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EXPONENTIAL AND HYPERBOLIC TYPES OF DISTRIBUTION IN MACRO SYSTEMS: THEIR COMBINED SYMMETRY AND FINITE PROPERTIES

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Запропоновано розширений ентропійний метод, що виявляє деякі нові зв'язки в організації макросистем, тим самим проливаючи світло на ряд існуючих питань теорії. Зокрема, показано, що тип розподілу всередині макросистеми визначається співвідношенням кінетичних властивостей її агентів – «носіїв» і «ресурсів». Якщо час релаксації менше у «носіїв» – формується експонентний тип розподілу, якщо менше у «ресурсів» – формується тип розподілу з важким хвостом.

Виявлено існування комбінованої симетрії цих двох типів розподілів, які можна розглядати як два різні статистичні трактування єдиного стану макросистеми. Розподіли реальних макросистем мають фінітні властивості - у них природним чином формуються праві межі. Запропонований метод враховує праві межі фінітних розподілів як продукт самоорганізації макросистем, координати яких визначаються на основі екстремального принципу.

Отримано аналітичні вирази для цих двох типів розподілів і їх спектрів, для яких знайдено вдалий спосіб параметричного запису через модальні характеристики. Отримано аналітичні вирази, що враховують фінітні особливості розподілів, де фігурують лише два параметри – середня кількість «ресурсів» та формпараметр як відношення модальної і граничної координат.

Цінність отриманих результатів полягає в тому, що вони проливають світло на ряд проблемних питань статистичної теорії макросистем, та містять набір зручних інструментів для аналізу двох типів розподілів з фінітними властивостями

Ключові слова: макросистема, ентропія, ентропійне моделювання, фінітні розподіли, гіперболічні розподіли, розподіли з важким хвостом

1. Introduction

The growing demand for quantitative predictions in natural, economic, humanitarian, and other fields, has prompted interest in the theory of macro systems as the ideological basis of these studies. Predicting the state of large systems with stochastic behavior of separate elements became possible owing to the tools developed in statistical physics. One such powerful tool is the extreme entropy principle, underlying classic distributions by Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac. All of them possess an exponential attenuation rate.

As practice reveals, the macro systems that are related to the aforementioned "non-physical" areas, in addition of quickly damped exponential distributions, often exhibit a different type of distributions, specifically a power (hyperbolic) distribution with a heavy tail. In contrast to the exponential distributions, with a solid theoretical basis, that one was discovered only as an empirical phenomenon.

Given this, there are at least two issues that are the focus of attention for several researchers. First, why is it that in one case there forms an exponential distribution type and in the other one a heavy tail distribution, and what is the connection between them? Second, how can we account for the finite features in real systems where, due to natural constraints, there forms the right bound of distributions? Consideration of finite properties is important for the proper calculation of statistical sum, especially when analyzing distributions with a heavy tail whose weight cannot be neglected.

This paper tackles the development of the method, which would make it possible, based on known extreme principles, to substantiate from a unified position the mechanism for the formation of different types of distributions, dominating most macro systems, as well as to determine the finite characteristics of these distributions. As it turned out, a given method makes it possible to obtain several nontrivial results, complementing the modern theory.

2. Literature review and problem statement

Using the entropy approach to analysis of the equilibrium state of macro systems often leads to the exponential type of distribution. It matches the known laws of Boltzmann, Gibbs, it describes urban traffic flows [1], the distribution of preferences in active systems [2].

At the same time, the phenomenon of hyperbolic distributions was originally discovered only as an empirical fact. This type, in addition to known Pareto distributions of Pareto, Zipf, Lotka, Bradford, Auerbach, governs, for example, the size of the fatigue micro defects in solid material, scales of turbulent eddies in the atmosphere, or the intensity of luminescence of star clusters in cosmos.

Ideological justification of the reasons of occurrence of heavy-tailed distributions has been given much attention. Without claiming completeness, one can draw some examples of the works where a hyperbolic distribution forms: as a reaction of a dynamic object to the impact of a random signal in the form of delta-correlated noise [3]; as a result of the limit transition of the function of hypergeometrical distribution [4], or a beta distribution function [5]; as a result of competitive behavior of agents of the system [6] (this very paper awakened the author's interest in conducting these studies). Article [7] shows that the hyperbolic type of distribution emerges as eigenvalues when solving a stationary Schrodinger equation. Authors of paper [8] succeeded, within a unified fractional differential approach, in describing both the "dispersion" (power) character of transfer in disordered semiconductors and the Gaussian (normal) character of transfer (transition to the normal distribution law occurs when an external electric field decreases). Additional variants are considered in studies [9–11], which explain the sources of forming the distributions with a heavy tail; [12] provides an overview of these variants.

A great variety of approaches testifies to the following. Even though the macro systems with a hyperbolic type of distribution are widespread, there is as yet no reliable theoretical grounds for explaining this statistical form of their organization.

Another feature of real systems is the finite character of their distributions. Consideration of finite properties is important for the proper calculation of a statistical sum, particularly when modeling statistics with a heavy tail whose weight, in contrast to the exponential type of distribution, cannot be neglected. In current practice, upper bounds are not computed, they are simply assigned instead; circumcision of tails is mainly artificial and is based on either common sense or on a requirement for better convergence towards the selected statistical model [13].

Based on the analysis of studies by other authors, one can conclude that solving such problems as the substantiation of the mechanism for forming different types of distributions, dominating most macro systems, as well as determining the finite properties of these distributions, is not always supported by a fundamental theoretical basis. A natural basis is possibly a variational method that is based on known extreme principles.

3. The aim and objectives of the study

The aim of this work is to create a tool to examine the conditions under which one or another type of distribution form in macro systems – exponential or hyperbolic with a heavy tail, as well as to find a theoretically substantiated technique for determining the finite properties of these distributions. In the applied aspect, this would make it possible to obtain, as a result, more effective tools for quantitative predictions of behavior of macro systems, belonging to the natural, economic, humanitarian, and other fields of knowledge.

To accomplish the aim, the following tasks have been set:

– to construct an advanced entropy method that would better account for the combinatorial configurations in the system, which would take into consideration the stage-wise character of relaxation processes, as well as the finite properties of distributions;

– to obtain, based on it, two types of finite distributions – fast fading exponential and extremely hyperbolic with a heavy tail, as well as to find a convenient form of their parametric notation;

– to find the relationship between exponential and hyperbolic distributions, to find a technique to determine the numerical values of their finite parameters.

4. Extended entropy approach

Four provisions underlie the extended entropy approach:

1) One selects a common property of most real macro systems. They are treated as objects in which at least two agents interact, specifically: – *a limited set of abstract "resources" are allocated among the finite set of abstract "carriers"* [10]. For example, energy is distributed among the molecules of gas, people – among cities, wealth – among people.

The status of carriers or resources – carriers or resources – is conditional and can be reversed. It is convenient to consider carriers to be a set whose single element can "possess" an arbitrary number of elements (resource portions) from another one. For example, a set of localities is the carriers, population are the resources. At the same time, urban dwellers act as the carriers of such resources as living space, or the amount of electricity consumed.

Note that the term "carriers" introduced here has nothing to do with the same one employed in the mathematical literature, meaning the closure of a set of function arguments (if such a closure is limited, the function is called *finite*).

2) One takes into consideration the peculiarities of finite distributions.

Real macro systems typically have a right bound of distributions. Despite this, in many sources the statistical sum is computed by integrating to infinity (for example, when deriving a distribution formula by Maxwell, Boltzmann, Planck [11]). However, this convenient technique is valid only for the rapidly decaying exponential dependences and is not at all suitable for heavy-tailed distributions.

When dealing with finite distributions, there emerges the uncertainty of the right bound (population size of the most inhabited city, or wealth of the society's richest member). Extended entropy method assumes that *the magnitude of maximum coordinate of the finite distribution (right bound) is generated within the system not arbitrarily. Similar to the distribution type, it is also a product of self-organization of the macro system and can also be determined on the basis of a predefined extreme principle.*

3) One assumes a broader view on the concept of the equilibrium state of the system.

In the current practice of entropy analysis, statistical weight is typically calculated based only on the number of splitting the set of carriers [1, 12, 13]. The extended entropy method implies counting the combinatorial configurations both within the set of carriers and within the set of resources two equal agents of the system.

In this case, the mutual process when elements of one set acquire the elements of another one typically occurs against the background of the dominant activity of one of them. For example, in social geography, resources (population) are more active in their movements than their carriers (cities). At the same time, in a closed thermodynamic system, it is the carriers that are more dynamic (relaxation time for the parameter of density is usually shorter than the relaxation time for the parameter of temperature (that is, energy, the resources) [14, 15]).

I shall further demonstrate that the *comparative kinetic activity of carriers and resources is the key factor that determines the type of statistical distribution in a macro system. The higher kinetic activity of carriers generates an exponential distribution, while the larger activity of resources predetermines the distribution with a "heavy tail".*

Thus, the empirical Auerbach law (distribution of the number of cities based on the number of their inhabitants) takes a *hyperbolic* form, due to the greater activity of people. In the law by Pareto, distribution of public wealth is due to the larger activity of money (resources) in comparison with the activity of the population (carriers). It is obvious that capital is more active than most people, and here, as well, a heavy tail distribution forms.

At the same time, the systems with more dynamic carriers form an exponential type of distribution (examples were given at the beginning of this paper).

4) The stage-wise character of the relaxation process is taken into consideration.

The non-physical macro systems, similar to objects studied in the physical kinetics, undergo a consistent shift of quasi-equilibrium states towards a complete equilibrium state. Such stages are predetermined by a difference in the relaxation time for each individual agent in the system.

5. Distribution entropies in systems with two agents

Let some closed system be composed of N carriers, among which the E quantity of resources. are allocated. Each carrier has its own individual portion of resource ε. Magnitude ε defines the value of coordinate *in the space of the individual states* of the carrier. For the case when there are several different resources, *the space of individual states* becomes multi-dimensional, and the individual state of the carrier is characterized by a point with multiple coordinates $(\varepsilon, \varsigma, \eta ...)$.

Consider the *one-dimensional and discrete* space of individual states. To this end, divide the range of possible change in coordinate ε into M equal intervals $\Delta\varepsilon$ with the coordinate values $\varepsilon_1, \varepsilon_2, ..., \varepsilon_M$ averaged within the interval. Based on this attribute, one selects M cells of the space of individual states. Thus, the cell with coordinate ε_i contains the number n_i of carriers, which possesses the amount of resources $E_i = n_i \cdot \varepsilon_i$. With respect to the accepted designations, the following conditions are satisfied:

$$\sum_{i=1}^{M} n_i = N \quad \text{– balance of carriers,} \tag{1}$$

$$\sum_{i=1}^{M} n_i \varepsilon_i = \sum_{i=1}^{M} E_i = E \quad \text{– balance of resources.} \tag{2}$$

The system implements with the greatest frequency the macro state that can be recreated by the maximum number of distinguishable microstates. The power (cardinal number) of the set of all microstates that are capable to recreate a given macro state is statistical weight W and its logarithm $S = \ln W$ is the statistical Boltzmann entropy.

The extended entropy method implies a joint analysis, not only of the entropy of carriers distribution, but also the entropy of distribution of the set of resources (sometimes called a *resource spectrum*). By assigning the appropriate index, it is possible to calculate statistical weight of each of the distributions as the number of ordered partitions of the finite set $R(\cdot)$ [17]:

$$W_H = R(n_1, n_2 ..., n_M) = \frac{N!}{\prod_{i=1}^{M} n_i!}; \tag{3}$$

$$W_P = R(\mathrm{E}_1, \mathrm{E}_2 ..., \mathrm{E}_M) = \frac{E!}{\prod_{i=1}^{M} E_i!}. \tag{4}$$

The factorial operation here at the amount of resources E is justified by the fact that, in line with the accepted designations, each term $E_i = n_i \varepsilon_i = i \cdot n_i \cdot \Delta\varepsilon$ consists of an integer number of portions $\Delta\varepsilon$, with the size of the latter depending on the will of the researcher.

Using the Stirling's approximation $\ln m! \approx m \cdot (\ln m - 1)$, one obtains a carrier distribution entropy, adjusted to the total number of elements from set N:

$$S_H(\mathrm{n}_1,\mathrm{n}_2\ldots\mathrm{n}_M)=\frac{\ln W_H}{N}\approx-\sum_{i=1}^{M}\left(\frac{n_i}{N}\ln\frac{n_i}{N}\right), \quad (5)$$

and the entropy of resources allocation, reduced to the volume of set E:

$$S_P(E_1,E_2\ldots E_M)=\frac{\ln W_P}{E}\approx-\sum_{i=1}^{M}\left(\frac{E_i}{E}\ln\frac{E_i}{E}\right). \quad (6)$$

The process of forming the equilibrium distribution in the system often proceeds in several stages, and therefore the *resulting equilibrium state can be regarded as the implementation of a complex experience*. If the carriers are more dynamic, the entropy of a complex experience equals [16]:

$$S_{HP}=S_H+S_{P|H},$$

where S_H is the entropy of distribution of carriers, $S_{P|H}$ is the conditional entropy of resources distribution (assuming the distribution of carriers is formed). If the resources are more dynamic, the indexes are swapped $S_{PH}=S_P+S_{H|P}$.

6. Exponential distribution as a consequence of the greater kinetic activity of carriers

Find the distribution of carriers $n_i=n(\varepsilon_i)$, at which entropy (5) under conditions (1) and (2) reaches a maximum. To this end, in accordance with the method of the Lagrange multipliers, one will search for an unconditional extremum of a certain new function $\mathrm{X}(\mathrm{n}_1,\mathrm{n}_2\ldots\mathrm{n}_M)$, which additively includes (5), as well as relations (1), (2), weighted by multipliers α, β:

$$\mathrm{X}(\mathrm{n}_1,\mathrm{n}_2\ldots\mathrm{n}_M)=-\sum_{i=1}^{M}\frac{n_i}{N}\cdot\ln\frac{n_i}{N}+$$
$$+\alpha\cdot\left(\sum_{i=1}^{M}\frac{n_i}{N}-1\right)+\beta\cdot\left(\sum_{i=1}^{M}\frac{n_i\cdot\varepsilon_i}{N}-1\right).$$

Known *exponential solution* matches its maximum:

$$n_i=C_H\cdot\exp(\beta\cdot\varepsilon_i). \quad (7)$$

The value for $C_H=N\cdot\exp(\alpha-1)$ is derived based in the normalization conditions. As regards the β multiplier, it is usually defined, in line with established practice, applying any external additional provisions.

It is of interest to find a convenient technique to independently define both distribution parameters C_H and β. It turns out that they are closely related to the modal characteristics E_{**}, ε_{**} in the appropriate allocation of resources $E_i=E(\varepsilon_i)$, or the so-called resource spectrum:

$$E_i=n_i\cdot\varepsilon_i=\varepsilon_i\cdot C_H\cdot\exp(\beta\cdot\varepsilon_i). \quad (8)$$

Modal parameters of discrete distribution (8) will be found by determining the extremum of its continuous analog

$$E(\varepsilon)=\varepsilon\cdot C_H\cdot\exp(\beta\cdot\varepsilon).$$

The derived maximum (modal) value $\mathrm{E}_{**}=n_{**}\cdot\varepsilon_{**}$, attainable at $\varepsilon=\varepsilon_{**}$, makes it possible to express parameters C_H and β in the form:

$$\beta=-\frac{1}{\varepsilon_{**}};$$

$$C_H=\mathrm{e}\cdot\frac{E_{**}}{\varepsilon_{**}}=\mathrm{e}\cdot n_{**},$$

where $\mathrm{e}=2.718\ldots$.

As a result, the allocation of resources (8), recorded via modal parameters, takes the form:

$$\frac{E_i}{E_{**}}=\frac{\varepsilon_i}{\varepsilon_{**}}\cdot\exp\left(1-\frac{\varepsilon_i}{\varepsilon_{**}}\right). \quad (9)$$

Dividing it by $\varepsilon_i/\varepsilon_{**}$, one finds the desired *exponential distribution* of the number of carriers (Fig. 1, *a*):

$$\frac{n_i}{n_{**}}=\exp\left(1-\frac{\varepsilon_i}{\varepsilon_{**}}\right). \quad (10)$$

Resource allocation (9) can be interpreted as a "resource spectrum", corresponding to the exponential distribution of the number of carriers (10). Thus, for example, for a perfect gas, expression (10) is the distribution of the number of molecules among energy levels, expression (9) is the energy spectrum (distribution of the amount of energy among the same energy levels).

7. Extreme hyperbolic distribution as a consequence of the greater kinetic activity of resources

Equilibrium distribution for the case of the greater activity of resources will be found based on the condition of an entropy maximum (6) under constraints (1), (2). The search for a *conditional* maximum of entropy (6) comes down to determining the *unconditional* maximum of function:

$$\mathrm{Y}(E_1,E_2\ldots E_M)=-\sum_{i=1}^{M}\frac{n_i\cdot\varepsilon_i}{E}\cdot\ln\frac{n_i\cdot\varepsilon_i}{E}+$$
$$+\lambda\cdot\left(\sum_{i=1}^{M}\frac{n_i}{E}-\frac{N}{E}\right)+\mu\cdot\left(\sum_{i=1}^{M}\frac{n_i\cdot\varepsilon_i}{E}-1\right),$$

where λ and μ are the Lagrange multipliers.

The following solution applies:

$$n_i=\frac{C_P}{\varepsilon_i}\cdot\exp\left(\frac{\lambda}{\varepsilon_i}\right). \quad (12)$$

One can show that parameters $C_P=E\cdot\exp(\mu-1)$ and λ are closely associated with mode ε_* and modal value n_* for distribution (12). Indeed, from condition

$$\frac{dn(\varepsilon)}{d\varepsilon}=0$$

for continuous function

$$n(\varepsilon)=\frac{C_P}{\varepsilon}\cdot\exp\left(\frac{\lambda}{\varepsilon}\right)$$

it follows:

$$\lambda=-\varepsilon_*,$$

$$C_P = n_* \varepsilon_* e,$$

where $e \approx 2.718...$

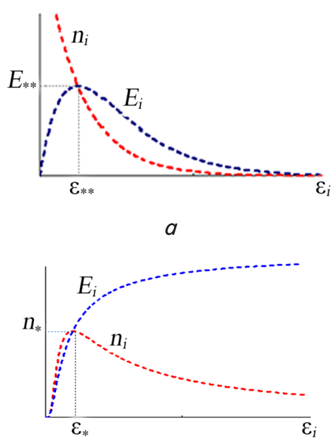
As a result, one obtains an expression for the equilibrium distribution of carriers provided that the relatively more active in a macro system is the second agent – resources. This is the so-called *extreme hyperbolic distribution law*, presented for the first time in [10]:

$$\frac{n_i}{n_*} = \frac{\varepsilon_*}{\varepsilon_i} \cdot \exp\left(1 - \frac{\varepsilon_*}{\varepsilon_i}\right). \qquad (13)$$

It has a "heavy tail"; the name is justified by the fact that its curve (Fig. 1, *b*) at a decrease in the coordinate of extremum ε_* approaches a purely hyperbolic dependence. It will be shown below that it is observed in real systems in the case of scarce resources.

Respective allocation of the resource volume (*resource spectrum*) will be derived from (13) by multiplying it by $\varepsilon_i / \varepsilon_*$:

$$\frac{E_i}{E_*} = \exp\left(1 - \frac{\varepsilon_*}{\varepsilon_i}\right). \qquad (14)$$



b

Fig. 1. Two types of distributions: *a* — exponential distribution $n_i = n(\varepsilon_i)$ — expression (10) and its resource spectrum $E_i = E(\varepsilon_i)$ — expression (9); *b* — extreme hyperbolic distribution $n_i = n(\varepsilon_i)$ — expression (13) and its resource spectrum $E_i = E(\varepsilon_i)$ – expression (14)

These figures (Fig. 1, *a, b*) show both types of the derived distributions and their corresponding resource spectra.

8. Form-parameter is an important characteristic of finite distributions

Real macro systems have a natural constraint $\varepsilon_M \le E$ (the part does not exceed the whole). As a result, distributions in these systems have a *finite character*. The finite distributions, in addition to the two modal parameters ε_{**}, n_{**} (or ε_*, n_*), have a third magnitude – maximum coordinate ε_M. It is the upper bound of integration when computing a statistical sum.

It is assumed that the magnitude of maximum coordinate ε_M of the finite distribution is formed within the system not arbitrarily. *It, as well as the distribution itself, is the product of the self-organization of a macro system and its value can also be found based on the appropriate extreme variational principles*. This problem is solved in the framework of the extended entropy method.

An analysis of finite distributions reveals that one can elegantly enough consider the impact the maximum coordinate ε_M by introducing such a notion as a *form-parameter*.

The form-parameter of the finite distribution *shall be understood as the ratio of its modal and maximal coordinates*. For the exponential and extreme hyperbolic distributions, the form-parameter is, respectively, equal to:

$$\psi = \frac{\varepsilon_{**}}{\varepsilon_M} \text{ and } \varphi = \frac{\varepsilon_*}{\varepsilon_M}. \qquad (15)$$

It can acquire values within $[0;1]$.

In this case, the fractions included in the laws of distribution (10) and (13), can be represented in the form:

$$\frac{\varepsilon_i}{\varepsilon_{**}} = \frac{i}{\psi \cdot M} \text{ and } \frac{\varepsilon_*}{\varepsilon_i} = \frac{\varphi \cdot M}{i}, \qquad (16)$$

which follows from totality

$$\varepsilon_i = i \cdot \Delta\varepsilon = i \cdot \frac{\varepsilon_M}{M},$$

where i is the number of the cell, $\Delta\varepsilon$ is its predefined size, $M = \varepsilon_M / \Delta\varepsilon$ is the total number of cells.

The importance of form-parameter is obvious from further constructs. It is the form-parameter that along with the magnitude of the mean portion of resource E/N is the second characteristic of the system that defines the shape of the distribution curve, as well as its corresponding spectrum.

9. Finite exponential distribution

9. 1. Modal parameters ε_{**} and n_{**}

Modal parameters for the exponential distribution law can be obtained by substituting expression (10) in formulae (1) and (2), and, with respect to representation (16), one obtains:

$$\varepsilon_{**} = \frac{E}{N} \cdot \frac{\sum_{i=1}^{M} \exp\left(-\frac{i}{\psi \cdot M}\right)}{\sum_{j=1}^{M} \frac{j}{\psi \cdot M} \cdot \exp\left(-\frac{j}{\psi \cdot M}\right)}, \qquad (17)$$

$$n_{**} = \frac{N}{\sum_{j=1}^{M} \exp\left(1 - \frac{j}{\psi \cdot M}\right)}. \qquad (18)$$

Here ψ is the exponential finite distribution form-parameter.

When one discovers that the sums contained herein have geometric and arithmetic-geometric progressions, one can record their analytical solutions (which can also be derived by passing from the discrete summation to integral calculations):

$$\frac{1}{M} \sum_{j=1}^{M} \exp\left(-\frac{j/M}{\psi}\right) \approx \psi \cdot \left(1 - \exp\left(-\frac{1}{\psi}\right)\right), \qquad (19)$$

$$\frac{1}{M}\sum_{j=1}^{M}\frac{j/M}{\psi}\cdot\exp\left(-\frac{j/M}{\psi}\right)\approx$$
$$\approx\psi\cdot\left(1-\exp\left(-\frac{1}{\psi}\right)\right)-\exp\left(-\frac{1}{\psi}\right). \qquad (20)$$

For the case of continuous distribution (at $M\to\infty$), these approximated formulae become precise.

I shall introduce designations:

$$B(\psi)=\frac{\psi\cdot\left(e^{1/\psi}-1\right)}{\psi\cdot\left(e^{1/\psi}-1\right)-1}; \qquad (21)$$

$$C(\psi)=\frac{e^{1/\psi}}{e^{1/\psi}-1}. \qquad (22)$$

Then, applying transforms (19), (20) and given that

$$\frac{1}{M}=\frac{\Delta\varepsilon}{\varepsilon_M}=\frac{\Delta\varepsilon\cdot\psi}{\varepsilon_{**}},$$

modal parameters (17), (18) can be reduced to the form:

$$\varepsilon_{**}=\frac{E}{N}\cdot B(\psi), \qquad (23)$$

$$n_{**}=N\cdot\frac{\Delta\varepsilon}{\varepsilon_{**}}\cdot\frac{C(\psi)}{e}. \qquad (24)$$

By deriving the mode ε_{**}, applying (23), one can, in line with designation (15), obtain the right bound of the finite distribution

$$\varepsilon_M=\varepsilon_{**}\cdot\frac{1}{\psi}.$$

Their charts are shown in Fig. 2.

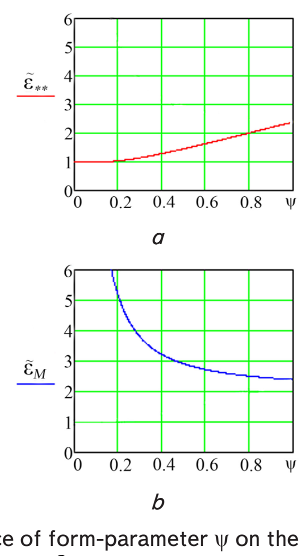


Fig. 2. Influence of form-parameter ψ on the magnitude: *a* – modal $\tilde{\varepsilon}_{**}=\frac{\varepsilon_{**}}{E/N}$ and *b* – maximal $\tilde{\varepsilon}_M=\frac{\varepsilon_M}{E/N}$ coordinates of the exponential finite distribution

9. 2. Parameters for the finite exponential distribution

Let me show that the finite exponential distribution of carriers (10) and the corresponding finite resource spectrum (9) depend on two parameters only – form-parameter ψ and the mean portion of resource E/N. To this end, substitute (23), (24) in original expressions (10) and (9); the result is a discrete form of the finite exponential distribution of carriers:

$$\frac{n_i/N}{\Delta\varepsilon}=\frac{1}{E/N}\cdot\frac{C(\psi)}{B(\psi)}\cdot\exp\left(-\frac{\varepsilon_i}{\varepsilon_{**}}\right), \qquad (25)$$

and its resource spectrum (resource distribution):

$$\frac{E_i/E}{\Delta\varepsilon}=\frac{\varepsilon_i}{(E/N)^2}\cdot\frac{C(\psi)}{B(\psi)}\cdot\exp\left(-\frac{\varepsilon_i}{\varepsilon_{**}}\right), \qquad (26)$$

where

$$\varepsilon_{**}=\frac{E}{N}\cdot B(\psi).$$

Values $B(\psi)$, $C(\psi)$, are determined from formulae (21), (22).

By applying limiting transition $\Delta\varepsilon\to 0$, to (25) and (26), one obtains expressions for the distributions of corresponding densities:

$$f_n(\varepsilon)=\lim_{\Delta\varepsilon\to 0}\frac{n_i/N}{\Delta\varepsilon} \text{ and } f_E(\varepsilon)=\lim_{\Delta\varepsilon\to 0}\frac{E_i/E}{\Delta\varepsilon}.$$

Their form almost coincides with (25), (26), but instead variable ε_i it contains variable ε.

The distributions of density $f_n(\varepsilon)$, $f_E(\varepsilon)$ and the limit of their integration $\varepsilon_M=\varepsilon_{**}/\psi$ depends on two parameters E/N – the mean value of an individual portion of resources (considered to be assigned) and form-parameter ψ, characterizing the finite properties of distribution of the macro system.

The value of a form-parameter is determined by the stage of the quasi-equilibrium state of the macro system, which is shown below.

9. 3. Estimation of the equilibrium value of form-parameter ψ

Represent the exponential finite distribution (10) and its resource spectrum (9) in relative variables. For this purpose, taking into consideration representation (16) and formulae (17), (18), transform these expressions to the form:

$$\frac{n_i}{N}=\frac{\exp\left(-\frac{i/M}{\psi}\right)}{\sum_{j=1}^{M}\exp\left(-\frac{j/M}{\psi}\right)}, \qquad (27)$$

$$\frac{E_i}{E}=\frac{\frac{i/M}{\psi}\cdot\exp\left(-\frac{i/M}{\psi}\right)}{\sum_{j=1}^{M}\frac{j/M}{\psi}\cdot\exp\left(-\frac{j/M}{\psi}\right)}. \qquad (28)$$

A growth of the upper limit of summation M leads to that the sums quickly converge. The curves that correspond to expressions (27), (28) are shown in Fig. 3.

These curves are represented as functions of the relative cell number i/M. Their only parameter is the form-parameter ψ, which can be interpreted as a *relative number of modal cell* i_{**}/M. Indeed, it follows from (15)

$$\psi=\frac{\varepsilon_{**}}{\varepsilon_M}=\frac{\Delta\varepsilon\cdot i_{**}}{\Delta\varepsilon\cdot M}.$$

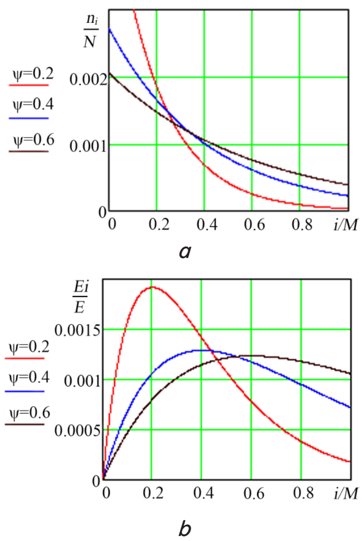


Fig. 3. Influence of form-parameter ψ on: *a* — exponential finite distribution (27); *b* — its corresponding spectrum (28)

Next, substituting (27) and (28) into formulae (5) and (6), one obtains dependences, expressed through form-parameter ψ, for the entropy of primary experience (distribution of carriers):

$$S_H(\psi)=-\sum_{i=1}^{M}\left(\frac{\exp\left(-\frac{i/M}{\psi}\right)}{\sum_{j=1}^{M}\exp\left(-\frac{j/M}{\psi}\right)}\right)\cdot\ln\left(\frac{\exp\left(-\frac{i/M}{\psi}\right)}{\sum_{j=1}^{M}\exp\left(-\frac{j/M}{\psi}\right)}\right), \quad (29)$$

and for the conditional entropy of subsequent experience (the allocation of resources taking into consideration the previously distributed carriers):

$$S_{P|H}(\psi)=-\sum_{i=1}^{M}\left(\frac{\frac{j/M}{\psi}\cdot\exp\left(-\frac{i/M}{\psi}\right)}{\sum_{j=1}^{M}\frac{j/M}{\psi}\cdot\exp\left(-\frac{j/M}{\psi}\right)}\right)\ln\left(\frac{\frac{j/M}{\psi}\cdot\exp\left(-\frac{i/M}{\psi}\right)}{\sum_{j=1}^{M}\frac{j/M}{\psi}\cdot\exp\left(-\frac{j/M}{\psi}\right)}\right). \quad (30)$$

Fig. 4 shows their charts.

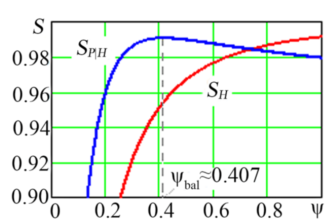


Fig. 4. Entropy of the distribution of carriers $S_H(\psi)$ and conditional entropy of resource allocation $S_{P|H}(\psi)$, normalized by the magnitude ln*M*

Fig. 4 shows that the maximum of entropy of initial experience $S_H(\psi)$ is achieved at the value of form-parameter of ψ=1, which corresponds to the initial stage – a stage of the quasi-equilibrium state (the equilibrium is reached only for the set of carriers as a more active agent). The system subsequently tends along a growth trajectory of the conditional entropy $S_{P|H}(\psi)$ to enter the state of the ultimate balance, which is attained as a result of the movement of the less active set of resources. The shows that the extremum $S_{P|H}(\psi)_{MAX}$ is reached at the value of $\psi_{bal}\approx 0.407$.

One can expect that the evolution trajectory of an exponential finite distribution corresponding to a macro system with the more dynamic carriers passes in the direction of form-parameter's values from ψ=1 to ψ≈0.407.

10. Finite extreme hyperbolic distribution

10. 1. Modal parameters ε_* and n_*

Modal parameters of the extreme hyperbolic distribution (13) will be obtained by substituting this expression alternately into formulae (1) and (2); hence, taking into consideration representation (16), one obtains:

$$\varepsilon_*=\frac{E}{N}\cdot\frac{\sum_{i=1}^{M}\frac{\varphi\cdot M}{i}\cdot\exp\left(-\frac{\varphi\cdot M}{i}\right)}{\sum_{i=1}^{M}\exp\left(-\frac{\varphi\cdot M}{i}\right)}, \quad (31)$$

$$n_*=\frac{N}{\sum_{i=1}^{M}\frac{\varphi\cdot M}{i}\cdot\exp\left(1-\frac{\varphi\cdot M}{i}\right)}, \quad (32)$$

where φ is the form-parameter of a finite extreme hyperbolic distribution (15).

In contrast to the exponential distribution, it is impossible to reduce discrete sums in given expressions to simple algebraic equations. However, there is a possibility to pass from discrete summation to the analytical integral calculations by making the cell size approach $\Delta\varepsilon\to 0$ (for the finite objects, it corresponds to an infinite number of cells $M\to\infty$):

$$\lim_{M\to\infty}\frac{1}{M}\cdot\sum_{i=1}^{M}\frac{\varphi\cdot M}{i}\cdot\exp\left(-\frac{\varphi\cdot M}{i}\right)=\varphi\cdot\int_0^{1/\varphi}\frac{1}{t}\cdot\exp\left(-\frac{1}{t}\right)\mathrm{d}t, \quad (33)$$

$$\lim_{M\to\infty}\frac{1}{M}\cdot\sum_{i=1}^{M}\exp\left(-\frac{\varphi\cdot M}{i}\right)=$$
$$=\frac{1}{\exp\varphi}-\varphi\cdot\int_0^{1/\varphi}\frac{1}{t}\cdot\exp\left(-\frac{1}{t}\right)\mathrm{d}t. \quad (34)$$

In these expressions, one identifies a link to the integral exponential function

$$E_1(x)=\int_x^{\infty}\frac{1}{t}\cdot\exp\left(-\frac{1}{t}\right)\mathrm{d}t$$

from a class of specialized functions [19], then

$$\int_0^{1/\varphi}\frac{1}{t}\cdot\exp\left(-\frac{1}{t}\right)\mathrm{d}t=E_1(0)-E_1\left(\frac{1}{\varphi}\right).$$

Introduce designations (Fig. 5, *a*):

$$C(\varphi)=\frac{1}{\int_0^{1/\varphi}\frac{1}{t}\cdot\exp\left(-\frac{1}{t}\right)dt}; \quad (35)$$

$$B(\varphi)=\frac{C(\varphi)}{\varphi\cdot\exp\varphi}-1. \tag{36}$$

Next, applying transforms (33), (34), with respect to

$$\frac{1}{M}=\frac{\Delta\varepsilon}{\varepsilon_M}=\frac{\Delta\varepsilon\cdot\varphi}{\varepsilon_*},$$

modal parameters (31), (32) are reduced to the form:

$$\varepsilon_*=\frac{E/N}{B(\varphi)}, \tag{37}$$

$$n_*=N\cdot\Delta\varepsilon\cdot\frac{C(\varphi)}{\varepsilon_*}\cdot\frac{1}{e}. \tag{38}$$

By deriving mode ε_*, from formula (37), it is possible, according to designation (15), to also obtain the right bound of finite distribution

$$\varepsilon_M=\varepsilon_*\cdot\frac{1}{\varphi}.$$

Their charts are shown in Fig. 5.

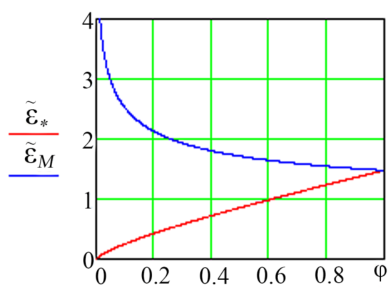


Fig. 5. Influence of form-parameter φ on the magnitude of: —modal $\tilde{\varepsilon}_*=\frac{\varepsilon_*}{E/N}$ and — maximum $\tilde{\varepsilon}_M=\frac{\varepsilon_M}{E/N}$ coordinates of the extreme hyperbolic finite distribution

10. 2. Parameters for the finite extreme hyperbolic distribution

I shall demonstrate that the finite distribution of carriers (13) and its corresponding finite resource spectrum (14) can be represented in the form that depends only on two parameters – form-parameter φ and the mean portion of resource E/N. To this end, substitute (37), (38) in original expressions (13) and (14), and obtain

$$\frac{n_i/N}{\Delta\varepsilon}=\frac{C(\varphi)}{\varepsilon_i}\cdot\exp\left(-\frac{\varepsilon_*}{\varepsilon_i}\right) \tag{39}$$

– an extreme hyperbolic finite distribution of carriers and

$$\frac{E_i/E}{\Delta\varepsilon}=\frac{C(\varphi)}{E/N}\cdot\exp\left(-\frac{\varepsilon_*}{\varepsilon_i}\right) \tag{40}$$

– its resource spectrum (allocation of the resource volume).

Here

$$\varepsilon_*=\frac{E/N}{B(\varphi)},$$

combinations $B(\varphi)$, $C(\varphi)$ are derived from formulae (21), (22).

By applying the limiting transition $\Delta\varepsilon\to 0$, to these expressions, one obtains the corresponding distribution of densities:

$$g_n(\varepsilon)=\lim_{\Delta\varepsilon\to 0}\frac{n_i/N}{\Delta\varepsilon} \text{ and } g_E(\varepsilon)=\lim_{\Delta\varepsilon\to 0}\frac{E_i/E}{\Delta\varepsilon},$$

whose form is almost the same as (39), (40), where, instead of variable ε_i, there is the variable ε. The limit of integration on the right is the maximum coordinate $\varepsilon_M=\varepsilon_*/\varphi$. Its value (Fig. 5) can be determined with respect to (15), deriving ε_* from formula (37).

The distributions of density $g_n(\varepsilon)$, $g_E(\varepsilon)$ and the limit of their integration $\varepsilon_M=\varepsilon_*/\varphi$ depend only on parameters E/N – the mean value of individual portion of resources (which is assigned), and form-parameter φ, characterizing the finite properties of distribution of the macrosystem.

The value for form-parameter φ depends on the stage of evolution of the quasi-equilibrium state of the macrosystem, which is described in detail below.

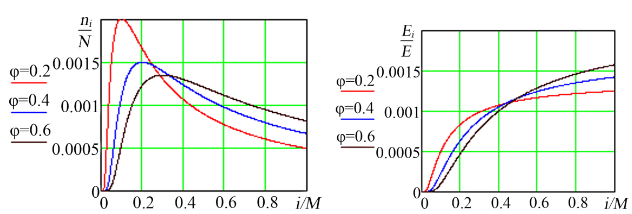
10. 3. Estimation of the equilibrium value of form-parameter φ

First, represent the extreme hyperbolic finite distribution (13) and its resource spectrum (14) in relative variables. For this purpose, transform these expressions with respect to representation (16), as well as formulae (31) and (32), and one obtains as a result:

$$\frac{n_i}{N}=\frac{\frac{\varphi}{i/M}\cdot\exp\left(-\frac{\varphi}{i/M}\right)}{\sum_{j=1}^{M}\frac{\varphi}{j/M}\cdot\exp\left(-\frac{\varphi}{j/M}\right)}, \tag{41}$$

$$\frac{E_i}{E}=\frac{\exp\left(-\frac{\varphi}{i/M}\right)}{\sum_{j=1}^{M}\exp\left(-\frac{\varphi}{j/M}\right)}. \tag{42}$$

An increase in the number of cells M leads to that these sums quickly converge. The curves that correspond to expressions (41) and (42) are shown in Fig. 6.



a *b*

Fig. 6. Influence of form-parameter on: *a* – extreme hyperbolic finite distribution (41); *b* – its spectrum (42)

These dependences are represented as functions of the relative number of cell i/M. A single parameter here is the form-parameter φ, which can also be interpreted as *the relative number of modal cell* i_*/M. Indeed, it follows from (15) that

$$\varphi=\frac{\varepsilon_*}{\varepsilon_M}=\frac{\Delta\varepsilon\cdot i_*}{\Delta\varepsilon\cdot M}.$$

Next, substitute expressions (41) and (42) in formulae (6) and (5), respectively. One obtains dependences for two entropies expressed via form-parameter ψ:

– the entropy of primary experience implying the resource allocation

$$S_P(\psi)=-\sum_{i=1}^{M}\left(\frac{\exp\left(-\frac{\varphi}{i/M}\right)}{\sum_{j=1}^{M}\exp\left(-\frac{\varphi}{j/M}\right)}\right)\cdot\ln\left(\frac{\exp\left(-\frac{\varphi}{i/M}\right)}{\sum_{j=1}^{M}\exp\left(-\frac{\varphi}{j/M}\right)}\right), \quad (43)$$

– conditional entropy of the subsequent experience implying the distribution of carriers, taking into consideration the event of the earlier allocated resources

$$S_{H|P}(\psi)=-\sum_{i=1}^{M}\left(\frac{\frac{\varphi}{i/M}\cdot\exp\left(-\frac{\varphi}{i/M}\right)}{\sum_{j=1}^{M}\frac{\varphi}{j/M}\cdot\exp\left(-\frac{\varphi}{j/M}\right)}\right)\ln\left(\frac{\frac{\varphi}{i/M}\cdot\exp\left(-\frac{\varphi}{i/M}\right)}{\sum_{j=1}^{M}\frac{\varphi}{j/M}\cdot\exp\left(-\frac{\varphi}{j/M}\right)}\right). \quad (44)$$

Fig. 7 shows their charts, normalized by the magnitude $\ln M$.

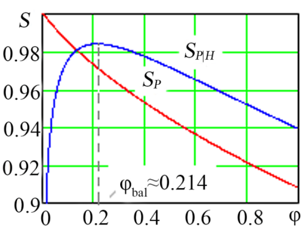


Fig. 7. Entropy of the distribution of "resources" $S_P(\varphi)$ and conditional entropy of the distribution of "carriers" $S_{H|P}(\varphi)$ normalized by the magnitude $\ln M$

Fig. 7 shows that the maximum entropy of the initial experience $S_P(\varphi)$ is achieved at $\varphi=1$. One can assume that this very value of form-parameter φ forms at the initial stage – the stage of the quasi-equilibrium state (when equilibrium is reached only for the set of resources as a more active agent). The system subsequently tends to enter the state of the ultimate equilibrium, taking into consideration the redistribution of less-active carriers, that is, along the growth trajectory of conditional entropy $S_{H|P}(\varphi)$. The chart demonstrates that extremum $S_{H/P}(\varphi)_{\max}$ is attained at value $\varphi_{bal}\approx 0.214$.

One can expect that the evolution trajectory of the finite extreme hyperbolic distribution, which corresponds to the macro system with more dynamic resources, passes in the direction of form-parameter's values from φ=0 to φ≈0.214.

Note that at φ=0 (which corresponds to the early quasi-equilibrium stage of the system's evolution), the extreme hyperbolic distribution (41) tends to the form of a purely hyperbolic distribution (Fig. 6, *a*).

11. Extreme hyperbolic distribution law and its relationship with real systems

The obvious question is: is there any reason to link the *extreme hyperbolic distribution law* to the empirically observed power distributions in real systems? The answer, most likely, should be in the affirmative, although at first glance, there are at least two arguments against it.

First, the modal character of theoretical curve of the extreme hyperbolic law allegedly contradicts the existing practical examples of purely hyperbolic distribution (some of them were mentioned above).

Second, in the obtained theoretical curve the right (descending) branch has a single exponent while real systems may exhibit distributions with a power degree different from unity.

Regarding the first objection. There are many examples of real distributions with a "heavy tail" that possess a mode. For example, distributions of the number of fatigue micro defects for their size [3], atmospheric aerosol particles for the magnitude of their diameters [20], shards of exploded ammunition for their mass [21], atmospheric turbulent pulsations for their intensity [4]. It is also known [22] that the complete empirical curve that describes the distribution of population in terms of income also exhibits a modal character rather than a monotonously decreasing hyperbola. The Pareto law actually approximates not the entire modal dependence, but only its right, downward, branch, while ignoring the information on the distribution of income of the poor.

The small value of the mode, that is, the ε_* coordinate in formula (13) results in an illusion of the monotonically decreasing hyperbolic dependence. As follows from (35), the mode is small at low value E/N – for the systems with a shortage of resources. The mode is also small when the values of form-parameter φ are low – at an early stage of the quasi-equilibrium state (Fig. 6).

If one constructs, under such conditions, a discrete diagram with a large enough sampling rate, the small mode can become unobservable, having been absorbed by the width of the first column of the diagram.

In this regard, one can assume that *many of the observed power distributions are actually the modal distributions with a "heavy tail", which were not considered in detail.*

It is important to note that the above does not apply to monotonously decreasing *rank distributions* where the argument is the rank – the number in the order of decreasing "weight". This paper considers the fundamentally different distributions where the argument is the *physical quantity* – an individual portion of resources. For such dependences, rank distributions are only a kind of shadow. Any modal distribution can always be reformatted into a monotonously decreasing rank correlation with a natural loss of information. It is also pertinent to note that the result of such conversion easily explains the known phenomenon – bend in the head and tail of a straight-line rank dependence, built in logarithmic coordinates.

As regards the second objection, one must say the following. Indeed, the exponent from a descending branch of the chart for the extreme hyperbolic law is always equal to unity, while in real systems it is often larger (rarely lower) than unity. There are two reasons, there are, to be exact, at least two sources for forming the non-unity distribution exponent.

The first reason (obvious) is related to the dimensionality of the argument of distribution. Indeed, in the extreme hyperbolic law, the argument is the size of the individual share of resources ε_i (energy, volume, ...). In practice, however, they often analyze distributions where the argument is the magnitude derived from resources (wavenumber, diameter, ...).

The second, less obvious reason is related to the fundamental impossibility of real systems to be isolated. Typically, each of them is included in the causal diagram with many other macro systems, mutually distorting *a priori* conditions for the formation of each other. For example, an economic system cannot exist outside of social or political system, while a demographic system could not be isolated from economic, or environmental. Such a statistical interaction between macro systems leads to the violation of condition for *a priori* equal probability of populating the "phase" cells, and hence to breaching the main postulate of statistical mechanics, which is an equal probability of microstates.

If the main postulate of statistical mechanics (the postulate of equal *a priori* probabilities) ceases to have effect, the entropy can no longer serve as a function, clearly describing the probability of a macro state of the system [23]. Such a task can be handled only by a more general function – entropy divergence, which includes entropy as a component. This paper shows that the equilibrium state of the system in the general case must be matched with the requirement for a conditional minimum of the entropy divergence, rather than the conventional requirement for a conditional maximum of entropy.

According to results from [23], the equilibrium distribution of the macrosystem, which is under conditions of statistical interaction, is a multiplicative combination of its natural isolated allocation and distribution of *a priori* probabilities of populating the cells $(p_1, p_2, ..., p_i, ..., p_M)$, generated under the influence of additional factors. Thus, instead of laws (10), (13), the following expressions are obtained, respectively:

$$\frac{n_i}{n_{**}} = \frac{p_i}{p_{**}} \cdot e^{1-\frac{\varepsilon_i}{\varepsilon_{**}}} \text{ and } \frac{n_i}{n_*} = \frac{p_i}{p_*} \cdot \frac{\varepsilon_*}{\varepsilon_i} e^{1-\frac{\varepsilon_*}{\varepsilon_i}},$$

where p_i is the *a priori* probability of populating the *i*-th cell, p_{**} and p_{**} are the *a priori* probabilities of populating the modal cells for exponential and hyperbolic laws, respectively. Paper [23] provides examples of how, in real systems, a given mechanism produces an exponent of power distributions different from unity.

All of the above suggests that many empirically observed power (with a heavy tail) distributions may in fact be undetected extreme hyperbolic distributions, which are formed in accordance with the entropy variational principle.

12. The combined symmetry of two kinds of distributions

Based on the extended entropy approach, I have received two pairs of distributions. *The first pair* is the exponential distribution of carriers (10) and its resource spectrum (9), *the second pair* is the extreme hyperbolic distribution of carriers (13) and the corresponding resource spectrum (14).

These dependences possess a combined (cross) symmetry, clearly observed at a logarithmic scale. Thus, the exponential distribution (10) is the mirror symmetry of curve (14) (Fig. 8, *a*), and the extreme hyperbolic distribution (13) is the mirrored symmetrical curve (9) (Fig. 8, *b*).

The ratio of symmetry is invariant relative to the combination of two transforms – mutual interchange between statuses of the system agents and their comparative kinetic activity (by analogy with the combined CP-symmetry in physics).

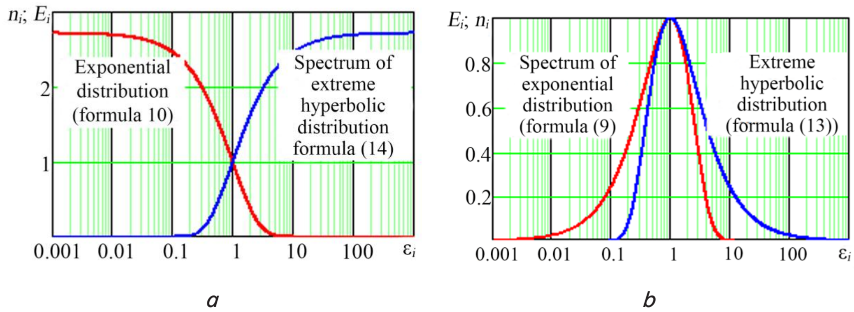


Fig. 8. The combined symmetry of the exponential and extreme hyperbolic types of statistics: *a* — distribution (10), spectrum (14); *b* — spectrum (9), distribution (13)

Currently, many authors perceive exponential and hyperbolic distributions as two independent forms of the existence of macro systems. But the above results suggest that these distributions can only regarded as two different statistical interpretations of the same equilibrium state. The choice of point of view depends on the circuit representation of the analyzed macrosystem – depending on the distribution of roles among its agents (carriers or resources) and their comparable activity.

12. Discussion of this study results

The benefit of the extended entropy method proposed here is that it has made it possible, owing to better accounting for the combinatorial configurations of the macrosystem's agents, to consider the stage-wise character in the formation of its equilibrium state, as well as to rigorously describe the finite properties of its distribution.

Similar to physical kinetics, distributions form against the background of the more active activity by one of the agents (which has a shorter relaxation time). The extended entropy method made it possible to find out that the real macro system, in the case of a larger kinetic activity of the carriers, forms the exponential type of distribution, and in the case of a greater activity of the resources – the so-called *extreme hyperbolic type* of distribution with a "heavy tail".

Using the proposed method, I have obtained expressions for these two distributions and corresponding resource spectra, and discovered their combined (cross) symmetry, which is invariant relative to the combination of two transforms – interchange of statuses between the system's agents and their relative activity. The existence of the combined symmetry allows me to consider these two different types of distributions just as two different statistical interpretations of the same equilibrium state.

Analysis of many empirically observed hyperbolic distributions suggests that they are actually the extreme hyperbolic distributions, formed in accordance with the entropy variational principle.

The advantage of the extended entropy method is also in that it makes it possible, if applied to actual macro systems, to methodically and consistently analyze the finite properties of their distributions. A given approach assumes that the maximum coordinate (as the distribution type itself) forms as a result of self-organization of the macrosystem; its value should also be searched for using an extreme criterion. It is shown that the finite distribution properties are conveniently related to the magnitude of a form-parameter, which is the ratio between its modal and maximum coordinates. I have determined the equilibrium values for the form-parameters of exponential and extreme hyperbolic distributions, which are, respectively, equal to $\psi_{bal} \approx 0.407...$ and $\varphi_{bal} \approx 0.214...$.

Specifically, this result could be applied to the estimation of the upper bound of distribution of the absolute velocity values of gas molecules at equilibrium motion. The Maxwell's law implies that the upper bound of this distribution is equal to infinity. This is justified from the computational point of view and is acceptable given the rapid attenuation of exponent. There are tasks, however, where one needs to know energy of the quickest molecule (for example, to determine the threshold temperature of the onset of chemical reaction). The equilibrium value for form-parameter $\psi_{bal} \approx 0.407...$, obtained in this work, could be interpreted as the ratio of energy possessed by a molecule, a representative of the majority, to the energy possessed by the fastest molecule in the finite version of Maxwell distribution. In this case, the ratio between the maximum and modal velocity values can be estimated as:

$$V_{max}/V_{mod} \approx 1/\sqrt{0.407} \approx 1.57.$$

This (at first glance) unexpected result agrees well within our intuitive understanding. Under equilibrium conditions, in a dense mass of randomly moving balls, one single ball among them cannot move at a very high speed, very different from the velocity of the majority.

In addition to a given result, one can also notice that the equilibrium magnitudes of form-parameters for both distributions $\psi_{bal} \approx 0.407...$ and $\varphi_{bal} \approx 0.214...$ exhibit a quantitative relationship with some known constants. Thus, their inverse values sufficiently enough match the Feigenbaum constants [26]: $1/\psi_{bal} \approx \alpha$ and $1/\varphi_{bal} \approx \delta$ (which are, respectively, $\alpha = 2.503...$, $\delta = 4.669...$), and their sum is

$$\psi_{bal} + \varphi_{bal} \approx 1/\Phi,$$

where $\Phi \approx 1.618...$ is the number of Phidias (golden ratio). The following approximation also holds:

$$(1-\varphi_{bal})^2 \approx 1/\Phi.$$

Among other things, there is a newly found fact here, not noticed by anybody before, which is beyond the scope of this paper. It turns out that the Feigenbaum constants are associated with the number of the golden ratio via a close correlation

$$\frac{1}{\alpha} + \frac{1}{\delta} \approx \frac{1}{\Phi}.$$

The study reported in this article is continuation of earlier work [14, 27]. However, they are limited to a particular case when a system has only one type of resources. Problems in which the same carriers possess several different kinds of resources require a more complex solving technique. That is due to that the distribution of carriers among the cells of one type of resource typically deform the *a priori* probability of populating them into cells for a different type of resource. Based on the results obtained in [25], the violation of the equality of *a priori* probabilities when populating the cells leads to a violation of the basic postulate of statistical physics, namely the condition for an equal probability of microstates. In this case, based on the results obtained in [25], a maximum of entropy can no longer act as a criterion for the equilibrium state of the system. One should use a more general criterion, specifically a minimum of entropic divergence.

Thus, this is the planned next step of the initiated research related to the analysis of macro systems with a poly-resource character of distributions.

13. Conclusions

1. The proposed extended entropy method considers a macro system to be an object at which a limited set of "resources" are allocated among the final set of "carriers". The method implies counting the combinatorial configurations not only on the set of "carriers", but also on the set of "resources", as two equal agents of the system. In this case, the equilibrium state is regarded as the implementation of complex experience with two possible variants of priority – first, "carriers" enter the equilibrium, followed by "resources", and vice versa. The criterion for a complete equilibrium state used here is a conditional maximum of entropy of the complex experience.

2. The result of solving an extreme problem for these two variants is the two types of distributions – exponential and extreme hyperbolic (with a heavy tail). Each of them is matched with its neighbor distribution (denoted as a spectrum), formed in the process of complex experience. I have found a universal and a very convenient technique to parametrically record all four ratios via their modal characteristics.

3. A technique for determining the finite characteristics of distribution is proposed. It is expected that the coordinate of its right (upper) bound forms not arbitrary, but rather a product of self-organization of the macrosystem; it can also be determined based on the extreme principle. A convenient parameter was established, taking into consideration the finite properties of distribution. The so-called form-parameter is the ratio between its modal and maximum coordinates. The equilibrium values for form-parameters of the exponential and extreme hyperbolic distributions were determined from the condition for a maximum of entropy of complex experience. They are, respectively, $\psi_{bal} \approx 0.407...$ and $\varphi_{bal} \approx 0.214...$ It was found that their inverse magnitudes are quantitatively close to the universal Feigenbaum constants.

4. Based on the results of the study conducted, a conclusion was drawn on that the comparative kinetic activity of carriers and resources is the key factor that determines the type of statistical distribution in a macro system. The higher kinetic activity of carriers generates exponential distribution, and the larger activity of resources predetermines the distribution with a heavy tail. In addition, I have justified the statement made by assumption that the empirically observed hyperbolic distributions in many macro systems are actually the extreme hyperbolic distributions, formed in accordance with the entropy variational principle.

5. It is shown that the distributions and spectra, related to the exponential and extreme hyperbolic type, possess combined symmetry, clearly observed at the logarithmic scale. Given this, it is concluded that the exponential distribution and the distribution with a heavy tail can be occa-

sionally regarded as two different statistical interpretations of the same equilibrium state. The choice of point of view depends on the circuit representation of the analyzed macrosystem, that is, on the distribution of roles of its agents (carriers or resources), as well as on their relative activity.

The relevance of the results obtained is predetermined by the existing need for effective methods to analyze macro systems, by the growing demand for quantitative and qualitative predictions of their behavior in different fields of activity.

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