no use of characteristic scales of independent variables

In accordance with the Fourier theorem, in the case of polynomial equations, all terms connected by plus or minus sign must have the same dimensionality. With the standard method of deletion of dimensions in such equations, the independent variables are normed by means of characteristic scales. All scales are constants and are taken out of the operator in each term of the equation. As a result, each term of equation is the product of a complex (parameter) of constant dimensional quantities and the operator over the independent variables with deleted dimensions. All parameters have the same dimensionalities. Further, all terms of equation are divided by the value of one of the parameters. As a result, all quantities acting as parameters become dimensionless as well. The parameter by the quantity of which all other parameters were divided remains equal to one. Thus, the mathematical model becomes dimensionless with a simultaneous reduction of the number of parameters. However, this method is not the only way to reduce their number. For example, this is demonstrated by comparison of the Coulomb law notation:

$$F = \varepsilon_0 \frac{q_1 \cdot q_2}{R^2}, \tag{1}$$

in SI and CGEE measurement systems. For example, the electric constant $\varepsilon_0=8.99\cdot10^9$ [$N\cdot m^2\cdot C^{-2}$] has a certain meaning and dimension in SI system while $\varepsilon_0=1$ and dimensionless in CGEE system. A similar picture can be observed in writing magnetic constant in the Ampère law in SI and CGEE systems, etc. This situation is caused by different sets of independent dimensions (independent variables) for the mentioned measurement systems.

With the standard method of deletion of dimensions, the possibility of changing the type of scales in the case of change in the mathematical model equations, for example, when it is detailed, is not considered. This is a limitation for reducing the number of model parameters determined by the π-theorem. For example, in work [14], when discussing the Fourier similarity criterion Fo which includes the time scale t^{Δ}

$$Fo=\frac{a\cdot t^{\Delta}}{(l^{\Delta})^2}, \tag{2}$$

it is noted that it is impossible to single out such a scale for a number of processes. If the periodic process is considered as a scale, its period can be adopted. And, for example, there is no characteristic scale for the process of nonstationary heating of a body. In this situation, a method of introduction of the similarity criterion (it is constant for the process being considered) can be applied under the time differential operator in a corresponding term of the energy conservation equation. Taking (2) into account, the following is obtained:

$$\frac{1}{Fo}\cdot\frac{\partial\overline{\theta}}{\partial\overline{t}}=\frac{\partial\overline{\theta}}{\partial(Fo\cdot\overline{t})}=\frac{\partial\overline{\theta}}{\partial\left[\dfrac{a\cdot t^{\Delta}\cdot\overline{t}}{(l^{\Delta})^2}\right]}=\frac{\partial\overline{\theta}}{\partial\left[\dfrac{a\cdot t}{(l^{\Delta})^2}\right]}=\frac{\partial\overline{\theta}}{\partial(Ho)}. \tag{3}$$

Here, $\overline{\theta},\overline{t}$ are dimensionless temperature and time, respectively.

In (3), it is taken into account that $t^{\Delta}\cdot\overline{t}=t$ is the running dimensional time, and (Ho) in the last expression denotes relative time (in a dimensionless form) called the number of homochronality. Comparison of the first and last expressions in (3) shows that if the dimensionless type of notation is retained, parameter in the form (1/Fo) is absent in the last expression.

Such transformations are only possible with the equation terms including variables not entering other members of the mathematical model. Consider for example equation of the law of conservation of momentum in the Navier-Stokes form without taking into account mass forces for a one-dimensional distributed model

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial x^2}, \tag{4}$$

which will have the following form under a standard approach to deletion of dimensions:

$$\mathrm{Sh}\cdot\frac{\partial\overline{u}}{\partial\overline{t}}+\overline{u}\frac{\partial\overline{u}}{\partial\overline{x}}=-\mathrm{Eu}\cdot\frac{\partial\overline{P}}{\partial\overline{x}}+\frac{1}{\mathrm{Re}}\cdot\frac{\partial^2\overline{u}}{\partial\overline{x}^2}. \tag{5}$$

Here, Sh, Eu, Re are the Strouhal, Euler and Reynolds criteria, respectively. The Sh criterion is the criterion of homochronicity for the process under consideration and can be introduced under the time-differential operator similarly to the Fo criterion in (3). However, such an operation fails for the Re criterion. It is even more difficult with the Eu criterion. In equation (5), pressure contains only one term in its composition as well. The Eu criterion in this term can be formally introduced under the pressure-differential operator. But in the case under consideration, equation (5) is presented in isolation and was only used to demonstrate a feasible procedure. When writing a mathematical model and investigating feasibility of modeling, boundary conditions have to be taken into account. In the initial condition, the time variable t=0. Consequently, it will be zero in both the form of $\overline{t}=t/t^{\Delta}$ made dimensionless by a standard method and the form of the relative variable (Ho). Therefore, putting F0 under the differential sign in the basic equation will not affect the time value in the initial condition. But the members in the marginal conditions that contain pressure do not have the Eu criterion in their composition. Therefore, putting it under the differential sign in the basic equation and the use of the pressure variable in a relative form contradicts the simply dimensionless pressure value in the boundary condition. A similar thing is observed for the rest of variables. For this reason, when performing the described procedure for Fo in [14], only self-similarity of the form representation (but not the physical situation of the process) is achieved. In no other case throughout the text of [14] such a procedure is repeated and only availability of studies in this direction is mentioned in the last sentence.

In contrast to the results of [14], work [5] points out that self-similarity of the representation form can be achieved for all members of the mathematical model in a simple particular case. The case of an ideal oscillatory system (a load suspended on a spring with no energy dissipation) is considered. The mathematical model has the form:

$$m\cdot\frac{d^2x}{dt^2}+k\cdot x=0;\ \text{at}\ t=0:\ x=\delta,\ \frac{dx}{dt}=0. \tag{6}$$

Here, *m, k,* δ are the load weight, spring stiffness, maximum deviation of the load from the equilibrium position (oscillation amplitude), respectively.

Upon performing deletion of dimensions in the following way, the model (6) is obtained as:

$$\frac{d^2\overline{x}}{d(\overline{t})^2}+\overline{x}=0;\ \text{at}\ \overline{t}=0:\ \overline{x}=1,\ \frac{d\overline{x}}{d\overline{t}}=0, \tag{7}$$

which really does not contain any parameters. Attention should be paid to the deletion of time dimensions by using an expression having time dimensionality instead of the scale t^{Δ}. An analogy with transformation of (3) and transition to relative dimensionless variables can be observed here.

Summing up, the above techniques which make it possible to reduce the number of parameters in mathematical models should be pointed out:

a) use of values of the expressions having corresponding dimensionality for deletion of dimensions instead of the characteristic scales;

b) change of the type of the measuring scales when the form of the equations under consideration changes;

c) transition to the dimensionless relative variables.

In the above examples, all techniques are used separately. It can be assumed that their joint use will ensure obtaining new results.

As already noted, in order to perform an adequate modeling, it is necessary to consider basic equations of the model together with the boundary conditions. But, on the one hand, this can be cumbersome for complex models and on the other hand, such a notation reflects just a particular case. Therefore, to demonstrate fundamental feasibility of obtaining new results, it seems rational to use a separate general equation. To this end, consider equation (5) once more. The dimensionless complexes (similarity criteria) are obtained for the characteristic scales of the described processes. They are marked with symbol «$^\Delta$» in the presented expressions:

$$\mathrm{Sh}=\frac{x^\Delta}{t^\Delta\cdot u^\Delta};\ \ \mathrm{Eu}=\frac{\Delta P}{\rho\cdot(u^\Delta)^2};\ \ \mathrm{Re}=\frac{x^\Delta\cdot u^\Delta}{\nu}. \tag{8}$$

Here ΔP, ρ, ν are the pressure variation between the characteristic points of the space under consideration, density, and viscosity of the medium in this space, respectively; x^Δ, t^Δ, u^Δ are the characteristic scales for deletion of dimensions of the coordinate, time, and velocity of the medium in the described process, respectively.

The standard method of transformation features correspondence of the physical nature of the characteristic scales and the quantities that become dimensionless with their help. The characteristic size is chosen as x^Δ, the characteristic velocity as u^Δ, etc. But the need for such a choice is not substantiated anywhere. In fact, only correspondence between the dimensionality of the scale and the quantity that is disdimensioned with its help is mandatory. The following expressions can be used as scales:

$$u^\Delta=\sqrt{\frac{\Delta P}{\rho}};\ \ x^\Delta=\nu\cdot\sqrt{\frac{\rho}{\Delta P}};\ \ t^\Delta=\nu\cdot\frac{\rho}{\Delta P}. \tag{9}$$

They have necessary dimensionalities and when substituted in (8), they transform all criteria into quantities identical to one. This substitution demonstrates feasibility of obtaining the number of parameters less than that prescribed by π-theorem when the expressions are dimensionless.

In the case under consideration for equation (5), self-similarity by all similarity criteria is achievable. It should be noted that expressions (9) are not artificial but can be related to each other or reflect certain relationships. The expression $x^\Delta=u^\Delta\cdot t^\Delta$ reflects relationship of speed, time and coordinate. The values in the expression for u^Δ reflect their interrelation in determining volume flows:

$$\dot{Q}=u\cdot S=\varepsilon\cdot S\cdot\sqrt{\frac{2\cdot\Delta P}{\rho}}\Rightarrow u=\varepsilon\cdot\sqrt{2}\cdot\sqrt{\frac{\Delta P}{\rho}}. \tag{10}$$

Let us consider the case when the process under investigation is described by an equation analogous to (4) but expanded by introduction of a term taking into account action of the field of mass forces. Following deletion of dimensions in the standard way, it will have the form similar to (5)

$$\mathrm{Sh}\cdot\frac{\partial\bar{u}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}=\frac{1}{Fr}\cdot n_x-\mathrm{Eu}\cdot\frac{\partial\bar{P}}{\partial\bar{x}}+\frac{1}{\mathrm{Re}}\cdot\frac{\partial^2\bar{u}}{\partial\bar{x}^2}, \tag{11}$$

differing from it by the appearance of a new member and one more criterion, Froude criterion (Fr). In this expression, n_x is an overload, a quantity showing how many times intensity of the active field of mass forces exceeds intensity of the field of gravitational forces (acceleration of gravity). The Sh, Eu and Re criteria remain unchanged and will have the form of (8). The Fr criterion is expressed using the same characteristic scales and quantities:

$$\mathrm{Fr}=\frac{(u^\Delta)^2}{g\cdot x^\Delta}, \tag{12}$$

where g is acceleration of gravity. The use of norming quantities in the form of (9), just as in the previous case, enables achievement of self-similarity for the Sh, Eu, Re criteria (convert them to one). But this does not happen for Fr. In order to achieve self-similarity for all criteria in the considered case, it is necessary and possible to change the form of the norming quantities:

$$u^\Delta=\sqrt[3]{\nu\cdot g};\ \ x^\Delta=\sqrt[3]{\frac{\nu^2}{g}};\ \ t^\Delta=\sqrt[3]{\frac{\nu}{g^2}};\ \ p^\Delta=\rho\sqrt[3]{(\nu\cdot g)^2}. \tag{13}$$

It should be noted that like in the previous case (9), the norming quantities (13) have not only corresponding dimensionalities but can also be interrelated reflecting general physical relationships for the coordinate (displacement) and flow rate (10):

$$x^\Delta=u^\Delta\cdot t^\Delta;\ \ u^\Delta=\sqrt{\frac{p^\Delta}{\rho}}. \tag{14}$$

The form of the norming quantities (13) shows that they consist just of the physical properties of the medium under investigation and the intensity of the field of mass forces. For example, typical dimensions are not included into them. This can serve as the basis for distorted modeling, i. e. without a necessary retention of geometric similarity.

The above examples show that various approaches to deletion of dimensions are possible. In many cases, the procedures that were considered make it possible to reduce the number of model parameters in comparison with that prescribed by the π-theorem. The best result is obtained by using not the scales selected on the basis of the characteristic dimensions and parameters of the processes under study but by constructing norming quantities of the (9) or (13) type. The use of characteristic scales implies retaining of the measurement system and only a multiple or partial change of dimensionality. For example, such characteristic scale as diameter is measured in meters and all other geometric dimensions are measured by the number of these diameters. The use of norming quantities of the (9), (13) form actually means introduction of a new measurement system. Such a system can be called natural coordinates. Such measurement system features change of natural coordinates when the problem of dimension deletion is changed. When using a complete mathematical model (with taking into account boundary conditions), one can expect changes in the norming quantities but the principle of deletion of dimensions remains in effect. In such formulation, the use of natural coordinates has something in common with the solution methods in the group theory.

Application of natural coordinates is not exhausted by the examples given. The Navier-Stokes equations are a particular expression of the generalized conservation law [15] which can be written in a vector form as:

$$\frac{\partial}{\partial t}(\rho\Phi)+\mathrm{div}(\rho u\Phi)=\mathrm{div}[\Gamma\cdot\mathrm{grad}(\Phi)]+S. \tag{15}$$

Equation (15) includes (in the order of notation) the following: nonstationary, convective, diffusion, and source terms. By substituting different (by the meaning of the problem) values of the dependent variable Φ, diffusion coefficient Γ and source term S for various purposes, one can obtain expressions for the laws of conservation of various quantities. For the law of conservation of momentum, it suffices to perform substitutions: $\Phi=u$ (velocity), $\Gamma=\mu$ (viscosity), $S=ng$ (the field of mass forces). For the energy equation: $\Phi=h$ (specific enthalpy), $\Gamma=k$ (coefficient of thermal conductivity). In general, the quantity Φ can denote weight concentration of chemical components, temperature, kinetic energy of turbulence or the scale of turbulence, etc. The continuity equation is also a variation of the conservation law and follows from (15): absence of a source at $\Phi=1$ and $S=0$:

$$\frac{\partial}{\partial t}(\rho)+\operatorname{div}(\rho u)=0. \tag{16}$$

The described procedure of deletion of dimensions can be applied to all equations of the (15) or (16) form.

5. The method for solving dimensionless ordinary nonlinear differential equations

Work [16] offers a method for approximate solution of the problems in which initial description of the investigated processes is based on distributed mathematical models. To do this, simplified models are used in a concentrated formulation that describe basic properties of the processes under consideration and are the core of solution of more complex models. Simplified models are built on the basis of integral averaging of distributed parameters using the coefficients determining completeness of their accounting. Ultimately, the simplified model is constructed in a dimensionless form similar to that described in the previous section and all coefficients are mapped as one integral. All these coefficients can be considered as the coefficients of stretch in the group methods of solution. Solution to the integral coefficient of the simplified model is considered as the sought one. In turn, the order of magnitude of the integral coefficient can be estimated by analytical calculations. But its more precise value is determined on the basis of few model experiments or exact solutions (both analytical and numerical) of the particular problems. A small number of necessary experiments or exact solutions are determined by the use of the above described method of deletion of dimensions and, as a consequence, by achievement of self-similarity for all parameters (or at least for their majority) of the process under consideration. Comparison of the resulting solutions with exact ones shows that they are in a good agreement with the relative errors (ε~5 %) permissible for engineering calculations at their considerable simplicity.

Such an approach can also be applied to an approximate solution in an analytical form of nonlinear ordinary differential equations of the automatic control theory having solutions of their linearized forms. Linearized equations are subject to deletion of dimensions as described above. This allows one to minimize the number of parameters (within the limit sufficient to achieve self-similarity for all parameters) and obtain solutions in the simplest form. Further, the original nonlinear equations are also subjected to deletion of dimensions using the norming values obtained by processing their linearized forms. The coefficients of stretch that transform solutions of the linearized equations into approximate solutions of the original nonlinear models are determined from a limited number of numerical realizations of the latter.

6. An example and estimation of adequacy of the approximate solutions of the nonlinear ordinary differential equations

As an example of application of the proposed method, consider solution of nonlinear differential equations describing motion of a mathematical pendulum in two cases:

1) in the absence of energy dissipation;
2) in the presence of environmental resistance.

The first case is interesting by feasibility of an analytic solution in terms of elliptic integrals (the Jacobi integral). This integral itself requires a numerical solution. But there is a parameter of the process under consideration (the oscillation period) which can be relatively easily determined based on the calculation of the sum of a rapidly converging series. Thereby, this quantity is a convenient example for evaluating effectiveness of the proposed approximate method.

The linearized form of the equation used to describe the pendulum motion in the second case, in the linear theory of automatic control, serves as the basis for the model of the second-order inertial link. Using the non-linear equation, the trajectory of the body motion is described in a wider range of coordinate variation but, unlike the case of linear formulation, it does not have a solution when using standard functions. Therefore, even an approximate solution of the nonlinear problem is of interest.

1) Motion of a mathematical pendulum in the absence of energy dissipation

On the basis of a nonlinear equation describing motion of a mathematical pendulum without dissipation of energy

$$m\frac{d^2x}{dt^2}+mg\sin(\alpha)=0, \tag{17}$$

the linearized model after taking into account initial conditions and a corresponding norming in a dimensionless form will be as follows:

$$\frac{d^2\bar{x}}{d\bar{t}^2}+\bar{x}=0 \text{ at } \left[\begin{array}{ll} \bar{t}=0 & \bar{x}=1; \\ \bar{t}=0 & \dfrac{d\bar{x}}{d\bar{t}}=0. \end{array}\right. \tag{18}$$

The variables and parameters that are standard for writing the equations of motion of the pendulum are used in (17). Linearization was performed in a same standard way by replacing $\sin(\alpha)\approx\alpha$. Here, $\alpha=x/r$ is the angle of deviation of the pendulum suspension from the equilibrium position, r is the length of the suspension thread. Norming and deletion of dimensions were carried out according to the method considered above:

$$x=\bar{x}\cdot x^{\Delta}, t=\bar{t}\cdot t^{\Delta},$$

where norming quantities were defined as follows:

$$x^{\Delta}=\delta;\ \ t^{\Delta}=\sqrt{r/g}. \tag{19}$$

In (19), δ is the coordinate of the body at its initial deviation from the equilibrium position (amplitude of oscillations), g is intensity of the field of mass forces (acceleration of gravity).

In the original equation (17), the function value depends on three quantities: $x=f(t, m, g)$. With the conventional method of deletion of dimensions, the function value depends on two quantities: $\bar{\bar{x}} = f(\bar{\bar{t}}, \omega)$, where ω is the natural frequency of oscillations. But the use of the proposed method of deletion of dimension has made it possible to obtain expression (18) in which function depends on just one variable, $\bar{x} = f(\bar{t})$. In fact, this means introduction of a new time unit: in solving problems, time will not be measured in seconds (minutes, hours) but in fractions of the oscillation period. Such simplification of the model (18) enables solution in an extremely simple form

$$\bar{x} = \cos(\bar{t}), \tag{20}$$

in which information about the source object (17) is reduced in the norming quantities (19) and the oscillation period for any ratio of the initial equation parameters will be a constant value equal to 2π.

Before solving the original equation (17), carry out deletion of dimensions for it with the help of norming quantities (19) obtained on the basis of the linearized model. As a result, the following is obtained:

$$\alpha_\delta \frac{d^2\bar{x}}{d\bar{t}^2} + \sin(\alpha_\delta \cdot \bar{x}) = 0. \tag{21}$$

Here, it is taken into account that $\alpha_\delta = \delta/r$ is the angular amplitude of the pendulum oscillations. Solution of (21) is sought on the basis of (20) as:

$$\bar{x} = \cos(\bar{t}/k). \tag{22}$$

In this expression, the quantity «k» appears as the coefficient of stretch correcting the argument value in the linearized problem. To determine this coefficient, consider the relationship known in the practice of solving equations of the (17) form as a series for determining the period of oscillations:

$$T_0 = \sqrt{\frac{r}{g}} \cdot 2\pi \cdot k_p;$$

$$k_p = 1 + \left(\frac{1}{2}\right)^2 b^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 b^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 b^6 \ldots \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 b^{2n}, \tag{23}$$

where $b = \sin(\alpha_\delta/2)$. The expression in the first part corresponds to the period of the pendulum oscillations described by the linearized equation. Coefficient «k_p» is a corrective coefficient and can be considered analogous to «k» from (22).

Use the value of «k_p» in (22) for determining of $\bar{x}$. For comparison and evaluation of the result, equation (21) was solved using the Runge-Kutta method of the fourth order. Calculations were performed in the range of the argument variation $\bar{t} = 0\ldots20$ by a step of $h=0.1$. The results obtained with the help of expression (22) and the numerical method were compared at each calculation point: an error related to the double amplitude (peak-to-peak value) was determined. The maximum error was chosen from among the errors for all points. The results for various initial angles of deviation (angular amplitudes) are given in Table 1. Fig. 1 shows examples of the graphs obtained on the basis of numerical solution of equation (21) (Runge-Kutta), analytic solutions of (20) (k=1) of the linearized equation (18) and (22) (k=1.1804) of the nonlinear equation (21) at an angular amplitude α_δ=90°. The choice of such amplitude for graphical illustration was determined by the maximum relative deviation ε_{max}= =1.19 % between the numerical and analytical solutions (22). But even in this condition, their graphs in Fig. 1 are practically indistinguishable. The graph for k=1 reflects the difference between solution of the original dimensionless equation (21) and its linearized form (18). In the vast majority of cases, the results show that the simple solution of (22) can be applied instead of solution in the form of elliptic integrals.

Table 1

Coefficients of stretch and errors of analytical solutions for different initial angles of deviation (of angular amplitudes) in absence of energy dissipation

α_δ, deg.	30°	60°	90°
k_p	1.0174	1.0732	1.1804
ε_{max},%	0.12	0.48	1.19

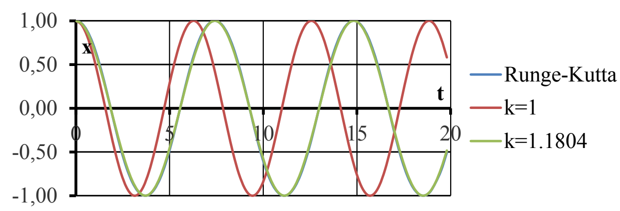


Fig. 1. The dimensionless x coordinates of the pendulum position as a function of the dimensionless time t obtained on the basis of various methods for solving non-linear dimensionless equation (21)

2) Motion of a mathematical pendulum in the presence of environmental resistance

Add a term that takes into account action of the forces of environment resistance to the initial equation (17):

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + mg\sin(\alpha) = 0. \tag{24}$$

Here, c is the coefficient of resistance. Further, like in Paragraph 1) of the section under consideration, linearize and delete dimensions for this equation. As a result, a pendulum motion model similar to (18) is obtained on its basis:

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \frac{1}{\bar{m}}\frac{d\bar{x}}{d\bar{t}} + \bar{x} = 0 \text{ at } \left[\begin{array}{ll} \bar{t}=0 & \bar{x}=1; \\ \bar{t}=0 & \dfrac{d\bar{x}}{d\bar{t}}=0, \end{array}\right. \tag{25}$$

where the norming quantities have the form:

$$x^\Delta = \delta;\ \ t^\Delta = \sqrt{r/g};\ \ m^\Delta = c\sqrt{r/g}. \tag{26}$$

Depending on the magnitude $\bar{m}$, solution of (25) has the form:

a) for $\bar{m} > 0.5$

$$\bar{x} = e^{-\frac{\bar{t}}{2\bar{m}}}\left[\cos\left(\frac{p}{2\bar{m}}\bar{t}\right) + \frac{1}{p}\sin\left(\frac{p}{2\bar{m}}\bar{t}\right)\right],$$

where

$$p = \sqrt{4\bar{m}^2 - 1}; \tag{27}$$

b) for $\overline{m} < 0.5$

$$\overline{x} = \frac{1}{2}\left(1+\frac{1}{p}\right)\cdot e^{(a_1\cdot\overline{t})} + \frac{1}{2}\left(1-\frac{1}{p}\right)\cdot e^{(a_2\cdot\overline{t})},$$

$$p = \sqrt{1-4\overline{m}^2};\quad a_1 = -\frac{(1-p)}{2\overline{m}};\quad a_2 = -\frac{(1+p)}{2\overline{m}}. \qquad (28)$$

By analogy with the operation performed in paragraph 1), introduce the coefficient of stretch «k»: in solutions of (27) and (28) of the linearized equation (25):

$$\overline{x} = e^{-\frac{\overline{t}/k}{2\overline{m}}}\left[\cos\left(\frac{p}{2\overline{m}}\cdot\frac{\overline{t}}{k}\right) + \frac{1}{p}\sin\left(\frac{p}{2\overline{m}}\cdot\frac{\overline{t}}{k}\right)\right], \qquad (29)$$

$$\overline{x} = \frac{1}{2}\left(1+\frac{1}{p}\right)\cdot e^{(a_1\cdot\frac{\overline{t}}{k})} + \frac{1}{2}\left(1-\frac{1}{p}\right)\cdot e^{(a_2\cdot\frac{\overline{t}}{k})}. \qquad (30)$$

Using (29) and (30), approximate solutions of the original nonlinear equation (24) can be obtained. To estimate the error of such solution, delete dimensions for the original nonlinear equation (24) using norming quantities (26). As a result, an equation analogous to (21) is obtained:

$$\alpha_\delta \frac{d^2\overline{x}}{d\overline{t}^2} + \frac{1}{\overline{m}}\frac{d\overline{x}}{d\overline{t}} + \sin(\alpha_\delta\cdot\overline{x}) = 0. \qquad (31)$$

Numerical solution is obtained using the fourth-order Runge-Kutta method in the range of the argument variation $\overline{t} = 0\ldots20$ with a step h=0.1. Comparison of the results and the search for errors were performed by analogy with the similar operation in paragraph a) with an addition of values $\overline{m}$, defining the type of solution of (29) or (30) The results for various initial angles of deviation are given in Table 2. Fig. 2 shows examples of the graphs obtained on the basis of various methods for solving the initial equation (24).

Table 2

Coefficients of stretch and errors of analytical solutions for various initial angles of deviation in the presence of environmental resistance

α_δ, deg.	30°		60°		90°	
$\overline{m}$	0.4	20	0.4	20	0.4	20
k	1	1.0126	1.105	1.0515	1.25	1.126
ε_{max},%	1.25	0.66	1.00	2.73	2.18	5.50

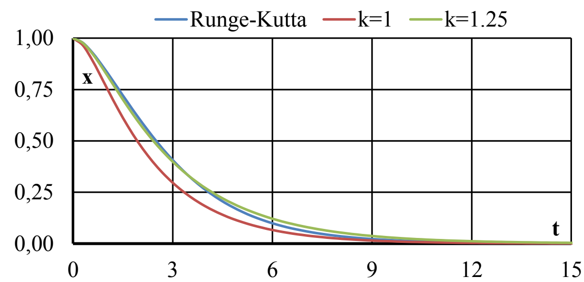
Fig. 2, *a* shows results for the case of $\overline{m} = 0.4$. Fig. 2, *b* shows results for the case of $\overline{m} = 20$. In both cases, the results were obtained on the basis of:

– numerical calculations using the fourth-order Runge-Kutta method;

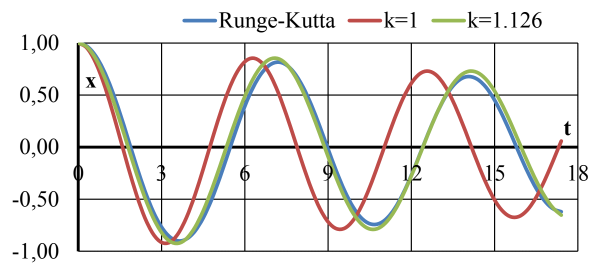
– solution of the linearized equation (25) at k=1 in the form of (28) for the case of $\overline{m} = 0.4$, (Fig. 2, *a*) and solution of equation (25) in the form of (27) for the case $\overline{m} = 20$, (Fig. 2, *b*);

– solution of the original nonlinear equation (24) in the form (30) with k=1.25 for the case of $\overline{m} = 0.4$ (Fig. 2, *a*) and solution of the equation (24) in the form of (29) for k=1.126 for the case of $\overline{m} = 20$ (Fig. 2, *b*).

The results are given for the case of initial deviation of the pendulum from the equilibrium position α_δ=90°. The choice of such amplitude for a graphical illustration as in the case of Fig. 1 was determined by the maximum relative deviation ε_{max}=5.5 % at this amplitude between the numerical and analytical solutions for the case of $\overline{m} = 20$.



a



b

Fig. 2. The dimensionless x coordinates of the pendulum position as a function of the dimensionless time t obtained on the basis of various methods for solving non-linear dimensionless equation (29): *a* =0.4; *b* =20

The deviations of the results of approximate analytical solutions in comparison with the numerical ones do not exceed or are equal to the permissible errors in engineering calculations (~5 %).

7. Discussion of the study results: features of the developed method for approximate solution of nonlinear ordinary differential equations

A method was proposed for deletion of dimensions in the mathematical model giving the number of variables less than that prescribed by the π-theorem. When using the proposed method, it is possible in a number of cases to exclude from consideration all similarity criteria or, in other words, to achieve self-similarity for them. In the framework of the procedure of deletion of dimensions, this is expressed in a transition from the criteria to the similarity numbers. Thus, information is reduced without its loss.

The limit reduction of the number of variables in the mathematical model facilitates application of analytical approximation dependences for approximate solutions. Solutions of the linearized forms of the original nonlinear equations with the use of coefficients of stretch characteristic for the group solution methods were proposed in the quality of such dependencies. This approach makes it possible to take into account physical nature of variation of the studied quantities in the case when solutions of nonlinear equations cannot be realized by using standard functions.

The limit reduction of the number of variables facilitates determination of the coefficients of stretch. The feature of their search in the proposed method of approximate solution of nonlinear equations consists in the use of not analytic transformations but numerical solutions or experimental studies. The minimum number of variables minimizes amount of the work required.

Although the proposed method is approximate, it enables obtaining of analytical solutions acceptable from the engineering point of view in the absence of their exact forms. The method functionality was illustrated by the example of study of pendulum motion which is a counterpart of inertial link of the second order in the theory of automatic control.

8. Conclusions

1. A method for deletion of dimensions in mathematical models based on the refusal to use characteristic scales of independent variables as the norming quantities has been developed. In the process of deletion of dimensions, norming quantities are formed from physical parameters of the processes under study which is equivalent to the introduction of a new coordinate system. This approach makes it possible to reduce number of similarity criteria to the values less than that prescribed by the π-theorem. In a number of cases, it becomes possible to completely exclude criteria from consideration, i. e. reduce dimensionality of the model.

2. In the case of a group approach to solving a dimensionless nonlinear model, it is proposed to choose solutions of their linearized forms as initial ones.

3. Functionality of the developed method for approximate solution of nonlinear ordinary differential equations was demonstrated by comparison of the results obtained with its help with the results of numerical calculations. The comparison was made using the example of study of pendulum motion which is a counterpart of an inertial link of the second order in the theory of automatic control. The deviations of the results of analytical solutions in comparison with the numerical ones do not exceed or are equal to the permissible errors in engineering calculations (~5 %). The proposed method can be used in the theory of automatic control when solving nonlinear differential equations describing behavior of other links. It is of interest to investigate the feasibility of applying the proposed method to an approximate solution of other types of differential equations, for example, partial differential equation.

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