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RAY TRACING SYNTHESIS OF IMAGES OF TRIANGULATED SURFACES SMOOTHED BY THE SPHERICAL INTERPOLATION METHOD

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Вирішується завдання візуалізації зворотним трасуванням (Ray Tracing) тріангульованих поверхонь, згладжених методом сферичної інтерполяції. Метод сферичної інтерполяції в основному був розроблений для інтерполяції тріангульованої поверхні з подальшою метою візуалізації цієї поверхні методом зворотного трасування. Такий підхід дозволяє поєднати метод зворотного трасування з накопиченою базою моделей з триангульованою поверхнею. Метод сферичної інтерполяції є універсальним і дозволяє також будувати плоскі і просторові гладкі криві, проведені через довільно задані точки. Пропонований алгоритм інтерполяції заснований на простій алгебраїчній поверхні – сфері і не використовує алгебраїчні поліноми третього і більш високих ступенів. Наведені аналітичні співвідношення для реалізації кожного етапу побудови інтерполюючої поверхні цим методом. Для візуалізації інтерполюючої поверхні розроблений ітераційний алгоритм (ІТА) обчислення точки перетину проекційного променя з цією поверхнею. Пропонований ІТА має можливість широкого розпаралелювання обчислень. Розроблено алгоритм побудови точок інтерполюючої поверхні, крок якого збігається з кроком ітераційного процесу обчислень, що дозволяє виконувати алгоритм візуалізації та побудови точки поверхні за один прохід ІТА. Результати досліджень підтверджені моделюванням процесу візуалізації в пакеті Wolfram Mathematica. Таким чином, виконано рішення задачі суміщення нових методів побудови гладких геометричних форм тріангульованих поверхонь і методу зворотного трасування, що в цілому дозволить підвищити реалістичність синтезованих сцен в комп'ютерній графіці

Ключові слова: зворотне трасування, проекційний промінь, моделювання кривих і поверхонь, квадрік, сферична інтерполяція

1. Introduction

In modern computer graphics, methods of rasterization and ray tracing are applied to synthesize images of 3D objects [1, 2]. The main line of studies in computer graphics consists in improving realism of synthesized scenes. When synthesized by rasterizing, any surface must necessarily be triangulated (be approximated by triangles) [1]. Such approach leads to distortion of the surface shape. To reduce negative effect of perception of a triangulated surface, various methods of illumination interpolation within a triangle are applied in the process of its synthesis. Gouraud shading and Phong shading are the most common methods [1, 2]. To date, studies are underway to improve these methods [3]. However, all these methods do not eliminate distortion of the surface geometry arising in the process of its triangulation. One of the possible ways to solve this problem consists in application of the ray tracing method for solving the image synthesis problems [2].

The ray tracing method has been actively developed in recent years. It enables synthesis of images of analytically described surfaces without their prior triangulation which significantly improves realism of the synthesized object images. At the same time, the database of object models and software products accumulated in computer graphics is focused on triangulated surfaces. Such representation is necessary in synthesis of surface images in classical computer graphics with application of the rasterization method.

Combination of the ray tracing method with the accumulated database of object models and software must be implemented while preserving features of high realism of the method.

This problem can be solved if interpolation of triangle illumination is eliminated and replaced by interpolation of a triangle with a certain curved surface. Such an approach makes it possible to improve realism of the synthesized polygonal object images.

Thus, search for new methods for modeling smooth geometric shapes of polygonal surfaces for the purpose of further imaging with application of the ray tracing method is still an urgent task at this stage of the computer graphics development.

2. Literature review and problem statement

Modeling of curved surfaces that interpolate points in space which are set by the triangulated surface topology is one of the critical goals of computer graphics. Let us consider some basic methods for modeling smooth geometric shapes of polygonal surfaces.

One of convenient mathematical methods for describing curved shapes of objects consists in their representation using pieces of parametric surfaces or splines. Finding the point of intersection of a projection ray (PR) with such surfaces is not an easy task. An algorithm is proposed in [4] for finding the point of PR intersection with non-uniform, rational B-splines (NURBS). This algorithm is based on the well-known Bezier cutting method. The author suggests conversion of the NURBS surface into rational Bezier sections at the preliminary preparatory stage of the algorithm. It has additional processing demands. A comprehensive approach is set out in [5]: the parametrically free forms of surfaces and the most suitable method for their imaging using the ray tracing method are studied. In general, when analyzing studies [4, 5], it is seen that the use of NURBS and other parametric surfaces significantly increases time of scene rendering by the ray tracing method and it is difficult to apply standard image synthesis algorithms using the ray tracing method. This feature of parametric description has made the researchers turn to algebraic surfaces. Construction of spherical splines for interpolation is considered in [6]. Because of complexity, it is proposed to use the method in solving the problems arising in geophysics and for realistic description of the Earth parameters. Use of local mixing of radially basis functions for constructing surfaces is proposed in [7]. When using this approach, difficulties arise in gluing at boundaries when interpolating triangulated surfaces. An interpolation surface for a triangle is constructed in [8] by mixing two algebraic surfaces of second orders. To increase variety of forms of the interpolation surface, it is proposed to mix two algebraic surfaces of the second and third orders. Disadvantage of this approach consists in complexity of calculation of such surfaces. The lack of an algorithm for finding the point of PR intersection with the surface makes it difficult to apply this method when constructing imaging systems (IS) for various purposes. The author of [9] proposes to use a complete second-order polynomial and perturbation functions but interpolation requires an initial triangulation grid of a high detail. Besides, formation of such a surface is a multi-stage process and if the surface geometry varies, the time of image synthesis increases. According to the author of this method, it can be used in systems of interactive space-oriented geometric modeling. Study [10] is closest to the topic under consideration. It uses the system of quadratic parametric polynomials and normals at the triangle vertices to construct an interpolating surface. This approach advantage consists in locality and the quadratic order of polynomial. As a drawback, it should be noted that to construct a smooth surface for such relatively simple figures as spheres, cylinders or cones, an initial triangulation grid of high detail is required. Studies [11–13] propose to use the method of spherical interpolation based on the use of the simplest quadric, a sphere. Solutions for constructing an interpolation surface by the method of spherical interpolation are presented in [11, 12]. With this approach, it is possible to solve the following problems. When using the existing base of models with a triangulated surface, it is possible to restore a smooth surface. In this case, there is no need for interpolation of illumination by Gouraud or Phong. When calculating glares, there is no need to calculate bidirectional reflectance distribution function (BRDF) and illumination perspective is also taken into account in calculation. When displaying, an undistorted shape of shadows is obtained. Texturing and many more operations are simplified. An example of constructing a curve by spherical interpolation and its imaging by ray tracing is considered in [13].

Thus, there is a problem in computer graphics consisting in combining highly realistic synthesis of images by the method of ray tracing with the accumulated base of models of objects with triangulated surfaces. The first part of the problem solution includes conduction of further studies of the interpolation methods based on the simplest algebraic surfaces, quadrics (for example, a sphere). It is necessary that this method be uniform (universal) both for constructing curves and surfaces. The second part of the problem solution involves development of main stages of imaging by the method of ray tracing of surfaces constructed using algebraic surfaces. These surfaces make it possible to most easily perform the PR tracing operation. The unresolved issues mentioned above restrain widespread use of the highly realistic method of ray tracing in computer graphics.

3. The aim and objectives of the study

The study objective is elaboration of stages of image synthesis by ray tracing of triangulated surfaces smoothed by spherical interpolation. This will make it possible to improve realism of image synthesis by the method of ray tracing with the use of the base of models of objects with triangulated surfaces accumulated in computer graphics.

To achieve this objective, the following tasks were solved:

– for the imaging purpose, construct an iterative algorithm (ITA) of calculating the point of PR intersection with an interpolating surface;

– develop stages of constructing an interpolating surface that would coincide with the step of the iterative calculation process to enable execution of the imaging algorithm and construction of points of the interpolating surface in a single ITA pass.

4. The main stages of the surface image synthesis

4. 1. The features of finding the point of the projection ray intersection with the interpolating surface

Methods and algorithms for constructing spatial curves and smoothing the triangulated surfaces by the spherical interpolation method are presented in [11–13].

An approach to constructing a surface by spherical interpolation is presented in [12]. In accordance with the method of this surface construction, only one possible point on the interpolating surface corresponds to each arbitrary point on the triangle surface. In what follows, vector quantities will be highlighted in bold. Note also that the points set in the Cartesian coordinate system (*XYZ* c.s.) can be defined as the radius vectors of these points.

Thus, any arbitrary point for such a surface can be written as a vector equation:

$$\mathbf{r} = \mathbf{r}_p + \mathbf{r}_s, \quad (1)$$

where $\mathbf{r}$ and $\mathbf{r}_p$ are the radius vectors respectively to an arbitrary point of the interpolating surface and its associated point $\mathbf{p}$ on the triangle surface. The second term $\mathbf{r}_s$ will be defined further.

Data for a triangulated surface are initial data. Coordinates of the triangle vertices and vectors in these vertices will be referred to these data. After completion of synthesis of the interpolating surface, all points of the triangle vertices must belong to the surface, and the vectors at the vertices must coincide with the normals to the surface at these points.

In accordance with the method of ray tracing, it is necessary to find the point of intersection of the projection ray with an interpolating surface constructed by the method of spherical interpolation [12]. Peculiarity of finding the point of intersection of PR with a surface constructed by the method of spherical interpolation is that it is impossible to apply a purely analytical method. One of the authors of study [14] proposes an iterative method for finding the point of intersection of PR with algebraic surfaces presented in an implicit form.

4. 2. Determining the interpolation region

In the process of modeling curved surfaces that interpolate the points in space set by vertices of an arbitrary triangulated surface, specifying of the interpolation region is a nontrivial task. Solution of this problem determines a number of parameters of the modeled surface: locality for each triangle of the triangulated surface, a one-to-one correspondence of surface points to points in a triangle, fulfillment of conditions of connectivity and smoothness, etc. Let us assume that the triangle for which surface is synthesized was previously selected in general by application of the scene rendering methods not considered in this article.

Fig. 1 presents geometric elements of the problem. In the space of the coordinate system (*c, s.*) *XYZ*, two adjacent triangles are shown which are usually set by the initial data:

– vertices of the triangles: $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$;

– normals (single) at the vertices: $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$,$\mathbf{n}_4$.

Like points, normals can be set or obtained by calculation taking into account locations of triangles. As an option, direction of the normals at a point can be calculated using the Gouraud method.

The center of gravity, point *C*, is shown in the triangle $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. Point *C* divides the triangle area into three equal areas with the medians meeting in this point. Determine the radius vector of the center of gravity of the triangle $\mathbf{p}_c=(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)/3$. Denote the normal to the plane of the triangle at the center of gravity, $\mathbf{n}_c$. Write the equation in the form of a scalar product for the triangle plane:

$$f(\mathbf{p}) = (\mathbf{p} - \mathbf{p}_c)\mathbf{n}_c. \quad (2)$$

The interpolation region was locally defined in [11, 12] for three set points (triangle vertices) and singled out in space using a sphere of greatest curvature and three planes.

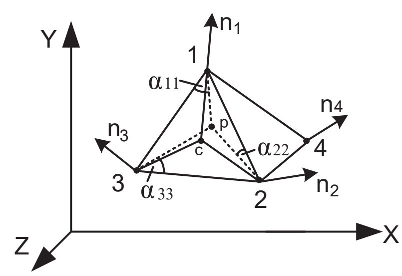

Fig. 1. Geometric elements of the problem and a system of angles for constructing the interpolating spheres

The sphere of greatest curvature bounds the space in such a way that any surface constructed by spherical interpolation over a chosen triangle will be inside this sphere. The sphere describing the triangle is chosen as such sphere. Determination of parameters of this sphere is a trivial task, so equation for this sphere can be immediately written in a vector form:

$$|\mathbf{p} - \mathbf{c}_0| - R = 0, \quad (3)$$

where $\mathbf{p}$ and $\mathbf{c}_0$ are the radius vectors to an arbitrary point on the surface and to the center of the sphere of greatest curvature, respectively, *R* is a scalar equal to the radius of this sphere.

To construct three planes, write down the vectors:

$$\mathbf{vc}_1 = \mathbf{n}_{12} \times (\mathbf{p}_2 - \mathbf{p}_1);\ \ \mathbf{vc}_2 = \mathbf{n}_{23} \times (\mathbf{p}_3 - \mathbf{p}_2);$$
$$\mathbf{vc}_3 = \mathbf{n}_{31} \times (\mathbf{p}_1 - \mathbf{p}_3), \quad (4)$$

where

$$\mathbf{n}_{12} = \mathbf{n}_1 + \mathbf{n}_2;\ \ \mathbf{n}_{23} = \mathbf{n}_2 + \mathbf{n}_3;\ \ \mathbf{n}_{31} = \mathbf{n}_3 + \mathbf{n}_1.$$

Finally, write down the relations for the three planes in the form of scalar products.

$$f_{12}(\mathbf{p}) = (\mathbf{p} - \mathbf{p}_1)\mathbf{vc}_1;\ f_{23}(\mathbf{p}) = (\mathbf{p} - \mathbf{p}_2)\mathbf{vc}_2;$$
$$f_{31}(\mathbf{p}) = (\mathbf{p} - \mathbf{p}_3)\mathbf{vc}_3. \quad (5)$$

It follows from relations (4), (5) that the planes are drawn perpendicular to the vectors (4), and the vectors ($\mathbf{n}_{12}$, $\mathbf{n}_{23}$, $\mathbf{n}_{31}$) and the triangle sides lie in these planes. Thus, the interpolation region for each triangle is limited in space by a sphere of greatest curvature (3) and planes (5). It is important to note here that the planes (5) also belong to adjacent triangles («neighbors» along their sides) which is necessary for matching the interpolating surfaces of adjacent and current triangles according to C_0 (surface continuity).

4. 3. The vector field of guides

Authors of [11] have proposed a method for constructing a vector field of guides (**vg**-vector guide) in the plane of a triangle arbitrarily set in space by points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$.

Write down equations for the planes that pass through the vertices of the triangle $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and the center of gravity perpendicular to the triangle plane.

$$f_{c1}(\mathbf{p})=(\mathbf{p}-\mathbf{p}_1)\mathbf{nc}_1;\quad f_{c2}(\mathbf{p})=(\mathbf{p}-\mathbf{p}_2)\mathbf{nc}_2;$$

$$f_{c3}(\mathbf{p})=(\mathbf{p}-\mathbf{p}_3)\mathbf{nc}_3, \tag{6}$$

where

$$\mathbf{nc}_1=\mathbf{n}_c\times(\mathbf{p}_1-\mathbf{p}_c);\quad \mathbf{nc}_2=\mathbf{n}_2\times(\mathbf{p}_2-\mathbf{p}_c);$$

$$\mathbf{nc}_3=\mathbf{n}_c\times(\mathbf{p}_3-\mathbf{p}_c).$$

The planes (6) divide the triangle into three equal areas over which vectors of the guide field can be built for any space point. Form a set of «reference signs» for each of the three areas which makes it possible to identify position of an arbitrary space point bounded by planes (5) and (6):

$$f_{sign1}(\mathbf{pc1})=\left(sign\left(f_{12}(\mathbf{pc1})\right),sign\left(f_{c1}(\mathbf{pc1})\right),sign\left(f_{c2}(\mathbf{pc1})\right)\right),$$

$$f_{sign2}(\mathbf{pc2})=\left(sign\left(f_{23}(\mathbf{pc2})\right),sign\left(f_{c2}(\mathbf{pc2})\right),sign\left(f_{c3}(\mathbf{pc2})\right)\right),$$

$$f_{sign3}(\mathbf{pc3})=\left(sign\left(f_{31}(\mathbf{pc3})\right),sign\left(f_{c3}(\mathbf{pc3})\right),sign\left(f_{c1}(\mathbf{pc3})\right)\right), \tag{7}$$

where

$$\mathbf{pc}1=(\mathbf{p}_{12}+\mathbf{p}_c)/2,\quad \mathbf{pc}2=(\mathbf{p}_{23}+\mathbf{p}_c)/2,$$

$$\mathbf{pc}3=(\mathbf{p}_{31}+\mathbf{p}_c)/2;$$

$$\mathbf{pc}_{12}=(\mathbf{p}_1+\mathbf{p}_2)/2,\quad \mathbf{pc}_{23}=(\mathbf{p}_2+\mathbf{p}_3)/2,$$

$$\mathbf{pc}_{31}=(\mathbf{p}_3+\mathbf{p}_1)/2.$$

In accordance with the method of [11], vectors of the guide field at any point on the medians drawn from the vertices to the C point are collinear with the normal drawn to the triangle plane at the point set by the radius vector $\mathbf{p}_c$.

Vectors of the guide field on the triangle boundaries are perpendicular to the sides and lie in planes (5).

The property of a simple ratio of three points inherent in affine transformations is the theoretical basis for constructing a vector field of guides for any point in each of the three regions of the triangle singled out by the planes in (7). Each vector of the field of guides [11] is written in the following form: **vg** (x'_g-x_g, y'_g-y_g, z'_g-z_g), where **p** (x_g, y_g, z_g) is an arbitrary point on the triangle surface (Fig. 1) which specifies beginning of the guide vector. The corresponding point **p'** (x'_g, y'_g, z'_g) lying on a plane parallel to the plane of the triangle sets the end of the guide vector. The method enables construction of a vector field of guides both locally for one triangle and adjacent triangles since there are common guides located in a common side and belonging to the fields of both triangles.

In accordance with the method, the entire set of space points lying on any vector of the field of guides have only one single point in the projection onto the triangle. The points of the interpolating surface actually lie on these vectors [12]. Many concepts and definitions used in this article are considered in detail in [11–13].

4. 4. Definition of a section of the projection ray in the interpolation field

At this stage of synthesis, the PR section which fell into the interpolation region is determined. This approach makes it possible to control accuracy and time of finding the point of PR intersection with an interpolating surface using iterative methods.

Let us write down the equation of a straight line in a parametric form (hereinafter, the projection ray, PR) which coincides with the observation vector [13, 14]:

$$\mathbf{p}(t)=\mathbf{h}+\mathbf{v}\cdot t, \tag{8}$$

where t is a parameter, **v** is the observation vector, **h** is the radius vector of the projection center.

Let us find a joint solution of equations (3), (8) with respect to the parameter t. Let it be t_1 and t_2. If the roots have a complex value, then go to the next PR or triangle. In the case of real values, substitute these roots in (8) to obtain values of the radius vectors for the points of PR intersection with the sphere of greatest curvature, $\mathbf{p}(t_1)$ and $\mathbf{p}(t_2)$.

Let us find a joint solution of equations for each of the planes (5) and the straight line (8) relative to the parameter t. Let it be t_3, t_4 and t_5. Exclude complex roots from the obtained roots and leave one from two or three equal roots. Leave only real roots for further analysis. In the case of three real roots, substitute them into (8) and obtain values of the radius vectors for the points of PR intersection with the surfaces (5): $\mathbf{p}(t_3)$, $\mathbf{p}(t_4)$ and $\mathbf{p}(t_5)$.

Let us form a set of «reference signs» which will make it possible to identify (indicate) position of an arbitrary space point with respect to the interpolation region bounded by the planes (5).

$$sign(\mathbf{p}_c)=\left(sign\left(f_{12}(\mathbf{p}_c)\right),sign\left(f_{23}(\mathbf{p}_c)\right),sign\left(f_{31}(\mathbf{p}_c)\right)\right). \tag{9}$$

If the condition of equality is met for any point **p** of space:

$$sign(\mathbf{p})=sign(\mathbf{p}_c), \tag{10}$$

then this point is in the interpolation region bounded by planes (5).

Analysis of the relative position of the segments of these points $\mathbf{p}(t_1)$–$\mathbf{p}(t_5)$ on the PR as well as positions of these points and segments in space of the sphere (3) and the planes (5), will be made further. The search logic is to find the PR segment which is simultaneously located in the region of the sphere (3) and the interpolation region bounded by the planes (5). The stages and a number of options for finding such a segment will be considered further.

Stage 1. Positions of two points of PR intersection, $p(t_1)$ and $p(t_2)$, with the sphere of greatest curvature (3) with respect to planes (5) is analyzed. The following options are possible.

Option 1. Condition (10) is satisfied for both points. For simplicity of further description, denote it (True, True). The segment search is completed and limited by points $p(t_1)$, $p(t_2)$.

Option 2. Condition (10) is not satisfied for both points. Denote it (False, False). Both points are excluded from further analysis.

Option 3. Condition (10) is met for the first point but not for the second one or vice versa. Denote it (True, False or False, True). The point for which condition (10) is fulfilled will participate in the further analysis.

Stage 2. Position of the three points $p(t_3)$, $p(t_4)$ and $p(t_5)$ with respect to the sphere (3) is analyzed. To do this, form function of the sphere:

$$f_s(\mathbf{p}) = |\mathbf{p} - \mathbf{c}_0| - R. \quad (11)$$

To identify position of an arbitrary space point with respect to the sphere, write down the condition:

$$signf_s(\mathbf{p}) = signf_s(\mathbf{c}_0). \quad (12)$$

There are some examples of possible options.

Option 4. Condition (12) is satisfied for two points from three. Denote it (True, True, False) or (False, True, True). The segment search is completed. In the first case, the found PR segment is bounded by points $p(t_3)$, $p(t_4)$ and by points $p(t_4)$, $p(t_5)$ in the second case.

Option 5. Condition (12) is satisfied for one point of the three. Denote it, for example, as (False, True, False). The point $p(t_4)$ obtained in this variant is supplemented, for example, with the point $p(t_1)$ obtained in step 1 (for example, option 3). In this case, a certain PR segment is bounded by points $p(t_1)$, $p(t_4)$.

With proper analysis, the found PR segment should remain alone. The ends of this segment are the sought entry and exit points in the interpolation region.

There is no need to analyze all possible options since this does not violate logic of further presentation of the material. But it is important to note that the number of these options, as simulation confirms, is final and small.

4. 5. Relationships for the construction of reference spheres

The main feature of the interpolation method is the use of such a quadric as a sphere for construction of a surface of arbitrary shape between the three points. To construct the surface, perform construction of reference spheres. Reference spheres are common for adjacent triangles and make it easy to achieve smoothness of C0 and C1 between splines of the constructed surface for each of the triangles. To do this, draw two reference spheres with centers o_{12} and o_{21} through the ends of the segments defined by sides of the triangle, for example, through the points $\mathbf{p}_1$, $\mathbf{p}_2$.

In this case, the center o_{12} of one sphere must be on a straight line passing through the point $\mathbf{p}_1$ and coinciding with the normal vector $\mathbf{n}_1$ at this point. The center o_{21} of the second sphere must be on a straight line passing through the point $\mathbf{p}_2$ and coinciding with the normal vector $\mathbf{n}_2$.

Ratios for equations of these straight lines in the vector form are as follows:

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{n}_1 \cdot t; \quad \mathbf{p} = \mathbf{p}_2 + \mathbf{n}_2 \cdot t. \quad (13)$$

Determine the radius vector for the middle of the side $\mathbf{p}_1$, $\mathbf{p}_2$:

$$\mathbf{p}_{12} = (\mathbf{p}_1 + \mathbf{p}_2)/2. \quad (14)$$

Write down the equation for a plane passing through the point (14) perpendicular to the segment $\mathbf{p}_1$, $\mathbf{p}_2$, in the form of a scalar product:

$$(\mathbf{p} - \mathbf{p}_{12})(\mathbf{p}_1 - \mathbf{p}_2) = 0. \quad (15)$$

Find joint solution of equations (13) for each straight line and the plane (15) with respect to t and substitute the obtained values into (13). Thus, the radius vector for the centers of the reference spheres $\mathbf{o}_{12}$, $\mathbf{o}_{21}$ is obtained.

Write down scalar values of the radii for the reference spheres:

$$r_{12} = |\mathbf{p}_1 - \mathbf{o}_{12}|; \quad r_{21} = |\mathbf{p}_2 - \mathbf{o}_{21}|.$$

Write down in the final form equations for the reference spheres in the vector form:

$$|\mathbf{p} - \mathbf{o}_{12}| - r_{12} = 0; \quad |\mathbf{p} - \mathbf{o}_{21}| - r_{21} = 0. \quad (16)$$

By analogy, remaining four reference spheres for the remaining two sides of the triangle can be constructed.

The reference spheres do not change during synthesis of the interpolating surface.

4. 6. Basic relations for the construction of interpolating spheres

Coordinates of vertices $\mathbf{p}_1$–$\mathbf{p}_3$, normals $\mathbf{n}_1$–$\mathbf{n}_3$ and curvature of the reference spheres (16) adjacent to the corresponding vertices of the triangle (Fig. 1) are the initial data for construction of interpolating spheres. Interpolating spheres are constructed for each triangle vertex in such a way that centers of these spheres lie on the straight lines passing through the triangle vertices of the triangle and coincide with the normals at these vertices.

In contrast to the reference spheres, interpolating spheres have a variable curvature which changes when position of the point $\mathbf{p}$ on the triangle surface changes. Curvature of the interpolating spheres is the source for formation of curvature of the interpolating surface for any arbitrary point on the triangle surface. To construct the interpolating spheres, angles α_1, α_2, α_3 at the triangle vertices (Fig. 1) are first determined. For example, the angle at vertex 1 is determined by relation:

$$\alpha_1 = \arccos\left(\left((\mathbf{p}_3 - \mathbf{p}_1)(\mathbf{p}_2 - \mathbf{p}_1)\right) / \left(|\mathbf{p}_3 - \mathbf{p}_1|\right)\left(|\mathbf{p}_2 - \mathbf{p}_1|\right)\right). \quad (17)$$

Construct a system of angles α_{11}, α_{22}, α_{33} for an arbitrarily chosen point $\mathbf{p}$ on the triangle surface (Fig. 1). By analogy with (17), the following relation is obtained, for example, for the angle α_{11},

$$\alpha_{11} = \arccos\left(\left((\mathbf{p}_3 - \mathbf{p}_1)(\mathbf{p} - \mathbf{p}_1)\right) / \left(|\mathbf{p}_3 - \mathbf{p}_1|\right)\left(|\mathbf{p} - \mathbf{p}_1|\right)\right). \quad (18)$$

Construct an interpolating sphere for each triangle vertex with curvature, for example, cs_1 at vertex 1, changing as follows:

$$cs_1 = cs_{13}(1 - u_1) + cs_{12} \cdot u_1, \quad (19)$$

where cs_{12}, cs_{13} are curvatures of the corresponding reference spheres, the parameter $u_1 = \alpha_{11}/\alpha_1$. Coordinates of the center $\mathbf{o}_1$ of the interpolating sphere at vertex 1 will have the form:

$$\mathbf{o}_1\left(x = \mathbf{p}_1(x) + \frac{\mathbf{n}_1(x)}{cs_1}, y = \mathbf{p}_1(y) + \frac{\mathbf{n}_1(y)}{cs_1}, z = \mathbf{p}_1(z) + \frac{\mathbf{n}_1(z)}{cs_1}\right). \quad (20)$$

Radius for this sphere is:

$$r_1 = |\mathbf{p}_1 - \mathbf{o}_1|. \quad (21)$$

By analogy, one can obtain relations of curvatures cs_2, cs_3, coordinates of the centers $\mathbf{o}_2$, $\mathbf{o}_3$, and the radii r_2, r_3 for the remaining two interpolating spheres. Finally, write down the relations for the three interpolating spheres:

$$|\mathbf{p}-\mathbf{o}_1|-r_1=0;\ \ |\mathbf{p}-\mathbf{o}_2|-r_2=0;\ \ |\mathbf{p}-\mathbf{o}_3|-r_3=0. \qquad (22)$$

Thus, a system of three interpolating spheres of different curvatures can be constructed for any arbitrary point on the triangle surface. If the point falls on the triangle side, then parameters of the interpolating sphere coincide with parameters of the reference sphere adjacent to the corresponding vertex.

4. 7. Relationships for constructing an interpolating surface

All stages of constructing an interpolating surface by spherical interpolation are described in detail in [12].

The ray tracing synthesis of an image of the interpolating surface constructed by spherical interpolation differs in that the radius vector $\mathbf{r}_p$ is specified in (12) and this vector must be determined during synthesis.

Let us assume that a PR segment was determined in subsection 4.4 in the interpolation region.

Let it be points $p(t_1)$, $p(t_2)$. At the same time, assume that the point $p(t_1)$ on the PR is located closer to the center of the projection h than the point $p(t_2)$. Write down the equation of a straight line for the part of the PR segment with these points being its ends. Then the equation for this line coinciding with the PR has the form:

$$\mathbf{pr}(t)=\mathbf{p}(t_1)+\left(\mathbf{p}(t_2)-\mathbf{p}(t_1)\right)\cdot t. \qquad (23)$$

To find the point $\mathbf{r}_p$, set an arbitrary point on the line and denote it:

$$\mathbf{pr}_j, \qquad (24)$$

where j is the index specifying number of the point on the PR.

Perform a test for belonging of a point (24) to any of the three i-regions bounded by the planes in (7). To evaluate the test, calculate the predicate.

$$\text{pred}i=\begin{vmatrix}1,\ f_{signi}(\mathbf{pr}_j)=f_{signi}(\mathbf{ps}i),\\ 0,\ \text{otherwise},\end{vmatrix} \qquad (25)$$

where i is the index of the region number, $i=(1, 2, 3)$.

If pred$i$=1, then the tested point belongs to the i-th region. If pred$i$=0, then the test point does not belong to the i-th region.

Further, it is necessary to find the vector of guides for the point (24). To this end through the point $\mathbf{pr}_j$, draw planes perpendicular to the triangle sides for example, $\mathbf{p}_1$, $\mathbf{p}_2$, in the region of which condition (25) is supposedly satisfied. Write down equation for this plane:

$$f_j(\mathbf{p})=(\mathbf{p}-\mathbf{pr}_j)(\mathbf{p}_2-\mathbf{p}_1). \qquad (26)$$

Draw a plane parallel to the triangle plane through the same point (24):

$$fn_j(\mathbf{p})=(\mathbf{p}-\mathbf{pr}_j)\mathbf{n}_c. \qquad (27)$$

In accordance with [11, 12], use planes (2), (5), (6), (26), (27) to find the only point, $\mathbf{r}_{pj}$ on the triangle plane corresponding to the point (24). To this end, use the property of the simple relation of three points inherent in affine transformations. Write down the vector of the guide for the point (24):

$$\mathbf{vg}_j=\mathbf{pr}_j-\mathbf{r}_{pj}. \qquad (28)$$

Draw a straight line parallel to the vector of the field of guides, **vg**, at this point through the obtained point $\mathbf{r}_{pj}$ on the triangle surface.

$$\mathbf{r}_j=\mathbf{r}_{pj}+\mathbf{vg}_j\cdot t. \qquad (29)$$

Substitute (29) into the equations of interpolating spheres (22) and find the values of t at which the line intersects the spheres.

There are two roots for each sphere. To select one of the roots correctly, use [12]. Denote these solutions for the point j: t_1, t_2, t_3. Substitute these solutions in (29) and obtain the coordinates (radius vectors) of the corresponding points on the interpolating spheres (22):

$$\mathbf{ps}_j(t_1),\mathbf{ps}_j(t_2),\mathbf{ps}_j(t_3). \qquad (30)$$

Determine a set of vectors:

$$\mathbf{s}_1=\mathbf{ps}_j(t_1)-\mathbf{r}_{pj},\ \ \mathbf{s}_2=\mathbf{ps}_j(t_2)-\mathbf{r}_{pj},\ \ \mathbf{s}_3=\mathbf{ps}_j(t_3)-\mathbf{r}_{pj}. \qquad (31)$$

To determine the point of the constructed surface, find relations of the barycentric coordinates b_1, b_2, b_3 for an arbitrary point **p** on the triangle surface:

$$\begin{aligned} b_1&=\left(|\mathbf{p}-\mathbf{p}_3|\sin\alpha_{33}\right)\left(|\mathbf{p}_2-\mathbf{p}_3|\right)/2s,\\ b_2&=\left(|\mathbf{p}-\mathbf{p}_1|\sin\alpha_{11}\right)\left(|\mathbf{p}_3-\mathbf{p}_1|\right)/2s,\\ b_3&=\left(|\mathbf{p}-\mathbf{p}_2|\sin\alpha_{22}\right)\left(|\mathbf{p}_1-\mathbf{p}_2|\right)/2s, \end{aligned} \qquad (32)$$

where s is the triangle area.

In the general case, expression for the resulting vector $\mathbf{r}_s$ in (1) whose beginning is at the point $\mathbf{r}_p$ and the end is on the interpolating surface, write down:

$$\mathbf{r}_s=\mathbf{s}_1b_1+\mathbf{s}_2b_2+\mathbf{s}_3b_3, \qquad (33)$$

where $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$ are the collinear vectors (31) coinciding with the vector of field of guides (28) and b_1, b_2, b_3 are barycentric coordinates (scalars). Subscripts denote their connection with the corresponding triangle vertices. The barycentric coordinates take into account the degree of influence of curvature of the corresponding interpolating sphere on formation of a constructed surface point [12].

4. 8. Construction of an iterative process and synthesis of the interpolating surface image

In the general case, it is necessary to find at least one point of intersection of the straight line (23) with the synthesized interpolation surface (1).

This point is calculated using an iterative algorithm which determines n_1 digit places of the sought number in each step of iteration. In this case, the number of steps will be equal to:

$$K=n/n_1, \qquad (34)$$

where n is the number of digits of the sought number. Assume that n and n_1 are multiples.

In addition, the iterative process (IP) should be constructed in such a way as to combine ITA with simultaneous construction of the sought surface point.

Hereinafter, to build the IP, turn to [14] in which the proposed parameter t is proposed to be presented in the form:

$$t_k = t_{k\text{-}1} + \eta_k \cdot \delta_k, \tag{35}$$

where k is the step number of the iterations, $k \in \{1, 2,..., K\}$; δ_k is a quantum of the k-th iteration step, $\delta_k = 2^{-n_1 \cdot k}$; η_k is a positive coefficient determining the number of quanta used at the k-th iteration step.

The goal of each k-th step is to define a segment on a straight line (23) (hereinafter, a selected segment) within which there may be a result of joint solution of equations (1), (23). It is important to note here that at each next ITA step, the selected segment is reduced by a factor of 2^{n_1}. The main stages of calculations performed in the k-th iteration step of ITA are considered below.

Stage 1. Substitute the parameter value in equation (23) in this form:

$$\mathbf{pr}\left(t_{k\text{-}1} + j \cdot \delta_k\right), \tag{36}$$

where t_{k-1} is the parameter calculated in the general case at the previous ITA step. Since we start consideration with k=1, then t_{k-1}=t_0=0. The parameter t in (23) sets length of the segment $p(t_2)$–$p(t_1)$ in relative units. At this step, this segment and all selected segments in subsequent steps will be divided into 2^{n_1} equal segments. The index j is introduced for numbering of points bounding equal segments.

Denote radius vectors of the points of the ends of these equal segments at the k-th step taking into account (23), (24):

$$\mathbf{pr}_j^k,$$

where

$$j \in \{0,1,2,\ldots J\}, J = 2^{n_1}. \tag{37}$$

In this case, when k=1, the value of quantum $\delta_k = 2^{-n_1}$. In view of (23), assume that the index j will be counted in the direction from the point $p(t_1)$ to the point $p(t_2)$.

Stage 2. Let us test for the belonging of points (37) to any of the three i-regions bound by planes in (7). To evaluate the test, calculate predicate (25) for each of the points (37) at each k-th step.

Stage 3. In the following, determine vectors of the field of guides for each point (37). To do this, use relations (26)–(28) and write down vector of the field of guides for all j points of the k-th step.

$$\mathbf{vg}_0^k, \mathbf{vg}_1^k, \ldots \mathbf{vg}_j^k, \ldots \mathbf{vg}_J^k. \tag{38}$$

Stage 4. Determine on the surface of each interpolating sphere (22) the points corresponding to the point $\mathbf{r}_{pj}$ on the triangle surface. Write down equation of a straight line passing through the point $\mathbf{r}_{pj}$ and parallel to vector (28) in the k-th step:

$$\mathbf{p}^k(t) = \mathbf{r}_{pj}^k + \mathbf{vg}_j^k \cdot t. \tag{39}$$

Next, find joint solutions of equations for the three interpolating spheres (22) and the straight line (39) with respect to t for each point j, for example, at k=1. There are two roots for each sphere. To select one of the roots correctly, use [12]. By analogy with (29), (30), substitute these solutions in (39) and obtain (taking into account k=1) coordinates (radius vectors) of the corresponding points on the surface of the interpolating spheres (22):

$$\mathbf{ps}_j^1(t_1), \ \mathbf{ps}_j^1(t_2), \ \mathbf{ps}_j^1(t_3). \tag{40}$$

Stage 5. Determine the points for the interpolating surface corresponding to the points $\mathbf{r}_{\mathbf{pj}}$. To do this, take into account (40) and find vectors (31) and the barycentric coordinates (32) corresponding to the points $\mathbf{r}_{pj}$. Finally, use (33) to define vectors $\mathbf{r}_{pj}$, the beginning of which is the point $\mathbf{r}_{pj}$ and the end belongs to the interpolating surface. Write down the set of these vectors for all points j:

$$\mathbf{r}_{s0}, \mathbf{r}_{s1}, \mathbf{r}_{s2}, \ldots, \mathbf{r}_{sj}, \ldots, \mathbf{r}_{sJ}. \tag{41}$$

In accordance with (1), sum up vectors (41) and the radius vector of the point $\mathbf{r}_{pj}$ to find the radius vector for all points j of the interpolating surface in the k-th step:

$$\mathbf{r}_j^k = \left(\mathbf{r}_{pj} + \mathbf{r}_{sj}\right)^k. \tag{42}$$

Stage 6. Substitute the radius vectors (37) and (42) in (2) and get two sets of scalar values with the j-index specifying number of the element of the sets:

$$f\left(\mathbf{pr}_j^k\right); f\left(\mathbf{r}_j^k\right). \tag{43}$$

Determine difference of these values:

$$F_j^k = f\left(\mathbf{pr}_j^k\right) - f\left(\mathbf{r}_j^k\right). \tag{44}$$

By analogy with [14], find elements of the set F_j^k as parameter-indicators (PI). Joint analysis of the values and signs of PI allows one to determine the strategy of movement in each iteration step. The value of η_k in (35) is determined in each ITA step by the results of F_j^k analysis. In doing this, two cases are possible.

Stage 7. Determining the selected segment.

Case 1. All elements of a set have the same sign. Then analysis of the F_j^k elements consists of the following steps.

1) The element of the set with the minimum value of modulus, i. e. $|F_j^k|_{\min}$ is determined. Let it be some j-th element.

2) The moduli of the first differences are determined.

$$\Delta F_j^k = |F_j^k - F_{j+1}^k|, \ \ \Delta F_{j-1}^k = |F_{j-1}^k - F_j^k|. \tag{45}$$

3) Indices of the smallest modulus of the difference defined in **p**. 2 are taken as indices of the selected segment. Let it be indices j and j+1. Name them accordingly: initial and final indices.

4) The coefficient η_k at this step is equal to:

$$\eta_k = (j)_k, \tag{46}$$

where $(j)_k$ is the initial index of the selected segment in the k-th step.

Case 2. The elements of the set F_j^k have different signs. This means that the PR intersects the interpolating surface in some part of it. As the analysis shows, the function of the interpolating surface spline within the interpolation region for one triangle can have no more than two adjacent extrema: maximum and minimum. Thus, the number of intersection points does not exceed three. For example, the relations between the elements of set (44) for two points of intersection can be:

$$F_0^k > 0;\;\; F_1^k > 0;\ldots;\;\; F_j^k > 0;\;\; F_{j+1}^k \le 0;$$

$$F_{j+2}^k < 0;\ldots;\;\; F_{j+5}^k \le 0;\;\; F_{j+6}^k > 0;\ldots;\;\; F_J^k > 0. \quad (47)$$

For relations (47), there are two selected segments with indices *j*, *j*+1 and *j*+5, *j*+6 and, accordingly, coefficients $\eta_k = (j)_k$ and $\eta_k = (j+5)_k$. In this case, further, two IPs can be performed for each of the selected segments if the surface is transparent and two intersection points must be found or one IP for the selected segment with a smaller index if the surface is opaque.

Stage 8. Calculate the parameter t_k in the iteration equation (35) taking into account that $\eta_k = \eta_1$, determined at the previous stage. Initial conditions at $k=1$ for IP: $t_{k-1} = t_0 = 0$. The obtained value of the parameter in this iteration step when substituted in (23), enables calculation of the radius vector of the initial point of the selected segment to be calculated in the *XYZ* c.s.

Calculations of each IP step in stages 1–8 proceed up to $k=K$. If relations (47) were not fulfilled in any of the iteration steps, then the point of PR intersection with the interpolating surface is absent. When relations (47) are fulfilled, at least in one step, the intersection point exists. Moreover, coordinates of this point are equal to the coordinates of any of the borders of the segment selected in the last step:

$$\mathbf{r}_j^K = \left(\mathbf{r}_{pj} + \mathbf{r}_{sj}\right)^K. \quad (48)$$

In accordance with the ray tracing method to calculate illumination, it is necessary to further determine the normal at point (48). Since construction of the normal is simply performed by a numerical method, definition of the normal is not considered in this study.

5. Discussion of results of simulating the synthesis of images of triangulated surfaces smoothed by the spherical interpolation method

Simulation was performed using the Wolfram Mathematica math package.

The task of simulation was to verify the provisions stated in the theoretical part, namely:

– verification of correct functioning of the imaging algorithm based on calculation of the point of PR intersection with an interpolating surface in the studied ITA;

– verification of the algorithm for constructing points of an interpolating surface with its step coinciding with the step of the iterative calculations process which makes it possible to perform the imaging algorithm and construction of points of the surface in a single ITA pass.

The simulation results fully confirmed correctness of all theoretical provisions of the study. For example, for subsections 4.1–4.7, along with construction of an interpolation surface using the package, an error estimate of the constructed surface was made. The following surfaces were chosen as reference surfaces: sphere, cylinder, cone and torus. For such diverse surfaces, error will be estimated as the relative error of deviation of the constructed surface points from the reference points lying on the guide straight line. Write down the relative error for our case in the following form:

$$\delta = \Delta / M, \quad (49)$$

where Δ is the absolute error of deviation of the points of the constructed surface from the reference points lying on the guide straight line; *M* is the parameter that has a maximum value for one or another reference surface. This is the maximum diameter of the reference surfaces.

Table 1 shows the results of estimating the maximum δ_{max} and average δ_{av} errors in accordance with (49). For a sphere, two options were considered for specifying polygons that define interpolation range. A tetrahedron: 4 polygons (4 triangles) and a cube: 6 polygons (12 triangles).

Table 1

Results of interpolation error estimation

Reference surfaces	Number of areas of triangle interpolation	δ_{max}	δ_{av}
Sphere (tetrahedron)	4/4	$3.7\cdot10^{-8}$	$9.258\cdot10^{-9}$
Sphere (cube)	6/12	$2.3\cdot10^{-16}$	$7.675\cdot10^{-17}$
Cylinder	8/16	0.0053	0.0014
Cone	8/8	0,0085	0,0023
Torus	48/96	0.015	0.0035

In comparison with the closest study [10], according to the results of simulation of the method of spherical interpolation, to restore surfaces of reference figures (Table 1), a smaller number of triangles are required as input data with the target error.

Fig. 2 shows the result of simulation of synthesis by ray tracing of images of triangulated surfaces smoothed by the spherical interpolation method.

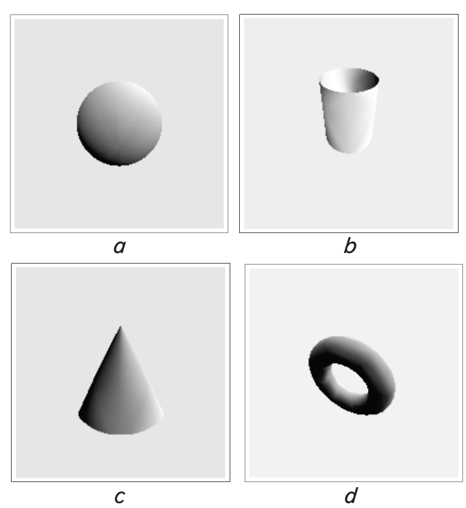

Fig. 2. The results of simulating the surfaces of reference figures according to the ratios presented in subsections 4.1–4.8: sphere (*a*), cylinder (*b*), cone (*c*), torus (*d*)

The results obtained differ from those known in such aspects:

– application of the simplest quadric (sphere) for interpolation of triangulated surfaces;

– the obtained interpolating surfaces satisfy the condition of surface continuity, C0, and the condition of continuity of the first derivative, C1;

– when constructing an interpolating surface passing through arbitrarily set vertices of triangles, it is unnecessary to use algebraic polynomials of the third and higher order;

– the method of spherical interpolation makes it possible to construct a smooth surface both locally for three points and for an ensemble of points arbitrarily located in space;

– simulation of the spherical interpolation method has shown the possibility of smoothing a triangulated surface with a target error at initial data with a smaller number of triangles;

– the proposed iterative process for imaging has the possibility of a wide parallelization of calculations. The ITA algorithm has the ability to customize the number of digits of the sought number determined in one step which is limited only by hardware.

As a disadvantage, it should be noted that, a sufficiently high performance of computational tools is necessary for implementation of this method, especially when constructing surfaces. It is supposed to aim further studies at the development of a purely analytical description of the method of spherical interpolation for smoothing the triangulated surfaces. Completed studies can be applied in various areas of computer graphics. For example, when designing imaging systems for simulators of vehicles for various purposes (aircraft simulators, etc.), when creating feature films using computer graphics, etc.

6. Conclusions

1. For synthesis by ray tracing of images of triangulated surfaces smoothed by spherical interpolation, an iterative algorithm (ITA) was developed. The proposed computational process could widely parallelize computations. In an iterative algorithm, it is possible to adjust the number of digits of the sought number determined in one step which is only limited by hardware.

2. The method of spherical interpolation of triangulated surfaces is based on the simplest algebraic surface, a sphere, and does not use algebraic polynomials of the third and higher orders. The stages of constructing an interpolating surface have been elaborated, they coincide with the step of the iterative process of computations making it possible to perform the imaging algorithm and construct points of the interpolating surface in a single ITA pass.

The stated results solve an acute problem of compatibility of the ray tracing method with the accumulated base of models and software tools focused on the rasterization method.

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