DEVELOPMENT OF THE METHOD FOR REDUCING A MODEL TO THE NON-DIMENSIONALIZED FORM

Запропоновано метод, що дозволяє перетворити математичну модель до обезрозміренного вигляду з меншою кількістю обезрозміренних величин, ніж пропонується пі теоремою (в англомовній літературі - теорема Бакінгема, у франкомовній – теорема Ваші). Процедура перетворень формалізована і для реалізації не потребує високої кваліфікації дослідника. Метод побудований на основі синтезу принципів стандартного підходу до обезрозмірювання і натуральних одиниць вимірювань. Рішення завдання обезрозмірювання базується на застосуванні матричних методів.

Прагнення до перетворення моделей в напрямку зменшення кількості величин, що входять до них, без втрати інформації стимулюється поруч з перевагами, що виникають при цьому. Реалізується можливість отримання заданого обсягу інформації при меншій кількості фізичних і численних експериментів, а також нових рішень.

В результаті досліджень отримано можливість зменшення кількості величин, що входять до перетвореної математичної моделі. Це відбувається за рахунок введення власних одиниць вимірювань (нормуючих величин) для кожної моделі індивідуально, що відповідає введенню натуральної системи координат. Процедура уніфікована і може бути застосована до будь-якої моделі, що властиво стандартним методам обезрозмірювання.

Працездатність методу продемонстровано на прикладі приведення до обезрозміренного вигляду моделі гідроудару в трубах. Розглянуто варіанти без урахування і з урахуванням дисипативних сил. У кожному випадку обезрозмірюванння виконано за допомогою стандартного методу і з використанням розробленої процедури. В обох випадках застосування запропонованого методу дозволило зменшити кількість обезрозміренних величин в порівнянні з результатами застосування стандартного підходу

Ключові слова: моделювання технічних систем, обезрозмірювання моделі, зниження мірності простору моделювання

O. Brunetkin
PhD, Associate Professor
Department of
thermal power automation processes
Odessa National Polytechnic University
Shevchenka ave., 1, Odessa, Ukraine, 65044
E-mail: a.i.brunetkin@gmail.com

O. Maksymova
PhD
Department of the Computer Systems and
Business Process Management*
E-mail: m.oxana.b @gmail.com

F. Trishyn
PhD, Associate Professor, Vice-rector on
scientific and educational work*
E-mail: fatrishyn@gmail.com
*Odessa National Academy of
Food Technologies
Kanatna str., 112, Odessa, Ukraine, 65039

1. Introduction

A comprehensive study of any process is closely connected with modeling. A variety of fields of science and technology, which use modeling, as well as a desire for a model to meet best the features of a problem, generates a large number of specific models and types of modeling. It is often difficult to choose a path leading to creation of the most appropriate model in each specific case. As a result, along with the accurate approach, required by the study, here appears some element of creativity, the heuristic approach in the process of development of an adequate model.

One of the methods that makes it possible to adjust the magnitude of adequacy, is the ability to reduce the model to a nondimensionalized form. The similarity theory is closely connected with this method. In this area, there is a basic Pi-theorem (in the English-language literature, it is the Buckingham theorem, in the French-language literature, it is the Vaschy theorem), fixing the possible number of nondimensionalized values in the convertible models. Nevertheless, the attempts are made to develop the methods that allow obtaining a fewer number of nondimensionalized magnitudes that Pi-theorem prescribes.

In the course of further development of nondimensionalization methods, a certain progress has been made. But the methods, used by researchers, are the result of the intuition of their developers, and do not mark the boundaries in the development of nondimensionalization theory. In a scientific approach, it is necessary to talk about the method as a coherent logical system. This provides a basis for further work in this direction and demonstrates the relevance of our research.

2. Literature review and problem statement

A decrease in magnitudes, considered in a model due to nondimensionalization, facilitates analysis of available solutions and causes of possible errors. In addition, fewer variables in a model help to obtain analytical solutions to

new problems, as well as decrease a required number of experimental studies (physical and numerical) by an order of magnitude.

When developing new methods, the number of nondimensionalized variables, predicted based on the Pi-theorem, are accepted as the starting point. In relation to this magnitude, one determines a degree of reduction of the number of nondimensionalized magnitudes, which is achieved when using the methods, proposed by various researchers. Thus, in papers [1, 2], the possibility of this kind of procedures is demonstrated. It is noted that a maximum possible decrease in the number of nondimensionalized magnitudes was achieved. However, it was not described how it was determined that more profound transformations of mathematical models (MM) in this direction are impossible.

Nondimensionalization of models can be used for analysis of the obtained solutions and analysis of their reliability. In paper [3], based on the analysis of nondimensionalized properties and models, there is an attempt to determine the reasons for inconsistency of the obtained results at intensification of the studied phenomena. The author believes that decreasing magnitudes in a model facilitates analysis of a problem. In this case, the standard method of nondimensionalization is used. As a result of decreasing the dimensionality of modeling space, the author managed to draw some generalizing conclusions. But although in the present case there is a decrease in the number of magnitudes in a model, in addition to variables and sought functions, there additionally remain a number of nondimensionalized magnitudes – similarity criteria. This is one of the factors, limiting the depth of possible analysis.

In a number of works, nondimensionalization of models in combination with other methods is used to obtain new solutions. Thus, in [4], the Laplace transforms are applied to nondimensionalized equations to simplify the obtained algebraic equations. The Laplace transform involves the use of linear equations, while nondimensionalization can be applied to homogeneous functions, which is a broader class of equations. Thus, the proposed sequence of operations introduces restrictions on possible transformations.

In article [5], based on the application of the theory of groups and nondimensionalized differential equations, their new solutions are sought. Moreover, nondimensionalization operations can be used to identify the group of homogeneous strains. In article [5], like in many other works, the nondimensionalization procedure is considered from the standpoint of the possibility of reducing the number of parameters in a model, but it is also based on the Pi-theorem. It does not make it possible to fully use the possibilities of reducing the dimensionality of modeling space and get all the advantages of a combination of the proposed methods.

From this point of view, paper [6] addresses the question what is the aim – to nondimensionalize variables or to reduce the number of model parameters? An unbiased opinion on this issue suggests that the aim is to reduce the number of parameters, and nondimensionalization is only a tool that makes it possible to reach just the same result in a number of cases. Not only the existence of different nondimensionalization procedures is mentioned, but also a complexity of their selection. Figuratively, the way is defined as "a narrow path between the Trap of Oversimplification and the Swamp of Overcomplication. It was proposed to use the methods of the theory of groups as a toolkit. To be more exact, a reduction of a model to a minimally parametric form is considered as

the problem of group bundle. Such an approach requires high mathematical skills of a researcher. It specifies the path but does not formalize the transformation process. In this case, the problem about the possibility of further reduction of a mathematical model is not discussed.

Representation in a nondimensionalized form makes it possible to use the MM properties for modeling the processes that are difficult to realize under experimental conditions [7]. This also facilitates the generalization of results, obtained in numerical and physical experiments [8]. The prospects of ensuring not only a geometrical similarity, but also a possibility of modeling physical properties of used working environments are noted. But in this case, the used methods do not enable going beyond the limits, prescribed by the Pi-theorem.

It is possible to expect that the development of nondimensionalization methods will further improve modeling processes. In some cases, [9], nondimensionalization of models is called "a problem of reduction to a minimally parametric form". But in this case, the issue of achieving self-similarity by parameters is not considered.

Systematization of the results of cited research makes it possible to draw a conclusion about effectiveness of the method of analysis, solutions and generalization of results, obtained when using MM, reduced to a nondimensionalized form. The results of the works of several authors indicate the possibility of decreasing the number of nondimensionalized variables to smaller magnitudes, set by the Pi-theorem. A deterrent to a further decrease in the number of nondimensionalized magnitudes in models is the lack of a common method of similar transformations and, as a consequence, uncertainty of the lower boundary of the possible number of such magnitudes.

The need to develop such a method determines the prospects of the present research.

3. The aim and objectives of the study

The aim of the study is to develop the method that ensures minimization of a number of nondimensionalized variables for the studied model.

To accomplish the set aim, the following objectives were set:

– to develop an algorithm, formalizing the process of nondimensionalization of the MM magnitudes with the view to minimizing their number compared to the results, prescribed by the Pi-theorem;

– to develop procedures for determining a lower boundary of the possible number of nondimensionalized values in a mathematical model;

– to ensure reproducibility (coverage, inclusion in composition) of results of the MM nondimensionalization, performed with the help of other methods.

4. Nondimensionalization method for mathematical models

Scheme for providing the models with self-similarity for criteria.

Let us consider the Pi-theorem. "Any equation, connecting N physical and geometrical values, dimensionality of which is expressed through n basic units of measurement

can be transformed into an equation of similarity $\pi=N\text{-}n$" [10]. By virtue of this theorem, due to nondimensionalization, a decrease in the number of magnitudes, included in a model, can be performed only by value "n". Thus, within the SI system, mechanical magnitudes can be described by only three measurement units: mass $[M]$, length $[L]$ and time $[T]$. Therefore, in correspondent models, the number of magnitudes, included in them, can be reduced only by three units. But on the other hand, in this definition, it is possible to consider a way to a further decrease in the number of magnitudes, included in a model.

It is known that currently used SI system or formerly used SGS and other similar systems are not based on any physical sense, but on metrological convenience. Applying the magnitudes from this system, written down in a dimensional form, in a system, it is necessary to agree to use the unified scales of dimensional magnitudes. But physical laws, with consideration of which model are constructed, display the relationship between magnitudes that they include without regard to their scales. As a result, recording physical laws using dimensional magnitudes leads to formation of dimensional physical constants, which take into consideration the scales of the currently used measurement system. These constants in various combinations, as well as variables, constitute a set "N" of physical and geometrical magnitudes, used in the statement of the Pi-theorem. It does not seem possible to decrease the number of variables without changing a model itself. Therefore, it is necessary to decrease the number of constants.

The standard procedure of nondimensionalization, displayed in the Pi-theorem, is associated with the tendency to minimize the number of constants. Introducing the normalization of dimensional variables by any characteristic values of the same nature, the standard scales of dimensional magnitudes, such as from the SI system, are reduced taking into consideration internal scales of the processes, described by a nondimensionalized model. In this case, the resulting criteria are complexes, formed by different scales. They themselves are the scales of the studied processes. A positive side of this procedure is consideration of the scales of proceeding processes in each model separately. In fact, this is the reason to decrease the number of magnitudes in a model through a combination of internal scales of the analyzed process. The drawback is the normalizing magnitudes of the same nature as nondimensionalized ones. The used dimensional magnitudes are selected not for physical reasons. For this reason, it is not possible to ultimately decrease the number of scales (criteria) up to their complete exclusion.

Another possible way of simplification of expressions at the expense of decreasing the magnitudes, included in them, is exclusion from consideration of a number of physical constants. This effect is pronounced when using natural measurement units. In these systems, the basic measurement units are selected not due to metrological considerations but using physical constants themselves. The constants, selected as the basic units, are equaled to unity and, based on this, all other magnitudes are subsequently expressed. The systems of units, constructed by M. Planck, H. Lewis, D. Hartree, P. Dirac, and others, are constructed in this way. For example, we will consider the expression of the Coulomb's law in various measurement units. In the SI system, electric constant in the Coulomb's law has the form $\varepsilon_0 = 8.99 \cdot 10^9$ $[N \cdot m^2 \cdot C^{-2}]$ Taking this into consideration, it is written as follows:

$$F = \varepsilon_0 \frac{q_1 \cdot q_2}{R^2} = 8.99 \cdot 10^9 \cdot \frac{q_1 \cdot q_2}{R^2}. \tag{1}$$

In the system of CGSE, where $\varepsilon_0 = 1$ was accepted as one of the main units, this law takes a simpler form:

$$F = 1 \cdot \frac{q_1 \cdot q_2}{R^2}. \tag{2}$$

The positive side of the procedure of this kind is the equality of physical constants to unity. Because constants are building blocks of similarity criteria, a number of criteria become equal to 1. In other words, self-similarity is achieved by correspondent criteria. However, the described procedure has a drawback. A certain natural system of units is convenient for a particular model. In other models, the magnitudes, determined on its basis, usually have the values that are inconvenient to use: they are too large or too little. In addition, physical constants are determined with some margin of error. As a result, for example, constants of mass, time of processes, determined on their basis, will have the errors, impermissible for practical use.

The proposed nondimensionalization method combines the positive aspects of the Pi-theorem (taking into account the scales of proceeding processes in each model separately) and of the introduction of natural measurement units (equality of physical constants to unity).

At the first stage, we will represent;

$$p_q = \overline{p}_q \cdot p_q^{\triangle}, \quad \forall q \in J_k. \tag{3}$$

Here $p, \overline{p}_q$ are the dimensional and nondimensionalized magnitudes of MM, respectively, $p_q^{\triangle}$ is the normalizing magnitude (scale), q is the number of variables in MM, k is the number of dimensional values. During the normally applied nondimensionalization procedure, the magnitude of the same kind as a nondimensionalized one is selected as a scale. Thus, geometric characteristics of the research space are normalized by the magnitude, corresponding to any characteristic dimensions, temperature – by the characteristic temperature and so on. This is a significant and unjustified restriction. Normalization can be made by the magnitude of the same nature as a normalized magnitude, or of the same dimension, but not necessarily of the same kind. For example, for geometric characteristics of space, the normalizing magnitudes, having dimensionality of length [L], depending on magnitudes, included in the MM, can take the form:

$$x^{\triangle} = \nu \cdot \sqrt{\frac{\rho}{\Delta P}} \ \text{ or } \ x^{\triangle} = \sqrt[3]{\frac{\nu^2}{g}}, \tag{4}$$

for speed, having dimensionality $[LT^{-1}]$:

$$u^{\triangle} = \sqrt{\frac{\Delta P}{\rho}} \ \text{ or } \ u^{\triangle} = \sqrt[3]{\nu \cdot g}. \tag{5}$$

Here, ΔP is the pressure drop, ν is the kinematic viscosity, ρ is the density, g is the free fall acceleration. With such approach, magnitude $x^{\triangle}$ displays its use as the scale for geometric characteristics, rather than its characteristic dimensions. Similarly for $u^{\triangle}$ as the scale of speed and other normalizing magnitudes.

Next, the procedure of nondimensionalization runs like the standard one:

– by removing normalizing magnitudes and physical constants beyond the sign of the operator and formation of complexes with the same dimensionality;

– nondimensionalization of complexes by dividing by one of them.

As a result, nondimensionalized complexes, which externally meet the similarity criteria, but differ from them in nature, are formed. Similarity criteria are formed from physical constants and scales of variables, which are unchanged characteristic magnitudes for the explored process: a characteristic size, time, speed, pressure, temperature, etc. For that reason, the criteria also have a constant form. Normalizing magnitudes are not selected in the proposed nondimensionalization method at the stage of transformation. Expressions (4), (5) are given to demonstrate the ability of their wider representation. The possibility to vary ratios with a view to representing obtained nondimensionalized complexes of the necessary form remains.

At the second stage, the condition of equality of all obtained nondimensionalized complexes to unity is set. A similar result, but only for some of the complexes and particular models, at the expense of some simplification, is achieved when introducing the natural reference frame for them. In this case, like in the case of the standard nondimensionalization procedure, the basic measurement units are constant and only change the form: there is a transition from the characteristic magnitudes of the process to physical constants, corresponding to this process. As a result, it is not possible to change the form of nondimensionalized complexes, which are the criteria.

In the proposed method, the nondimensionalization procedure is constructed from the opposite. The desired form of nondimensionalized complexes is assigned (in the present case, equal to one), and this is achieved by varying the type of normalizing magnitudes.

Formalization of the procedure for ensuring self-similarity for criteria.

At the first stage, we will designate normalizing magnitudes as $p_q^\triangle$ (3), without determining their specific form. These magnitudes have dimensionality of scalable magnitudes. They include normalization: for a function in the considered model, for spatial-temporal coordinates of the model, for other variables of the model, for edge conditions of the model. In the process of the standard nondimensionalization procedure on their basis, as well as using physical constants c_p, nondimensionalized complexes, which have the form of products of power functions, are constructed:

$$\pi_h = \prod_{q=1}^{q=l} (p_q^\triangle)^{\alpha_q} \times \prod_{p=1}^{p=m} (c_p)^{\beta_p}, \qquad (6)$$

where l is the number of dimensional magnitudes, included in the model, m is the number of physical constants, included in the model. Similarity criteria have the same form.

At the second stage, all complexes (6) are equal to unity:

$$\pi_h = \prod_{q=1}^{q=l} (p_q^\triangle)^{\alpha_q} \times \prod_{p=1}^{p=m} (c_p)^{\beta_p} = 1 \quad \forall h \in J_n, \qquad (7)$$

where h is the number of the formed nondimensionalized complexes.

As a result, we have the system of h equations with l unknown. In this case, $h<l$. In addition to $(l–h)$ variables, m

physical constants take part in formation of h normalizing magnitudes $p_q^\triangle$.

To solve (7), we will take logarithms of these equations and obtain the system of linear homogeneous algebraic equations $A \cdot \vec{M} = 0$ or

$$\left(\sum_{q=1}^{q=l} \alpha_q \cdot \ln(p_q^\triangle) + \sum_{p=1}^{p=m} \beta_p \cdot \ln(c_p) = \ln(1) = 0 \right)_h, \qquad (8)$$

where

$$A = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1l} & \beta_{11} & \cdots & \beta_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{h1} & \cdots & \alpha_{hl} & \beta_{h1} & \cdots & \beta_{hm} \end{bmatrix},$$

A is the matrix of exponents α_q and β_p;

$$\vec{M} = \begin{bmatrix} \ln(p_1^\triangle) & \cdots & \ln(p_l^\triangle) & \ln(c_1) & \cdots & \ln(c_m) \end{bmatrix}$$

is the vector-column of all dimensional magnitudes of the model (scales and physical constants).

Elements in the lines of matrix A are located in the order, corresponding to the list:

$$p_{1_y}, \cdots, p_{n_y}, p_{1_s}, \cdots, p_{n_s}, p_{1_z}, \cdots, p_{n_z}, p_{1_\Omega}, \cdots p_{n_\Omega}, c_1, \cdots c_m. \qquad (9)$$

In the tuple of magnitudes (9):

– the first n_y positions are allocated for the sought functions;

– n_s positions correspond to spatial-temporary coordinates of the studied process;

– n_z magnitudes-parameters of the process;

– n_Ω boundary conditions;

– m physical constants.

In this case, $(n_y+n_s+n_z+n_\Omega)=l$.

Using the Gauss-Jordan Elimination Method, matrix A can be transformed into the form of:

$$A \rightarrow [E \vdots B], \qquad (10)$$

where E is the identity matrix of dimensions $(r \times r)$; $r - \text{rank}[A]$. Lines with linearly dependent elements from matrix A are eliminated; B is the matrix of exponents γ_{ij} of dimensions $[r \times (l+m-r)]$.

In general form, matrix $[E \vdots B]$ can be represented as follows:

$$\begin{matrix} \pi_1 \\ \cdots \\ \pi_i \\ \cdots \\ \pi_r \end{matrix} \begin{bmatrix} 1 & \cdots & & 0 & \| & \gamma_{11} & \cdots & \gamma_{1(l+m-r)} \\ & & & & \| & & & \\ \cdots & & 1 & \cdots & \| & \cdots & \gamma_{ij} & \cdots \\ & & & & \| & & & \\ 0 & \cdots & & 1 & \| & \gamma_{r1} & \cdots & \gamma_{r(l+m-r)} \end{bmatrix}. (11)$$

Using matrix $[E \vdots B]$, it is easy to write down the solution to the system and analyze the results.

Based on (11), normalizing magnitudes for the first r elements from tuple (9) (corresponding to identity part of matrix $[E \vdots B]$) can be expressed through the other elements of the tuple (9) in the correspondent powers γ_{ij}:

$$p_i^\triangle = \prod_{v=(l+m+1)-r}^{v=(l+m)} p_v^{-\gamma_{iv}}. \qquad (12)$$

With their help, normalized (nondimensionalized) magnitudes (3), which ensure meeting condition (7): $\pi_h = 1$. It follows from it that complete self-similarity by criteria (equality of all $\pi_h = 1$) can be achieved at $\mathrm{rank}[A] = n_y + n_s$.

5. Examples of reducing models to the nondimensionalized form in problems on the dynamics of engineering systems

To demonstrate the workability of the proposed nondimensionalization method, the MM of the hydraulic impact in a pipe is explored as an example in two versions: without and with taking into consideration the dissipative term. In each case, the process of nondimensionalization in the accepted way and with the use of the proposed method is considered.

5. 1. Nondimensionalization of MM, recorded without taking into account a dissipative term

In the classical statement of N. Y. Zhukovsky [11], this model takes the form:

$$\begin{cases} -\dfrac{\partial P}{\partial x} = \rho \cdot \dfrac{\partial \omega}{\partial t}; \\ -\dfrac{\partial P}{\partial t} = \rho \cdot c^2 \cdot \dfrac{\partial \omega}{\partial x}. \end{cases} \tag{13}$$

Edge conditions:
– initial

$$\begin{cases} t = 0 & \begin{aligned} \omega &= \omega_o; \\ P &= 0; \end{aligned} \end{cases}$$

– boundary

$$\begin{cases} x = 0 & P = 0; \\ x = l & \omega = 0, \end{cases} \tag{14}$$

where P is the pressure in the flow; ω is the flow rate; x, t are the coordinates by the length of the pipe and the time of the process, respectively; ρ is the density of fluid, flowing along the pipe, c is the sound velocity in fluid; ω_o is the initial flow rate; l is the length of the pipe.

Since the model is linear, instead of absolute pressure, we consider its deviation P from the initial value, accepted as equal to 0.

5. 1. 1. The generally accepted nondimensionalization model

When using normalizations $P^\triangle$, $\omega^\triangle$, $t^\triangle$, $x^\triangle$, nondimensionalized magnitudes of the correspondent variables are written down as:

$$\bar{P} = \frac{P}{P^\triangle}; \quad \bar{\omega} = \frac{\omega}{\omega^\triangle}; \quad \bar{t} = \frac{t}{t^\triangle}; \quad \bar{x} = \frac{x}{x^\triangle}. \tag{15}$$

Subsequently, when they are used in the model (13), (14), dimensional complexes are separated before operators. Operators are written down in the nondimensionalized form. According to the Fourier theorem, dimensional complexes within one equation have the same dimensionality. Then nondimensionalized complexes are formed within each equation by dividing

all dimensional complexes by one of these. All equations and, accordingly, a model get a nondimensionalized form:

$$\begin{cases} -\dfrac{\partial \bar{P}}{\partial \bar{x}} = \pi_1 \cdot \dfrac{\partial \bar{\omega}}{\partial \bar{t}}; \\ -\dfrac{\partial \bar{P}}{\partial \bar{t}} = \pi_2 \cdot \dfrac{\partial \bar{\omega}}{\partial \bar{x}}; \end{cases} \tag{16}$$

edge conditions:
– initial

$$\begin{cases} \bar{t} = 0 & \begin{aligned} \bar{\omega} &= 1; \\ \bar{P} &= 0; \end{aligned} \end{cases}$$

– boundary

$$\begin{cases} \bar{x} = 0 & \bar{P} = 0; \\ \bar{x} = 1 & \bar{\omega} = 0. \end{cases} \tag{17}$$

$$\pi_1 = \frac{\rho \cdot \omega^\triangle \cdot x^\triangle}{t^\triangle \cdot P^\triangle}; \quad \pi_2 = \frac{\rho \cdot c^2 \cdot \omega^\triangle \cdot t^\triangle}{x^\triangle \cdot P^\triangle}; \quad \pi_3 = \frac{\omega_o}{\omega^\triangle}; \quad \pi_4 = \frac{l}{x^\triangle}, \tag{18}$$

where π_1, π_2, π_3, π_4 are the nondimensionalized complexes.

Subsequently, based on the heuristic approach, the number of complexes decreases and the form of complexes π_i is simplified. The result depends on complexity of a model and experience of a researcher. Assuming than, under boundary conditions $\pi_3 = 1$, $\pi_4 = 1$, the values of normalizing values $\omega^\triangle = \omega_o$, $x^\triangle = l$ are determined. Normalizing magnitude for time can be determined from the ratio of characteristic magnitudes of the process: $t^\triangle = l/c$. For $P^\triangle$, in the reduced model, a characteristic magnitude is absent, but can be introduced artificially. Let us assume $P^\triangle = P_o$. We can accept pressure in the system before the beginning of development of hydraulic impact as P_o. Substituting the values of $\omega^\triangle$, $x^\triangle$, $t^\triangle$, $P^\triangle$ in the remaining complexes (18), we will obtain:

$$\pi_1 = \frac{\rho \cdot \omega^\triangle \cdot x^\triangle}{t^\triangle \cdot P^\triangle} = \frac{\rho \cdot \omega_o \cdot \cancel{l} \cdot c}{\cancel{l} \cdot P_o} = \frac{\rho \cdot \omega_o \cdot c}{P_o};$$

$$\pi_2 = \frac{\rho \cdot c^2 \cdot \omega^\triangle \cdot t^\triangle}{x^\triangle \cdot P^\triangle} = \frac{\rho \cdot c^{\cancel{2}} \cdot \omega_o \cdot \cancel{l}}{\cancel{l} \cdot P_o \cdot \cancel{c}} = \frac{\rho \cdot \omega_o \cdot c}{P_o}. \tag{19}$$

Comparison of complexes (19) shows their equality $\pi_1 = \pi_2 = \pi$. In nondimensionalized model (16), (17), there remained only one nondimensionalized complex – similarity criterion.

At this stage, the process of transformation of a model usually finishes. In the studied case, characteristic magnitude of the process t^Δ for normalizing the time variable, which is missing under boundary conditions, was selected successfully. It is not always possible to do it. In a similar situation, two similarity criteria π_1 and π_2 would remain in the studied model.

5. 1. 2. The proposed nondimensionalization method

According to (9), a tuple of dimensional magnitudes from model (13), (14) is constructed, in which elements 1–2 correspond to n_y, elements 3–4↔n_s, elements 5–6↔n_Ω, elements 7–8↔m of physical constants of the process. Based on (8), matrix A is formed from exponents at correspondent variables in expressions for nondimensionalized complexes (18):

$$
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & \vdots & 5 & 6 & 7 & 8 \\
P & \omega & x & t & \vdots & \omega_o & l & \rho & c
\end{array}
$$

$$
\begin{array}{c}
\pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4
\end{array}
\left[
\begin{array}{ccccccccc}
-1 & 1 & 1 & -1 & \vdots & 0 & 0 & 1 & 0 \\
-1 & 1 & -1 & 1 & \vdots & 0 & 0 & 1 & 2 \\
0 & -1 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & \vdots & 0 & 1 & 0 & 0
\end{array}
\right]. \tag{20}
$$

For matrix (20), $\mathrm{rank}[A]=4$. Therefore, at this stage, it is possible to speak about the possibility of achieving self-similarity by all criteria before making transformations.

After applying the Gauss-Jordan algorithm, the transformed matrix has the form of:

$$
\begin{array}{cccccccc}
P & \omega & x & t & \vdots & \omega_o & l & \rho & c
\end{array}
$$

$$
\begin{array}{c}
\pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4
\end{array}
\left[
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & \vdots & -1 & 0 & -1 & -1 \\
0 & 1 & 0 & 0 & \vdots & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \vdots & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & \vdots & 0 & -1 & 0 & 1
\end{array}
\right]. \tag{21}
$$
$$
\begin{array}{ccc}
\quad\quad\quad E & \vdots & B
\end{array}
$$

With the help of matrix (21), normalizing magnitudes are formed in the following way:

– any line from (21), for example π_1, is selected. In part [E] in this line, «1» is located in the column, corresponding to magnitude P. For it, normalization $P^\triangle$ is determined;

– with the help of magnitudes from this line, located in part [B] of matrix (21), the kind of normalization $P^\triangle$ is formed. It is constructed according to (12).

The specified values act as exponents with an opposite sign for the magnitudes of edge conditions and physical constants, designating correspondent columns in this part of the matrix.

According to this algorithm:

– from line $\pi_1 \to P^\triangle = \omega_o^1 \cdot \rho^1 \cdot c^1 = \omega_o \cdot \rho \cdot c$;
– from line $\pi_2 \to \omega^\triangle = \omega_o^1 = \omega_o$;
– from line $\pi_3 \to x^\triangle = l^1 = l$;
– from line $\pi_4 \to t^\triangle = \dfrac{l^1}{c^1} = \dfrac{l}{c}$. $\tag{22}$

Substitution of normalizing values (22) in (18) transforms all nondimensionalized complexes into equal magnitudes of $\pi=1$. In other words, self-similarity is achieved by all similarity criteria. Results of (22) were obtained without heuristic searches based on a formal procedure that can be performed by a researcher of any skill level.

5. 2. Nondimensionalization of MM, recorded with consideration of a dissipative term

A more complicated variant of the MM of the process of hydraulic impact can be represented by the model (13), (14) with addition of linearized dissipative term. In this case, the model is presented, for example, in [12]:

$$
\begin{cases}
-\dfrac{\partial P}{\partial x} = \rho \cdot \left(\dfrac{\partial \omega}{\partial t} + 2a \cdot \omega \right); \\
-\dfrac{\partial P}{\partial t} = \rho \cdot c^2 \cdot \dfrac{\partial \omega}{\partial x};
\end{cases} \tag{23}
$$

edge conditions:
 – initial

$$
\begin{cases}
t = 0 & \quad \omega = \omega_o; \\
& \quad P = 0;
\end{cases}
$$

– boundary

$$
\begin{cases}
x = 0 & P = 0; \\
x = l & \omega = 0,
\end{cases} \tag{24}
$$

were a is the resistance coefficient

5. 2. 1. Common nondimensionalization method

As a result of using transformation and normalization $P^\triangle$, $\omega^\triangle$, $t^\triangle$, $x^\triangle$, we will obtain the model in the nondimensionalized form:

$$
\begin{cases}
-\dfrac{\partial \overline{P}}{\partial \overline{x}} = \pi_1 \cdot \dfrac{\partial \overline{\omega}}{\partial \overline{t}} + \pi_3 \cdot \overline{\omega}; \\
-\dfrac{\partial \overline{P}}{\partial \overline{t}} = \pi_2 \cdot \dfrac{\partial \overline{\omega}}{\partial \overline{x}};
\end{cases} \tag{25}
$$

edge conditions:
 – initial

$$
\begin{cases}
\overline{t} = 0 & \quad \overline{\omega} = 1; \\
& \quad \overline{P} = 0;
\end{cases}
$$

– boundary

$$
\begin{cases}
\overline{x} = 0 & \overline{P} = 0; \\
\overline{x} = 1 & \overline{\omega} = 0,
\end{cases} \tag{26}
$$

where

$$
\pi_1 = \pi_2 = \frac{\rho \cdot c \cdot \omega_o}{P_o}; \quad \pi_3 = \frac{\rho \cdot 2a \cdot l}{P_o}; \quad \pi_4 = \pi_5 = 1. \tag{27}
$$

In comparison with model (13), (14), appearance of a new term in the equation led to appearance of one more nondimensionalized complex – similarity criterion π_3.

5. 2. 2. The proposed nondimensionalization method

Nondimensionalized complexes for model (23), (24) are written down in general form like (18):

$$
\pi_1 = \frac{\rho \cdot \omega^\triangle \cdot x^\triangle}{t^\triangle \cdot P^\triangle}; \quad \pi_2 = \frac{\rho \cdot c^2 \cdot \omega^\triangle \cdot t^\triangle}{x^\triangle \cdot P^\triangle};
$$

$$
\pi_3 = \frac{\rho \cdot a \cdot \omega^\triangle \cdot x^\triangle}{P_o}; \quad \pi_4 = \frac{\omega_o}{\omega^\triangle}; \quad \pi_5 = \frac{l}{x^\triangle}. \tag{28}
$$

A tuple of dimensional values is built form model (23), (24) and matrix A, similar to (20), is formed:

$$
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & \vdots & 6 & 7 & 8 & 9 \\
P & \omega & x & t & a & \vdots & \omega_o & l & \rho & c
\end{array}
$$

$$
\begin{array}{c}
\pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5
\end{array}
\left[
\begin{array}{cccccccccc}
-1 & 1 & 1 & -1 & 0 & \vdots & 0 & 0 & 1 & 0 \\
-1 & 1 & -1 & 1 & 0 & \vdots & 0 & 0 & 1 & 2 \\
-1 & 1 & 1 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & \vdots & 0 & 1 & 0 & 0
\end{array}
\right]. \tag{29}
$$

From (29), at all linearly independent lines, it follows that $\mathrm{rank}[A]=5$. To solve this system, 5 variables are necessary. At 4 available (P, ω, x, t), for modeling it is necessary to separate another magnitude, which would be used in this

capacity. In the explored case, we separate, for example, a – resistance coefficient. In addition to (15), it can be determined from the ratio $\bar{a} = a/a^{\triangle}$, where $\bar{a}$ is the nondimensionalized magnitude of resistance coefficient. As a result of it, normalizing magnitude $a^{\triangle}$ will appear in complex π_3 in (28) instead of the dimensional magnitude "a". After applying the algorithm of Gauss-Jordan, the transformed matrix has the form of:

$$
\begin{array}{c}
\\
\pi_1 \\
\pi_2 \\
\pi_3 \\
\pi_4 \\
\pi_5
\end{array}
\begin{array}{cccccccccc}
P & \omega & x & t & a & \vdots & \omega_o & l & \rho & c \\
\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \end{array}\right. & & & & & \vdots & \begin{array}{cccc} -1 & 0 & -1 & -1 \end{array} & & & \\
0 & 1 & 0 & 0 & 0 & \vdots & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \vdots & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \vdots & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & \vdots & 0 & 1 & 0 & -1
\end{array}
\qquad (30)
$$

$$
\underbrace{}_{E} \qquad \underbrace{}_{B}
$$

Similarly to (22), normalizing magnitudes were obtained from (30):
 – from line $\pi_1 \rightarrow P^{\triangle} = \omega_o^1 \cdot \rho^1 \cdot c^1 = \omega_o \cdot \rho \cdot c$;
 – from line $\pi_2 \rightarrow \omega^{\triangle} = \omega_o^1 = \omega_o j$;
 – from line $\pi_3 \rightarrow x^{\triangle} = l^1 = l$;
 – from line $\pi_4 \rightarrow t^{\triangle} = \dfrac{l^1}{c^1} = \dfrac{l}{c}$;
 – from line $\pi_5 \rightarrow a^{\triangle} = \dfrac{c}{l}$. $\qquad (31)$

With the use of (31), the original MM (23), (24) in the nondimensionalized form will be written down as:

$$
\begin{cases}
-\dfrac{\partial \bar{P}}{\partial \bar{x}} = \dfrac{\partial \bar{\omega}}{\partial \bar{t}} + 2\bar{a} \cdot \bar{\omega}; \\[2mm]
-\dfrac{\partial \bar{P}}{\partial \bar{t}} = \dfrac{\partial \bar{\omega}}{\partial \bar{x}};
\end{cases}
\qquad (32)
$$

edge conditions:
 – initial

$$
\begin{cases}
\bar{t} = 0 & \quad \bar{\omega} = 1; \\
& \quad \bar{P} = 0;
\end{cases}
$$

 – boundary

$$
\begin{cases}
\bar{x} = 0 & \quad \bar{P} = 0; \\
\bar{x} = 1 & \quad \bar{\omega} = 0.
\end{cases}
\qquad (33)
$$

As in the previous case, all nondimensionalized complexes – similarity criteria $\pi = 1$. Self-similarity is achieved by all criteria. But another magnitude $\bar{a}$. appeared in the transformed model (32) in addition to nondimensionalized variables $\bar{P}$, $\bar{\omega}$, $\bar{x}$, $\bar{t}$. On the one hand, it was introduced to (29) as a variable. On the other hand, in the process of solving a specific problem, it remains a constant magnitude like the similarity criterion. It is its special feature. Ultimately, what is important is that when using the proposed method, in contrast to the generally accepted method of nondimensionalization, it was possible to decrease the number of magnitudes that determine the transformed model. Thus, in the conventional method, the model includes 6 magnitudes: $\bar{P}$, $\bar{\omega}$, $\bar{x}$, $\bar{t}$, as well as π_1, π_3 from (27). In case of application of the proposed method, 5 such magnitudes remain: $\bar{P}$, $\bar{\omega}$, $\bar{x}$, $\bar{t}$, as well as $\bar{a}$.

6. Discussion of results of the solution, based on the developed method

Application of different methods of nondimensionalization to identical MM potentially should provide a uniform result. That is why effectiveness of such procedure should be assessed by the number of nondimensionalized magnitudes, included in models after the procedure of transformations.

Let us consider model (13), (14). It is composed of $N=8$ dimensional variables (20) at $n=3$ main measurement units $[M]$, $[L]$ and $[T]$. Based on Pi-theorem, a nondimensionalized model must include $\pi = N-n = 8-3 = 5$ nondimensionalized magnitudes. This result was obtained due to the use of the common procedure: $\bar{P}, \bar{\omega}, \bar{t}, \bar{x}$ and $\pi_1 = \pi_2 = \pi$ from (19). The application of the proposed nondimensionalization method made it possible, based on the formalized procedure, to get the form of the normalizing magnitudes (22), leading to a further decrease in the number of nondimensionalized variables. As a result, $\pi_1 = \pi_2 = 1$, was obtained for magnitudes from (19), which corresponds to achievement of self-similarity according to the criteria of similarity. This result is maximally possible in such procedures. This was due to consideration of the MM structure in the nondimensionalization process.

The MM of hydraulic impact makes it possible to show the possibility to achieve self-similarity by using other methods. This happens due to its simplicity. It follows from (19), that the product $\rho \cdot \omega_o \cdot c$ has dimensionality of pressure. Using this expression as normalizing magnitude $P^{\triangle}$ leads to $\pi_1 = \pi_2 = 1$, like in the previous case. This result is trivial and is possible in this case because there is only one criterion (19). If there are more criteria and complex relationships between them in a model, a limit decrease in the number of nondimensionalized magnitudes is theoretically possible, but very difficult in practice. The proposed method of nondimensionalization does not have this drawback due to formalization of the nondimensionalization procedure.

In MM (23), (24), while taking into account the dissipative forces, compared to (13), (14), an additional term and another dimensional magnitude (resistance coefficient) appear, which make altogether $N=9$. In this case, the number of basic measurement units did not change and remained $n=3$. As a result, based on the Pi- theorem, a nondimensionalized model must include $\pi = N-n = 9-3 = 6$ of nondimensionalized values. This result was obtained after application of the generally accepted procedure $\bar{P}$, $\bar{\omega}$, $\bar{t}$, $\bar{x}$, as well as $\pi_1 = \pi_2$ and π_3 from (27).

The application of the proposed method of nondimensionalization made it possible, as in the previous case, based on a formalized procedure, to obtain the form of normalizing magnitudes (31), resulting in self-similarity by all similarity criteria.

The nondimensionalized MM of hydraulic impact in pipes, selected for discussion of the ways of implementation of the proposed nondimensionalization method due to its simplicity, does not enable demonstration of all capacities of the proposed approach. Thus, paper [13] shows without a description of the method the results of applying the proposed method to more complex models based on the Navier-Stokes equation or even more general equations of conservation (energy, momentum, and substance). The possibility of the "distorted" modeling was also shown in paper [13].

7. Conclusions

1. We developed the algorithm, formalizing the process of the MM nondimensionalization with the view to minimizing their number as compared with the results, prescribed by the Pi- theorem. In the explored examples, the number of nondimensionalized magnitudes, determining the explored model, is by unity less than the number, prescribed the by Pi-theorem.

2. Based on the developed procedure, the lower boundary of the possible number of nondimensionalized magnitudes of the mathematical model was determined. At reaching self-similarity by all criteria, their number is determined by the sum of the sought functions of the model and its spatial-temporal coordinates. This state has been achieved for the given example of the model of hydraulic impact in pipes without taking into consideration dissipative forces. Number of nondimensionalized magnitudes is minimally possible, equal to 4 and is determined by the sum of the two sought functions (P, ω) and two spatial-temporal coordinates (x, t).

3. The possibility to obtain all results, prescribed by the Pi-theorem, using the nondimensionalized magnitudes, obtained with the help of the described transformation procedure, was shown. The nondimensionalization method (3) in all the explored cases is the same. The difference is only in the form of records of normalizing magnitudes (12). For this reason, both original results and the results, prescribed by the Pi-theorem, can be obtained in the process of transformations, depending on the form of recording the normalizing magnitudes.

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