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SEARCH FOR THE CONDITIONS FOR THE OCCURRENCE OF AUTO-BALANCING IN THE FRAMEWORK OF A PLANAR MODEL OF THE ROTOR MOUNTED ON ANISOTROPIC VISCOUS-ELASTIC SUPPORTS

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Застосований емпіричний критерій стійкості основного руху. Встановлено, що в залежності від сил в'язкості в опорах у ротора одна чи три критичні швидкості. Автобалансування, відповідно, настає: при перевищенні єдиної критичної швидкості; між першою і другою та над третьою критичною швидкостями. Оцінена точність отриманих результатів обчислювальними експериментами

Ключові слова: ротор на анізотропних опорах, пасивний автобалансир, автобалансування, критерій настання автобалансування, критичні швидкості ротора

Применен эмпирический критерий устойчивости основного движения. Установлено, что в зависимости от сил сопротивления в опорах у ротора одна или три критические скорости. Автобалансировка, соответственно, наступает: при превышении единственной критической скорости; между первой и второй, и над третьей критической скоростью. Оценена точность полученных результатов вычислительными экспериментами

Ключевые слова: ротор на анизотропных опорах, пассивный автобалансир, автобалансировка, критерий наступления автобалансировки, критические скорости ротора

1. Introduction

Passive auto-balancers are used for balancing high-speed rotors [1, 2]. Correcting weights balance the rotor on the so-called main (steady-state) motions and no balance on the secondary motions. From a mathematical point of view, for operability of the auto-balancer, it is necessary and sufficient that the main motions are stable and the secondary motions are unstable.

Stability of motions of rotary machines with auto-balancers was traditionally studied by Liapunov's methods [2–9].

In the case of isotropic supports, when the coordinate system in use is synchronously rotating with the rotor, the equations of motion and steady motions are stationary. Stability is studied with application of the theory of stability of stationary solution of the systems of autonomous nonlinear differential equations [2–5]. It is unnecessary to use additional assumptions about the smallness ratios between parameters.

By Liapunov, motion stability is also studied with the use of the small-parameter methods [6–9]. The approach is especially relevant in the case of anisotropic supports since the differential equations of motion are nonstationary in this case. The approach introduces additional assumptions about smallness ratios between parameters. Therefore, the stability conditions obtained have limitations as to their accuracy and the field of application.

The search for and study of stability of steady motions of the rotor-auto-balancer system is a complex mathematical problem [2–9]. The task becomes even more complicated for the auto-balancers with many weights when taking into account the forces of resistance, anisotropy of supports, etc.

Taking into consideration the features of motion dynamics of rotary machines with automatic balancers makes it possible to formulate empirical criteria for stability of main motions or occurrence of auto-balancing [2, 10, 12]. The criteria enable determination of the conditions for the occurrence of auto-balancing with no resort to the study of stability of the main motions by Lyapunov's method, which minimizes labor-intensiveness [2, 10–12].

It is vital to show effectiveness of the criterion for the main motion stability in determining conditions for the occurrence of auto-balancing of the rotor mounted on anisotropic elastic-viscous supports. It is important to estimate accuracy of the results obtained.

2. Literature review and problem statement

Let us consider in more detail how deeply the possibility of rotor balancing was analytically investigated in the framework of planar models.

Stability of main motions of the rotor mounted on isotropic elastic-viscous supports and balanced by a passive auto-balancer was studied in [2–5] with application of a mobile coordinate system.

In [3], the rotor is balanced by a two-ball auto-balancer. It was established that the rotor has one critical speed coinciding with the frequency of natural oscillations of the rotor in absence of resistance forces (natural frequency). Auto-balancing onset takes place at supercritical rotor speeds.

In [4], the rotor is balanced by a two-pendulum auto-balancer. It was established that depending on the values of the system parameters, the rotor has one or three critical speeds. One critical speed always coincides with the natural frequency. Two additional speeds arise in the vicinity of the natural frequency and somewhat exceed this frequency. In the case of one critical speed, auto-balancing occurs at supercritical speeds. In the case of three critical speeds, auto-balancing occurs between the first and the second and above the third critical speeds. The effect of splitting the natural frequency into three critical speeds arises only at small forces of viscous resistance.

In [5], the rotor is balanced by a multi-ball auto-balancer. A characteristic equation was obtained for studying stability of families of main motions. Up to its notations, this equation coincides with the equation obtained in [4].

In [2], it was suggested to study stability of the main motions with the use of generalized rotor coordinates and total imbalances (rotor and auto-balancers) in the correction planes (of auto-balancers). For the main motions, these generalized coordinates are zero. This approach was realized within the framework of a planar model of the rotor mounted on isotropic elastic-viscous supports and balanced by a multi-ball (multi-roller, multi-pendulum) auto-balancer. A characteristic equation obtained in this work coincides with the characteristic equations derived in [4, 5], up to their notations.

The approach to studying the auto-balancing phenomenon in the case of auto-balancers with many weights proposed in [2] was further developed in [6]. It was shown that the dynamics of ball, roller and pendulum auto-balancers is described by similar differential equations. A method was proposed for derivation of differential equations describing the process of auto-balancing (relative to the generalized rotor coordinates and imbalances) with the use of differential equations of the system motion.

The small-parameter method (dynamic system synchronization [7]) was used in analytical study of rotor balancing by only two-ball (two-pendulum) auto-balancers [6–8]. The weight mass to rotor mass ratio was taken as a small parameter.

A rotor mounted on isotropic elastic supports and performing a planar motion was considered in [6]. The results of [3] were confirmed in the part that auto-balancing occurs at super-resonant speeds of the rotor.

A rotor mounted on anisotropic elastic supports was considered in [7]. Existence of three critical speeds of the rotor was established. The first and the third speeds coincide with respective first and second natural frequencies of the rotor. The second critical speed is between the first two. Auto-balancing arises at the rotor motion with the speeds between the first and the second and above the third critical speeds. It is worth to note that presence of three critical speeds is associated with anisotropy of the supports and not with the splitting of the natural frequency at small resistance forces in the supports. Balancing of a working grinding wheel by a two-ball auto-balancer was also studied in [7].

A planar model of a rotor mounted on isotropic elastic-viscous supports was considered in [8] for the case of small forces of viscous resistance. The rotor was balanced by a two-ball auto-balancer. The results obtained in [4] concerning splitting of the natural frequency into three critical velocities were confirmed. To this end, stability was studied with the use of higher small-parameter approximations.

Advantages of the method of dynamical system synchronization include the possibility of studying auto-balancing for rotors mounted on anisotropic supports. The drawbacks of the method include a considerable increase in labor input of calculations when taking into account the resistance forces in supports. The method was not designed to study families of steady motions. The results obtained are only suitable for executing the introduced smallness ratios between parameters.

Rotor balancing by using the empirical criterion for the occurrence of auto-balancing was investigated in [2, 10].

The results obtained in [3, 7] were confirmed in [2]. Balancing of a working grinding wheel with the help of passive auto-balancers taking into account various resistance and cutting forces was also studied.

Balancing of a grinding wheel of a hand grinder mounted on a tripod by means of passive auto-balancers was studied in [10].

There are empirical criteria for the occurrence of auto-balancing [2] and the main motion stability [2], a generalized criterion for the main motion stability [11], and a generalized criterion for the occurrence of auto-balancing [12]. The history of criteria development was described in [12]. In accordance with the criteria, potentiality of auto-balancing onset is determined by the reaction of the rotor to elementary imbalances applied in the correction planes (of the auto-balancers).

It should be pointed out that asymptotic methods solve mathematical problems just approximately [13]. Therefore, a question arises: how accurate are the results they provide. A similar question concerns the results obtained by empirical criteria.

Thus, within the framework of the planar model of a rotor mounted on anisotropic elastic-viscous supports, the effect of finite and large forces of viscous resistance acting in supports on the region of auto-balancing onset was not analytically studied. No critical rotational speeds of the rotor (i. e. the speeds causing onset or disappearance of auto-balancing in case of their transition) have been found. This is studied below with application of an empirical criterion for stability of the main motion. Accuracy of determining critical rotor speeds is estimated by a computational experiment.

3. The aim and objectives of the study

The study objective was to obtain, within the limits of the planar model, the conditions under which a passive auto-balancer of any type will balance a rigid rotor mounted on anisotropic elastic-viscous supports.

To achieve this objective, it was necessary to solve the following tasks:

– construct a planar model of a rotor mounted on anisotropic supports and balanced by a passive auto-balancer and derive differential equations of the system motion;

– find the conditions for the occurrence of auto-balancing by applying the empirical criterion for stability of the main motion;

– conduct a computational experiment to evaluate accuracy of determining critical speeds, i. e. boundaries of the regions of auto-balance occurrence.

4. The study methods

To find the conditions for the occurrence of auto-balancing, an empirical criterion for stability of the main motion was used [2]. The criterion was intended to answer the question: under what conditions a certain main motion (possibly one from a multiparametric family of main motions) will be stable.

Let us use the criterion for the case of rotor balancing with one passive auto-balancer.

Empirical (engineering) criterion for stability of the main motion. Let the rotor – auto-balancer system performs a certain main motion. Consider motion of the C point in which longitudinal axis of the rotor intersects the auto-balancer correction plane. Let the weights in the auto-balancer elementarily deviate from the main motion and take a new position relative to the rotor. At the same time, an elementary imbalance appears. Over time, motion of the rotor with its auto-balancer will establish. The C point in the new motion will deviate from the main motion by a $\vec{r}_C(t)$ vector where t is time.

According to the criterion, in order that the main motion is stable, it is necessary and sufficient that the C point deviated under the action of any elementary imbalance from the main motion on the average per one rotation opposite to the vector of the elementary imbalance.

The criterion can be mathematically written as follows:

$$\overline{r}_{Cu}=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\vec{u}\cdot\vec{r}_C(t)dt<0, \tag{1}$$

where ω is a constant angular rotation speed of the rotor, $\vec{u}$ is a unit vector directed along the vector of elementary imbalance.

The criterion is applied in the following sequence:

1) a physical-mechanical model of the rotor with an auto-balancer of a certain type is described;

2) differential equations of the rotor motion are derived on the assumption that the correction weights are stationary relative to the rotor;

3) steady motion of the rotor is sought for;

4) functional of the criterion for the occurrence of auto-balancing is derived;

5) the conditions for the occurrence of auto-balancing are found proceeding from the condition of the functional negativity.

To verify the results obtained, a computational experiment should be conducted. To do this, the differential equations of motion are reduced to a normal form. As initial conditions, the values of generalized coordinates and velocities corresponding to a certain main motion are to be used. Equations of motion are integrated numerically. The conclusion on stability of the main motion is made on the condition that the system continues to perform one of the main motions. If the system leaves the main motions, then a conclusion of instability of the main motions is made.

5. The conditions for stability of the main motions of the rotor mounted on anisotropic elastic-viscous supports and balanced by a passive auto-balancer

5. 1. Description of the rotor and auto-balancer model

Fig. 1 shows the diagrams explaining fastening and motion of the rotor. The fixed X, Y axes are directed along the main directions of rigidity of the supports and in such a way that the X axis is along the minimum rigidity and the Y axis is along the maximum rigidity (Fig. 1, *a*). The rotor motion is defined as the sum of two motions (Fig. 1, *b*): the translational motion together with the center of the mass in the C point and rotational motion around the center of mass at a constant angular speed ω. Position of the center of mass of the rotor is determined by the x, y coordinates.

The static imbalance of the rotor is created by a point mass m_0 located at a distance R from the longitudinal axis of the rotor (the C point). Its radius forms an angle ωt with the X axis.

The auto-balancer consists of N identical correction weights: pendulums, balls or rollers. The mass of the auto-balancer body is attributed to the mass of the rotor. As is customary in the theory of passive auto-balancers, we assume that the weights on the track do not interfere with each other's motion. The effect of gravity on the motion of weights is neglected. Mass of one weight is m. The center of mass of the weight moves along a circle of radius R with its center on the longitudinal axis of the rotor (Fig. 1, *b*). In the case of balls or rollers, we assume that they roll along the track without slippage. Position of the weight number j defines angle φ_j between the X axis and the radius of the weight's center of mass, $/\,j=\overline{1,N}\,/$. Motion of the weight relative to the auto-balancer body is impeded by a force of viscous resistance having modulus

$$F_j=b_W v_j^{(r)},\quad /\,j=\overline{1,N}\,/, \tag{2}$$

where b_w is the coefficient of the force of viscous resistance, $v_j^{(r)}=R\,|\,\varphi_j'-\omega\,|$ is the modulus of velocity of motion of the center of mass of the weight number j relative to the auto-balancer body, and the stroke after the value denotes the time t derivative.

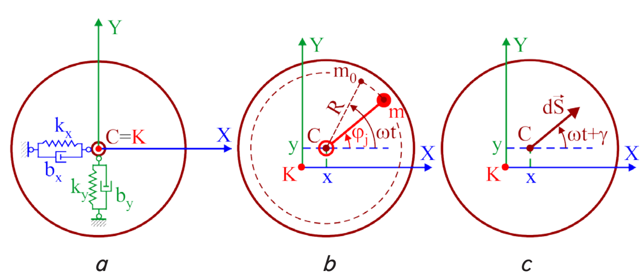

Fig. 1. The planar model of the rotor – auto-balancer system: the rotor mounted on anisotropic elastic-viscous supports (*a*); kinematics of the rotor motion, unbalanced mass and weights (*b*); kinematics of motion of elementary imbalance (*c*)

The mass of the system and the total imbalance of the rotor

$$M_\Sigma=M+Nm+m_0,$$

$$S_x = mR\sum_{j=1}^{N}\cos\phi_j + m_0 R\cos\omega t,$$

$$S_y = mR\sum_{j=1}^{N}\sin\phi_j + m_0 R\sin\omega t. \qquad (3)$$

If the system performs the main motion (one of the multiparametric family for $N\geq 3$), the weights balance the rotor, and $S_x = S_y = 0$.

Let the system perform any main motion. Suppose the correction weights elementarily deviate from it resulting in appearance of constant elementary imbalance (Fig. 1, *c*). There is ($\omega t+\gamma$) angle between the X axis and the $d\vec{S}$ vector of elementary imbalance.

5. 2. Differential equations of motion

Differential equations of motion of a rotor with a ball, roller or pendulum auto-balancer take the form [2, 9]:

$$M_\Sigma x'' + b_x x' + k_x x - \\ -mR\sum_{j=1}^{N}(\phi_j''\sin\phi_j + \phi_j'^2\cos\phi_j) - m_0 R\omega^2\cos\omega t = 0,$$

$$M_\Sigma y'' + b_y y' + k_y y + \\ +mR\sum_{j=1}^{N}(\phi_j''\cos\phi_j - \phi_j'^2\sin\phi_j) - m_0 R\omega^2\sin\omega t = 0,$$

$$\kappa m R^2\phi_j'' + b_W R^2(\phi_j' - \omega) + \\ +mR(-x''\sin\phi_j + y''\cos\phi_j) = 0, \quad /\, j=\overline{1,N}\,/, \qquad (4)$$

where κ is the coefficient depending on the weight type. For a mathematical pendulum, κ=1; for a ball, κ=7/5; for a roller, κ=3/2 [9].

Natural rotor oscillation frequencies in absence of resistance forces are

$$\omega_x = \sqrt{k_x / M_\Sigma}, \quad \omega_y = \sqrt{k_y / M_\Sigma}. \qquad (5)$$

Introduce:

– characteristic scales

$$\tilde{\omega} = \omega_x, \quad L = NmR / M_\Sigma; \qquad (6)$$

– dimensionless variables

$$\xi = x / L, \quad \eta = y / L, \quad \tau = \tilde{\omega} t \left(\frac{d}{dt} = \tilde{\omega}\frac{d}{d\tau}\right); \qquad (7)$$

– dimensionless parameters

$$n = \frac{\omega}{\tilde{\omega}}, \quad \mu_\xi = \frac{b_x}{2M_\Sigma\tilde{\omega}}, \quad \mu_\eta = \frac{b_y}{2M_\Sigma\tilde{\omega}}, \quad \varepsilon = \frac{L}{\kappa R},$$

$$\mu_w = \frac{b_w}{\kappa m\tilde{\omega}}, \quad n_\eta = \frac{\omega_y}{\omega_x}, \quad \chi = \frac{m_0}{Nm}, \quad \sigma = \frac{1}{N}. \qquad (8)$$

Then the equations of motion take the following dimensionless form:

$$\ddot{\xi} + 2\mu_\xi\dot{\xi} + \xi - \\ -\sigma\sum_{j=1}^{N}(\ddot{\phi}_j\sin\phi_j + \dot{\phi}_j^2\cos\phi_j) - \chi n^2\cos n\tau = 0,$$

$$\ddot{\eta} + 2\mu_\eta\dot{\eta} + n_\eta^2\eta + \\ +\sigma\sum_{j=1}^{N}(\ddot{\phi}_j\cos\phi_j - \dot{\phi}_j^2\sin\phi_j) - \chi n^2\sin n\tau = 0,$$

$$\ddot{\phi}_j + \mu_w(\dot{\phi}_j - n) + \varepsilon(-\ddot{\xi}\sin\phi_j + \ddot{\eta}\cos\phi_j) = 0, \quad /\, j=\overline{1,N}\,/, \qquad (9)$$

where the dot over the quantity denotes the dimensionless time τ derivative.

Introduce the dimensionless total imbalance of the rotor

$$s_\xi = S_x / (M_\Sigma L) = \sigma\sum_{j=1}^{N}\cos\phi_j + \chi\cos n\tau,$$

$$s_\eta = S_y / (M_\Sigma L) = \sigma\sum_{j=1}^{N}\sin\phi_j + \chi\sin n\tau. \qquad (10)$$

For the main motion,

$$\xi = \eta = s_\xi = s_\eta = 0.$$

Introduce new variables

$$z_0 = \xi,\ z_1 = \dot{\xi} = \dot{z}_0,\ z_2 = \eta,\ z_3 = \dot{\eta} = \dot{z}_2,$$

$$z_4 = \phi_1, z_5 = \dot{\phi}_1 = \dot{z}_4, \ldots, z_{2j+2} = \phi_j, z_{2j+3} = \\ = \dot{\phi}_j = \dot{z}_{2j+2}, \ldots, z_{2N+2} = \phi_N, z_{2N+3} = \dot{\phi}_N = \dot{z}_{2N+2}. \qquad (11)$$

Introduce matrix and vector

$$A = \begin{pmatrix} 1 & 0 & -\sigma\sin z_4 & \cdots & -\sigma\sin z_{2N+2} \\ 0 & 1 & \sigma\cos z_4 & \cdots & \sigma\cos z_{2N+2} \\ -\varepsilon\sin z_4 & \varepsilon\cos z_4 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -\varepsilon\sin z_{2N+2} & \varepsilon\cos z_{2N+2} & 0 & 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} -2\mu_\xi z_1 - z_0 + \sigma\sum_{j=1}^{N} z_{2j+3}^2\cos z_{2j+2} + \chi n^2\cos n\tau \\ -2\mu_\eta\dot{\eta} - n_\eta^2\eta + \sigma\sum_{j=1}^{N} z_{2j+3}^2\sin z_{2j+2} + \chi n^2\sin n\tau \\ -\mu_w(z_5 - n) \\ \vdots \\ -\mu_w(z_{2N+3} - n) \end{pmatrix}. \qquad (12)$$

Then the system of equations (9) in a normal form will be as follows:

$$\dot{z}_0 = z_1,\ \dot{z}_2 = z_3,\ \dot{z}_{2j+2} = z_{2j+3}, \quad /\, j=\overline{1,N}\,/,$$

$$(\dot{z}_1, \dot{z}_3, \dot{z}_5, \ldots, \dot{z}_{2N+3})^T = A^{-1}B. \qquad (13)$$

The system of equations in the form (13) with coefficients from (8) will be used for carrying out computational experiments. The occurrence of auto-balancing will be observed for variables:

$$z_0,\ z_2,\quad s_\xi = \sigma\sum_{j=1}^{N}\cos z_{2j+2} + \chi\cos n\tau,$$

$$s_\eta = \sigma\sum_{j=1}^{N}\sin z_{2j+2} + \chi\sin n\tau. \qquad (14)$$

For the main motion, these variables are zero.

5. 3. Determination of conditions for the occurrence of auto-balancing by an empirical criterion

Projections on the X, Y axes of the vector of elementary (dimensionless) imbalance and the unit vector directed as an imbalance:

$$ds_\xi = ds\cos(n\tau+\gamma),\quad ds_\eta = ds\sin(n\tau+\gamma),$$

$$u_\xi = \cos(n\tau+\gamma),\ u_\eta = \sin(n\tau+\gamma). \qquad (15)$$

The differential equations of the rotor motion with a constant elementary imbalance are obtained from the first two equations of system (9):

$$\ddot{\xi}+2\mu_\xi\dot{\xi}+\xi = ds\cos(n\tau+\gamma),$$

$$\ddot{\eta}+2\mu_\eta\dot{\eta}+n_\eta^2\eta = ds\sin(n\tau+\gamma). \qquad (16)$$

Note that these equations do not depend on the type of weights in the auto-balancer.

Solution of system (16) has several components. But in the presence of even small resistance forces, they all tend to zero, except for the partial solution of this system. This component will determine deviation of the C point when motion is steady-state. The partial solution has the form:

$$\xi = \frac{n^2 ds}{(1-n^2)^2+4\mu_\xi^2 n^2}\times$$
$$\times[(1-n^2)\cos(n\tau+\gamma)+2\mu_\xi n\sin(n\tau+\gamma)],$$

$$\eta = \frac{n^2 ds}{(n_\eta^2-n^2)^2+4\mu_\eta^2 n^2}\times$$
$$\times[(n_\eta^2-n^2)\sin(n\tau+\gamma)-2\mu_\eta n\cos(n\tau+\gamma)]. \qquad (17)$$

The integrand of the criterion for occurrence of auto-balancing has the form:

$$\vec{u}\cdot\vec{r}_C = \xi u_\xi + \eta u_\eta =$$
$$= \frac{n^2 ds}{(1-n^2)^2+4\mu_\xi^2 n^2}[(1-n^2)\cos(n\tau+\gamma)+2\mu_\xi n\sin(n\tau+\gamma)]\times$$
$$\times\cos(n\tau+\gamma)+\frac{n^2 ds}{(n_\eta^2-n^2)^2+4\mu_\eta^2 n^2}\times$$
$$\times[(n_\eta^2-n^2)\sin(n\tau+\gamma)-2\mu_\eta n\cos(n\tau+\gamma)]\sin(n\tau+\gamma).$$

The condition for the occurrence of auto-balancing

$$\frac{n}{2\pi}\int_0^{2\pi/n}(\xi u_\xi+\eta u_\eta)d\tau =$$
$$= \frac{n^2 ds}{2}\left[\frac{1-n^2}{(1-n^2)^2+4\mu_\xi^2 n^2}+\frac{n_\eta^2-n^2}{(n_\eta^2-n^2)^2+4\mu_\eta^2 n^2}\right]<0.$$

This condition is equivalent to the following:

$$p(n) = (1-n^2)(n_\eta^2-n^2)(1+n_\eta^2-2n^2)+$$
$$+4n^2[(1-n^2)\mu_\eta^2+(n_\eta^2-n^2)\mu_\xi^2]<0. \qquad (18)$$

Condition (18) in a dimensional form:

$$P(\omega) = (\omega_x^2-\omega^2)(\omega_y^2-\omega^2)(\omega_x^2+\omega_y^2-2\omega^2)+$$
$$+\omega^2[(\omega_x^2-\omega^2)\beta_y^2+(\omega_y^2-\omega^2)\beta_x^2]<0, \qquad (19)$$

where

$$\beta_x = b_x/M_\Sigma,\quad \beta_y = b_y/M_\Sigma. \qquad (20)$$

Take condition (19) to determine critical speeds of the rotor. In a case of deviation from these velocities, the main motion acquires or loses stability. It is worth to note that the condition (19) does not depend on the type of weights in the auto-balancer.

5. 4. Analysis of the condition for the occurrence of auto-balancing

1. The case of equirigid supports.

$$\omega_y = \omega_x. \qquad (21)$$

The condition for the occurrence of auto-balancing takes the form:

$$P(\omega) = (\omega_x^2-\omega^2)[2(\omega_x^2-\omega^2)^2+\omega^2(\beta_x^2+\beta_y^2)]<0. \qquad (22)$$

Auto-balancing occurs at velocities greater than the only critical speed coinciding with the natural frequency of the rotor oscillations in absence of resistance forces:

$$\omega>\omega_1,\quad \omega_1=\omega_x=\sqrt{k_x/M}\,. \qquad (23)$$

The forces of viscous resistance acting in supports do not affect the critical speed and the range of speeds at which auto-balancing occurs.

2. The case of absence of viscous resistance forces and anisotropy in supports.

$$\beta_x=\beta_y=0,\ \omega_x\neq\omega_y. \qquad (24)$$

The condition for the occurrence of auto-balancing (19) takes the form:

$$(\omega_x^2-\omega^2)(\omega_y^2-\omega^2)(\omega_x^2+\omega_y^2-2\omega^2)<0. \qquad (25)$$

It can be seen that there are three critical speeds. In a case of transition these speeds, the main motion acquires or loses stability.

$$\omega_1=\omega_x,$$
$$\omega_2=\sqrt{(\omega_x^2+\omega_y^2)/2},$$
$$\omega_3=\omega_y,\quad \omega_1<\omega_2<\omega_3. \qquad (26)$$

Auto-balancing can occur at velocities between the first and the second, and above the third critical speeds:

$$\omega\in(\omega_1,\,\omega_2)\cup(\omega_3,+\infty). \qquad (27)$$

Note that an additional critical rotor speed ω_2 appears. It is between the natural frequencies of the rotor. Appearance of an additional speed is caused by mounting of an auto-balancer on the rotor. In the transition of this speed changes the behavior of the weights. At slightly lower velocities, the weights balance the rotor but if the velocities are slightly larger, no balancing occurs.

3. The case of small viscous resistance forces in anisotropic elastic-viscous supports.

Assuming the coefficients β_x, β_y of the viscous resistance forces are small quantities, we can approximately find decompositions of the critical velocities from these parameters:

$$\omega_1 \approx \omega_x \sqrt{1+\beta_x^2/(\omega_y^2-\omega_x^2)} \approx$$
$$\approx \omega_x\{1+\beta_x^2/[2(\omega_y^2-\omega_x^2)]\},$$

$$\omega_2 \approx \sqrt{\frac{\omega_x^2+\omega_y^2}{2}}\sqrt{1+(\beta_y^2-\beta_x^2)/(\omega_y^2-\omega_x^2)} \approx$$
$$\approx \sqrt{\frac{\omega_x^2+\omega_y^2}{2}}\left[1+\frac{\beta_y^2-\beta_x^2}{2(\omega_y^2-\omega_x^2)}\right],$$

$$\omega_3 \approx \omega_y \sqrt{1-\beta_y^2/(\omega_y^2-\omega_x^2)} \approx$$
$$\approx \omega_y\{1-\beta_y^2/[2(\omega_y^2-\omega_x^2)]\}. \tag{28}$$

It is seen from (28) that small viscous resistance forces in the supports increase the smallest and reduce the greatest critical speeds of the rotor. The change in the second critical speed depends on the ratio between the coefficients β_x, β_y.

4. The general case (of anisotropic elastic-viscous supports). The following is found from (19)

$$P(\omega_x) = \omega_x^2(\omega_y^2-\omega_x^2)\beta_x^2 > 0,$$

$$P(\omega_y) = -\omega_y^2(\omega_y^2-\omega_x^2)\beta_y^2 < 0,$$

$$P(\sqrt{(\omega_x^2+\omega_y^2)/2}) = -(\omega_y^4-\omega_x^4)(\beta_y^2-\beta_x^2). \tag{29}$$

The following estimates are valid:

$\forall\omega \le \omega_x,\ \ P(\omega) > 0$ – no auto-balancing;

$\forall\omega \ge \omega_y,\ \ P(\omega) < 0$ – auto-balancing occurs.

For further analysis, write condition (19) as a polynomial:

$$\tilde{P}(\Omega) = c_0\Omega^3 + c_1\Omega^2 + c_2\Omega + c_3 < 0, \tag{30}$$

where

$$\Omega = \omega^2,\ c_0 = -2,\ c_1 = 3(\omega_x^2+\omega_y^2) - \beta_\xi^2 - \beta_\eta^2,$$

$$c_2 = -[(\omega_x^2+\omega_y^2)^2 + 2\omega_x^2\omega_y^2 - (\omega_x^2\beta_y^2 + \omega_y^2\beta_x^2)],$$

$$c_3 = \omega_x^2\omega_y^2(\omega_x^2+\omega_y^2). \tag{31}$$

With respect to Ω, the polynomial (30) can have one or three real positive roots. Therefore, the rotor has one or three critical speeds. In the case of one critical speed, auto-balancing occurs when this speed is exceeded. In the case of three critical velocities, auto-balancing occurs between the first and the second and above the third critical velocity.

The polynomial (30) has three different real roots if its discriminant is greater than zero [2, 13]

$$-4c_1^3c_3 + c_1^2c_2^2 - 4c_0c_2^3 + 18c_0c_1c_2c_3 - 27c_0^2c_3^2 > 0. \tag{32}$$

All roots of the polynomial (30) will be positive if conditions of the Routh-Hurwitz criterion [2] are additionally met

$$c_0 < 0,\ \ c_1 > 0,\ \ c_2 < 0,\ \ c_3 > 0,\ \ c_1c_2 < c_0c_3. \tag{33}$$

Conditions (32) and (33) are cumbersome for analysis in a general form.

5. The case of large viscous resistance forces in supports. It is seen from (31) that when viscous resistance forces in supports (β_x, β_y) grow, the conditions c_1>0, c_2<0 are violated and therefore only one critical speed remains at the rotor.

For large viscous resistance forces in the supports (β_x, β_y>>1), condition (19) takes approximately (accurately to the principal summands) the following form:

$$P(\omega) \approx \omega^2[(\omega_x^2-\omega^2)\beta_y^2 + (\omega_y^2-\omega^2)\beta_x^2] < 0 \tag{34}$$

from which the following approximate value of critical rotor speed is found:

$$\omega_1 \approx \sqrt{\frac{\omega_x^2\beta_y^2+\omega_y^2\beta_x^2}{\beta_x^2+\beta_y^2}} \approx \sqrt{\frac{\omega_x^2b_y^2+\omega_y^2b_x^2}{b_x^2+b_y^2}}. \tag{35}$$

It is clear from (29) and (35) that:

– increase in b_y results in going (from above) of the critical rotor speed to the lowest natural frequency ω_x;

– an increase in b_x results in going (from below) of the critical rotor speed to the largest natural frequency ω_y.

5. 5. Computational experiments

Computational experiments were performed for a two-ball auto-balancer (n=2). In all experiments, differential equations (13) were integrated with the following parameter values:

– criterion parameters that do not affect the region of the occurrence of auto-balancing κ=1, σ=0.5, χ=0.5;

– criterion parameter which affects the auto-balancing region n_η=7.

The initial conditions for integrating the system (13): $z_{0,1,2,3}$=0, z_4=2.094, z_5=n, z_6=4.189, z_7=n correspond to the main motion.

The case of small forces of viscous resistance in supports. The results of integration are listed in Table 1.

In absence of viscous resistance forces in supports ($\mu_\xi = \mu_\eta = 0$), the dimensionless critical speeds are roots of the polynomial (18):

n_1=1, n_2=5, n_3=7.

At μ_ξ=0.25, μ_η=0.5, dimensionless critical velocities speeds are:

n_1=1.003, n_2=5.041, n_3=6.925.

Table 1

Influence of parameters ε and μ_w on accuracy of determining critical speeds of the rotor

Parameters		Dimensionless critical speeds		
invariable	variable	n_1	n_2	n_3
g=0, n_η=7, κ=1, α=π/6, δ=0, σ=0.5, d=0.5, μ_ξ=0.25, μ_η=0.5	ε=0.001, μ_w=0.25	1.0÷1.05	5.0÷5.05	7.45÷7.50
	ε=0.001, μ_w=1	1.0÷1.05	5.0÷5.05	7.0÷7.05
	ε=0.001, μ_w=5	1.0÷1.05	5.0÷5.05	6.90÷6.95
	ε=0.01, μ_w=0.5	1.0÷1.05	5.0÷5.05	8.3÷8.35
	ε=0.01, μ_w=5	1.0÷1.05	5.0÷5.05	7.05÷7.10
	ε=0.01, μ_w=25	1.0÷1.05	5.0÷5.05	6.90÷6.95
	ε=0.1, μ_w=0.5	1.1÷1.15	5.10÷5.15	9.95÷10.0
	ε=0.1, μ_w=1	1.05÷1.10	5.15÷5.20	9.75÷9.80
	ε=0.1, μ_w=2	1.0÷1.05	5.10÷5.15	9.70÷9.75
	ε=0.1, μ_w=25	1.0÷1.05	5.05÷5.10	7.15÷7.20
	ε=0.1, μ_w=50	1.0÷1.05	5.0÷5.05	7.0÷7.05

The computational experiment confirms that for small viscous resistance forces in supports:

– the rotor has three critical speeds with the first and the third speeds-being close to the two natural rotor oscillation frequencies and the second speed is between them;

– small forces of viscous resistance in supports increased the lowest critical speed of the rotor and can reduce the highest velocity speed.

The computational experiment has made it possible to establish that:

– accuracy of determining the first two critical speeds of the rotor is practically not affected by parameters ε and μ_w;

– accuracy of determining the third critical speed grows with decreasing ε (the mass of the auto-balancer with respect to the mass of the rotor) and with increasing μ_w (the forces of viscous resistance to weight motion).

The case of large viscous resistance forces in supports. The results of integration are listed in Table 2.

Table 2

Influence of ε and μ_w parameters on accuracy of determining (single) critical rotor speed for large values of μ_ξ and μ_η parameters

Parameters		Dimensionless critical speed n_1	
invariable	variable	theoretical	experimental
g=0, n_η=7, κ=1, α=π/6, δ=0, σ=0.5, d=0.5, ε=0.01, μ_w=0.5÷5.0	μ_ξ=1.0, μ_η=2.0	1.044	1.0÷1.10
	μ_ξ=2.5, μ_η=5.0	1.411	1.40÷1.45
	μ_ξ=0.5, μ_η=1.5	1.011	1.0÷1.05
–//– μ_w=0.5	μ_ξ=3.0, μ_η=2.0	6.278	8.650÷8.70
	μ_ξ=5.0, μ_η=2.5	6.431	9.45÷9.50
	μ_ξ=5.0, μ_η=5.0	5.0	11.70÷11.75
–//– μ_w=0.5, ε=0.001	μ_ξ=3.0, μ_η=2.0	6.27780	6.45÷6.50
	μ_ξ=5.0, μ_η=2.5	6.43059	6.60÷6.65
	μ_ξ=5.0, μ_η=5.0	5.0	5.10÷5.15
–//– μ_w=5.0, ε=0.01	μ_ξ=3.0, μ_η=2.0	6.27780	6.30÷6.35
	μ_ξ=5.0, μ_η=2.5	6.43059	6.45÷6.50
	μ_ξ=5.0, μ_η=5.0	5.0	5.0÷5.05

The computational experiment has confirmed that for large viscous resistance forces in supports:

– the rotor has one critical speed and the main motion is stable when this speed is exceeded;

– an increase in μ_η results in going (from above) of the critical rotor speed to the smallest natural frequency ω_x;

– an increase in μ_ξ can result in going (from below) of the critical rotor speed to the largest natural frequency ω_y;

– at equal μ_ξ and μ_η, their growth results in going of the critical rotor speed to an additional critical speed.

The computational experiments have made it possible to establish that:

– accuracy of determining the critical rotor speed close to ω_x is practically not affected by the ε and μ_w parameters;

– accuracy of determining the critical speed close to $\sqrt{(\omega_x^2+\omega_y^2)/2}$ or ω_y increases with decrease in ε (the mass of the auto-balancer with respect to the rotor mass) and with increase in μ_w (the forces of viscous resistance to the weight motion).

6. Discussion of the obtained conditions for occurrence of auto-balancing

1. Within the framework of the planar model of a rotor mounted on anisotropic elastic-viscous supports and balanced by a pendulum (roller, ball) auto-balancer:

– differential equations of motion are nonlinear and depend on the auto-balancer type;

– differential equations describing steady motion of the rotor with an elementary imbalance caused by deviation of weights from the main motion are linear and do not depend on the type of auto-balance.

2. As a result of analytical studies, it was confirmed that for isotropic elastic supports (with no viscosity), the rotor has a unique critical speed coinciding with the natural frequency. Auto-balancing occurs at supercritical speeds of the rotor.

With anisotropic elastic supports (with no viscosity), the rotor has three critical speeds. The first and the third speeds coincide with the natural frequencies of the rotor oscillation. The second critical speed is between the first two velocities. Auto-balancing occurs at speeds between the first and second and above the third critical speeds.

The additional critical speed (the second speed) appears when the auto-balancer is mounted on the rotor. In the transition of this speed behavior of the auto-balancer changes: the auto-balancer reduces imbalance of the rotor at slightly lower speeds and increases it at somewhat higher speeds.

3. As a result of analytical studies, it has been established that for isotropic elastic supports, emergence of viscous resistance forces in supports (including anisotropic ones) does not affect the value of the critical speed of the rotor and the condition for the occurrence of auto-balancing.

In the case of anisotropic elastic supports, with the appearance of small viscous resistance forces in supports, the smallest and the highest critical speeds of the rotor grow. The change of the second critical speed depends on the relationship between the b_x, b_y coefficients.

At large forces of viscous resistance in supports, the rotor has only one critical speed and when it is exceeded, auto-balancing occurs. If the coefficient of viscous resistance force in a support of smaller rigidity is larger than this coefficient in a more rigid support, the critical speed approaches a lower natural frequency.

If the coefficient of viscous resistance forces in a less rigid support is less than this coefficient in a more rigid support, then the criticalspeed approaches a larger natural frequency.

4. As a result of the computational experiments, it was established that accuracy of determining speeds increases with:

– reduction of the auto-balancer mass with respect to the mass of the rotor;

– an increase in the forces of viscous resistance to weight motion.

As the forces of viscous resistance to motion of weights decrease, the first and the second critical speeds practically do not change and the third one grows.

The empirical criterion for stability of the main motion is an effective method for determining conditions under which auto-balancers of a certain type can balance a certain rotor. This criterion correctly describes the qualitative behavior of the rotor – auto-balancer system: it determines the number of critical speeds and the regions of auto-balancing.

The criterion has drawbacks inherent to approximate Liapunov's methods of studying motion stability. It gives approximate formulas for critical speeds that are uniformly unsuitable for the entire range of variation of the system parameters. Also, it does not enable study of the influence of the forces resisting motion of weights on the conditions for onset of auto-balancing.

For the future, it is planned to determine conditions for onset of auto-balancing of rotors in concrete machines with the help of empirical criteria for the occurrence of auto-balancing and stability of the main motion.

7. Conclusions

The empirical criterion for stability of the main motion is an effective method for determining conditions under which an auto-balancer of a certain type can balance a certain rotor.

1. Within the framework of a planar model of a rotor mounted on anisotropic elastic-viscous supports and balanced by a pendulum (roller, ball) auto-balancer:

– differential equations of motion are nonlinear and depend on the auto-balancer type;

– differential equations describing steady motion of the rotor with an elementary imbalance caused by deviation of weights from the main motion are linear and do not depend on the auto-balancer type.

2. For equirigid supports, regardless their viscosity, the rotor has a unique critical speed. Auto-balancing occurs at supercritical rotor speeds.

For anisotropic purely elastic supports (with no viscosity), the rotor has three critical speeds. The first and the third speeds coincide, respectively, with the first and the second resonant speeds of the rotor. The second, additional critical speed, is between the resonant speeds. Auto-balancing can occur at speeds between the first and the second and above the third critical speeds.

With appearance of small viscous resistance forces in supports, the smallest and the greatest critical speeds of rotor increase. The change in the second critical speed depends on the relationship between the coefficients of support viscosity.

For large forces of viscous resistance in supports, the rotor has the only critical speed and when it is exceeded, auto-balancing takes place. Depending on anisotropy of the support viscosity, the critical speed may be closer to the lower or higher frequency of the rotor's natural oscillations in absence of resistance forces in the supports.

3. The criterion correctly describes the qualitative behavior of the rotor – auto-balancer system: it determines the number of critical speeds and the regions of the auto-balancing onset. Accuracy of determining critical speeds (the boundaries of the regions of auto-balancing onset) increases with:

– reduction of the auto-balancer mass with respect to the rotor mass;

– growth of forces of viscous resistance to the motion of correction weights.

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