19. A numerical analysis of non-linear contact tasks for the system of plates with a bolted connection and a clearance in the fixture / Atroshenko O., Bondarenko O., Ustinenko O., Tkachuk M., Diomina N. // Eastern-European Journal of Enterprise Technologies. 2016. Vol. 1, Issue 7 (79). P. 24–29. doi: https://doi.org/10.15587/1729-4061.2016.60087
20. Thinwalled structures: analysis of the stressedstrained state and parameter validation / Tkachuk M., Bondarenko M., Grabovskiy A., Sheychenko R., Graborov R., Posohov V. et. al. // Eastern-European Journal of Enterprise Technologies. 2018. Vol. 1, Issue 7 (91). P. 18–29. doi: https://doi.org/10.15587/1729-4061.2018.120547
21. Numerical methods for contact analysis of complex-shaped bodies with account for non-linear interface layers / Tkachuk M. M., Skripchenko N., Tkachuk M. A., Grabovskiy A. // Eastern-European Journal of Enterprise Technologies. 2018. Vol. 5, Issue 7 (95). P. 22–31. doi: https://doi.org/10.15587/1729-4061.2018.143193
22. Tkachuk M. A numerical method for axisymmetric adhesive contact based on kalker's variational principle // Eastern-European Journal of Enterprise Technologies. 2018. Vol. 3, Issue 7 (93). P. 34–41. doi: https://doi.org/10.15587/1729-4061.2018.132076

UDC 534.1:539.3
DOI: 10.15587/1729-4061.2019.155854

MODELING THE ELASTIC IMPACT OF A BODY WITH A SPECIAL POINT AT ITS SURFACE

V. Ol'shanskii
Doctor of Physical and Mathematical Sciences, Professor*
E-mail: OlshanskiyVP@gmail.com
O. Spol'nik
Doctor of Physical and Mathematical Sciences, Professor*
E-mail: alexspo@ukr.net
M. Slipchenko
PhD, Associate Professor*
E-mail: Slipchenko1982@gmail.com
V. Znaidiuk
PhD*
E-mail: karlsonman@gmail.com
*Department of Physics and Theoretical Mechanics
Kharkiv Petro Vasylenko National Technical University of Agriculture
Alchevskikh str., 44, Kharkiv, Ukraine, 61002

Розглянуто пружний прямий удар по плоскій границі нерухомого півпростору тіла, обмеженого в зоні контактної взаємодії поверхнею обертання, порядок якої менший двох. Особливість задачі полягає в тому, що для вибраного випадку нескінченна кривизна граничної поверхні в точці первісного контакту, з якої розпочинається процес динамічного стискання тіл у часі. Крім основних припущень не хвильової квазистатичної теорії пружного удару твердих тіл, тут використано також відомий розв'язок статичної вісесиметричної контактної задачі теорії пружності. Процес удару з невеликою початковою швидкістю поділено на два етапи, а саме на динамічне стискання і динамічне розтискання. Для кожного з них побудовано аналітичний розв'язок нелінійного диференціального рівняння відносного зближення у часі центрів мас тіл. Розв'язок нелінійної задачі з початковими умовами для диференціального рівняння другого порядку на першому етапі виражено через Ateb-синус, а на другому – через Ateb-косинус. Для спрощення розрахунків складено окремі таблиці вказаних спеціальних функцій, а також запропоновано компактні апроксимації їх елементарними функціями. Встановлено, що похибка аналітичних наближень обох спеціальних функцій менша одного відсотка. Виведено також замкнені вирази для обчислень максимальних значень: стискання тіл, сили удару, радіуса кругової площадки контакту та тиску, який обмежений у центрі цієї площадки. Розглянуто числовий приклад, пов'язаний з ударом жорсткого пружного тіла по гумовому півпростору. Задачі такого типу виникають при моделюванні динамічної дії кусків твердої мінеральної сировини на гуму, при падінні їх на футеровані гумою валки вібраційного класифікатора. Внаслідок порівняння розрахованих параметрів удару, одержано гарну узгодженість числових результатів, до яких призводять побудовані аналітичні розв'язки та інтегрування нелінійного рівняння на комп'ютері. Цим підтверджена вірогідність побудованих аналітичних розв'язків задачі удару, які дають розгортку короткочасного процесу в часі

Ключові слова: пружний удар, особлива точка на поверхні контакту, періодичні Ateb-функції

1. Introduction

The impact interaction between solid bodies typically occurs over a short period of time and is accompanied by large dynamic loads, which could result in the possible destruction of structures' elements. Thus, it is only natural that the simplest theories for calculating the canonical bodies for strength upon impact are highlighted in the resistance of materials [1, 2]. They consider an impact to be instantaneous and, rather than the magnitude of force, apply its momentum. Actually, they consider not the process of a mechanical impact, but its consequences, that is a post-impact motion.

More complex variations in the theories of a mechanical impact are outlined in many publications, among which we emphasize [3–6].

Paper [3] sets out the synthesis of wave and quasi-static theories, as well as certain experimental methods for studying impact processes. Monograph [4] described the impact theory that was developed by the author of that work. It also contains a critical review of known experimental results. Study [5] is a literary review of achievements in the impact theory in the past century. Monographic publication [6] addressed studying the performance of different structures upon impact and compared the theory with experiment.

The above publications consider the development of the impact process over time, similar to [7], take into consideration local deformations, using a solution to the contact problem from the theory of elasticity. Thus, studying the impact processes has for a long time attracted attention of scientists and is a relevant task in mechanics.

2. Literature review and problem statement

The theory, initiated in [7], plays a key role in the research into impact processes of solid bodies. It was modernized in works [8, 9], where it was also suggested that the bodies subjected to an impact are, in a contact zone, restricted by surfaces of the second order, but they additionally considered oscillatory processes.

In addition to the above publications, a review of publications on a mechanical impact was performed in [5]. There was a marked tendency to an increase in the flow of publications that address an analysis of the dynamics of layered composite structures during impact.

Paper [10] describes analysis of performance, upon impact, of layered plates and cylindrical shells using the method of finite elements. The positive point is the presence of a large number of numerical results and geometric illustrations. The disadvantage of the work is the lack of simple calculation formulae, convenient for engineering calculations, because the research was conducted by purely numerical methods.

Study [11] considered an impact at a little velocity along a plane with the preliminary load by pressure. Structures of this type are common in engineering and are widely used. In the work, a body that hits was an elastic ball; no cases were considered for bodies that hit with a more complex shape because it complicates the theory.

Paper [12] examined the performance of layered plates at low-velocity impact using the method of finite elements. However, the work lacks analytical solutions. The obtained results are not universal, because they relate to specific numerical parameters for plates. For the case of change in the parameters, it is necessary to carry out new calculations, which is a disadvantage of numerical methods.

The influence of compressing forces on the performance of composite panels under impact was studied in [13]. It was established that the presence of such forces increases dynamic deflections. However, the problem was solved in a linear statement that rules out a possibility to model a change in the form of equilibrium upon impact.

Study [14] examined the dynamics of a composite plate with a hole upon impact. The authors calculated dynamic concentration of stresses in the zone of a hole. The analysis was carried out by numerical methods, but the study also lacks analytical solutions, which would make it possible to analyze the influence of different factors (size and shape of the hole, the properties of a material, etc.) on the magnitude of stress concentrations.

The above papers [10–14] considered a mechanical impact of bodies restricted by smooth surfaces, which have, at a contact point, a limited curvature; typically, it was a ball. However, in practice, there are other shapes of elastic bodies exposed to an impact. Only paper [4] considered the case of a tighter contact between bodies at dynamic compression, constrained by boundary surfaces that have an order higher than two. However, remain unexplored are the cases of an impact between bodies, whose boundary surfaces' order is less than two (ogive, cone, and others). These bodies have a particular point at the contact surface. Therefore, existing theories for the case of such bodies are not applicable and require new mathematical models; construction of new theories related to the impact between bodies with a particular point at their surfaces is a promising task.

3. The aim and objectives of the study

The aim of this study is to derive and verify formulae to calculate the variability in the parameters for an impact compression of bodies over time, in the presence of a particular point at the surface of one of the bodies.

To accomplish the aim, the following tasks have been set:

– to derive a formula to calculate a coefficient in the equation of impact;

– to build a solution to the equation of impact at compression of bodies;

– to build a solution to the equation of impact at decompression of bodies;

– to perform calculations and run a comparative analysis of numerical results.

4. Materials and methods to study an elastic impact of bodies with a particular point at the contact surface

When carrying out a mathematical modeling of the process of the dynamic compression of solid bodies, we shall use assumptions from work [7] where it was believed that the entire kinetic energy of relative motion is converted into potential energy of the elastic deformations of bodies in the zone of their interaction while disregarding other forms of energy, specifically the energy of elastic waves, thermal energy, etc. This imposes certain limitations on the velocity of an impact, which, in accordance with [4], should not exceed 4 m/s. Therefore, we build a mathematical model of a purely elastic impact, employing the theory of nonlinear differential equations and special functions.

Determining coefficients in the equation of impact. When deriving the basic equation of impact, we shall, in addition to basic provisions for theory [7], use a known solution to the axisymmetric contact problem from the elasticity theory [15]. According to this solution, the convergence between the centers of masses of elastic bodies x, one of which is a half-space, and the second is restricted by the surface of rotation $z=Ar^{3/2}$, ($A>0$) (Fig. 1), is described by expression [15]:

$$x=\frac{3}{2}J_1 A a^{3/2}, \tag{1}$$

in which:

$$a=\left[\frac{P(Q_1+Q_2)}{3AJ_2}\right]^{2/5};\quad Q_1=\frac{1-\mu_1^2}{E_1};\quad Q_2=\frac{1-\mu_2^2}{E_2};\qquad (2)$$

$$J_1=\int_0^1\frac{\sqrt{\xi}d\xi}{\sqrt{1-\xi^2}};\quad J_2=\int_0^1\frac{\xi^2\sqrt{\xi}d\xi}{\sqrt{1-\xi^2}};$$

E_1, μ_1, E_2, μ_2 are, accordingly, the modulus of elasticity and the coefficient of transverse deformation of bodies' materials; P is the force of bodies' compression; a is the radius of the contact area.

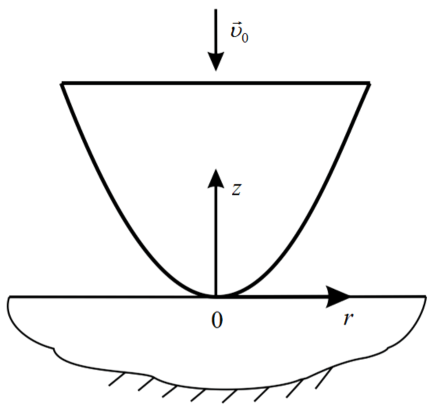

Fig. 1. Schematic of bodies collision

Pressure distribution at it is governed by law [15]:

$$p(r)=1{,}25\cdot\frac{P}{\pi a^2}f\left(\frac{r}{a}\right),\qquad (3)$$

where

$$f\left(\frac{r}{a}\right)=\sqrt{\frac{r}{a}}\int_{r/a}^{1}\frac{d\xi}{\xi\sqrt{\xi(1-\xi^2)}}.\qquad (4)$$

Integrals J_1 and J_2 are represented through gamma-function $\Gamma(z)$, tabulated in [16, 17], because, according to reference [18]:

$$J_1=\int_0^{\pi/2}\sin^{1/2}\phi\, d\phi=2^{-1/2}\frac{[\Gamma(3/4)]^2}{\Gamma(3/2)};$$

$$J_2=\int_0^{\pi/2}\sin^{5/2}\phi\, d\phi=2^{3/2}\frac{[\Gamma(7/4)]^2}{\Gamma(7/2)}.$$

These quadratures were obtained through a transition in (2) to the new variable of integration $\xi=\sin\varphi$.

If one takes into consideration the tabular value [16]: $\Gamma(7/4)\approx 0{,}9190625$, then

$$\Gamma(3/4)=\frac{4}{3}\Gamma(7/4)\approx\frac{4}{3}\cdot 0{,}9190625\approx 1{,}2254167;$$

$$\Gamma(3/2)=\frac{1}{2}\sqrt{\pi};\quad \Gamma(7/2)=\frac{15}{8}\sqrt{\pi}.$$

Thus, approximately, $J_1\approx 1.198140$; $J_2\approx 0.718884$.

Values for J_1 and J_2, but with a less accuracy, were also calculated in [15].

Expression (4) at $r\to 0$ has the uncertainty of type $|0\cdot\infty|$. To reveal it, compute the integral in (4). Through a transition to the new variable: $\xi=\eta^2$, we obtain:

$$f\left(\frac{r}{a}\right)=2\sqrt{\frac{r}{a}}\int_{\sqrt{r/a}}^{1}\frac{d\eta}{\eta^2\sqrt{1-\eta^4}}.$$

This integral is expressed through incomplete elliptic integrals $F(\delta;\zeta)$, $E(\delta;\zeta)$ of first and second kind, respectively, because in [18]:

$$\int_u^1\frac{d\eta}{\eta^2\sqrt{1-\eta^4}}=\frac{1}{\sqrt{2}}\left[F(\delta;\zeta)-2E(\delta;\zeta)\right]+\frac{1}{u}\sqrt{1-u^4}$$

and

$$\delta=\arccos u,\quad \zeta=\frac{1}{\sqrt{2}}.$$

Thus, pressure distribution is described by expression:

$$p(r)=2{,}5\cdot\frac{P}{\pi a^2}\left\{\sqrt{\frac{r}{2a}}\left[\begin{array}{l}F\left(\arccos\sqrt{\frac{r}{a}};\frac{1}{\sqrt{2}}\right)-\\ -2E\left(\arccos\sqrt{\frac{r}{a}};\frac{1}{\sqrt{2}}\right)\end{array}\right]+\sqrt{1-\frac{r^2}{a^2}}\right\}.\qquad (5)$$

Since

$$F\left(0;\frac{1}{\sqrt{2}}\right)=E\left(0;\frac{1}{\sqrt{2}}\right)=0,$$

then at the edge of the contact area, at $r=a$, pressure $p(a)=0$. At the center of the area, the pressure is maximum. There,

$$p(0)=2{,}5\cdot\frac{P}{\pi a^2}=2{,}5\cdot p_{cp}.$$

Therefore, maximum pressure is 2.5 times greater than the average over the zone. Graph of pressure distribution along the radius of the zone is shown in Fig. 2.

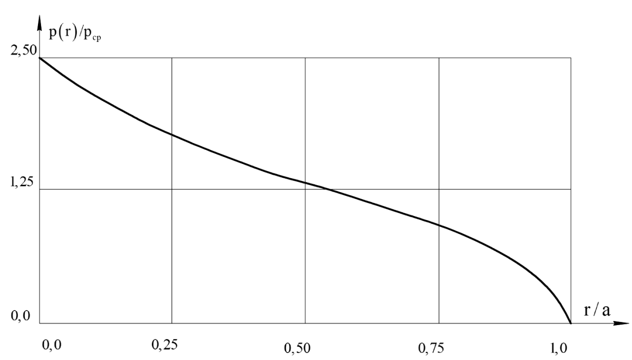

Fig. 2. Distribution of contact pressure along the radius of the zone

In addition to [15], other authors later built the generalized solutions to the static contact problem [19].

The following formula to calculate the strength of an impact follows from (1), (2):

$$P=\beta x^{5/3}.$$

Here

$$\beta=\frac{3AJ_2}{Q_1+Q_2}\left(\frac{2}{3AJ_1}\right)^{5/3}\approx\frac{0,81180}{A^{2/3}(Q_1+Q_2)}.$$

If the mass of a body that hits is equal to M, then the impact process, according to [7], will be described by a differential equation:

$$M\ddot{x}=-P=-\beta x^{5/3}, \tag{6}$$

where a dot above x indicates a time-dependent derivative t.

For the further integration of equation (6), we shall use it in the form:

$$\dot{x}\frac{d\dot{x}}{dx}=-\frac{\beta}{M}x^{5/3}. \tag{7}$$

Construction of the solution to the equation of impact at compression of bodies. It is matched by time $t\in(0;t_c)$. Differential equation (7) is to be solved under initial conditions:

$$x(0)=0,\quad \dot{x}(0)=\upsilon_0, \tag{8}$$

we denote via symbol υ_0 the initial velocity of collision between bodies.

Upon integrating (7), taking into consideration (8), we obtain:

$$\dot{x}=\sqrt{\upsilon_0^2-\frac{3}{4}\frac{\beta}{M}x^{8/3}};\quad \int_0^x\frac{dy}{\sqrt{\upsilon_0^2-\frac{3}{4}\frac{\beta}{M}y^{8/3}}}=t. \tag{9}$$

At the time of maximum compression, radicand $x=x_c$ in (9) is equal to zero. Therefore, the maximum compression is:

$$x_c=\left(\frac{4M\upsilon_0^2}{3\beta}\right)^{3/8}.$$

Through a transition to the new variable in the integration $\xi=\frac{y}{x_c}$, the integral in (9) is reduced to the following:

$$\int_0^{x/x_c}\frac{d\xi}{\sqrt{1-\xi^{8/3}}}=\frac{\upsilon_0 t}{x_c},$$

where the upper bound is the Ateb-sine [20–23]. Thus,

$$x(t)=x_c\cdot\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0 t}{x_c}\right). \tag{10}$$

A time-dependent change in the force of an impact is governed by law:

$$P(t)=\beta x_c^{5/3}\left[\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0 t}{x_c}\right)\right]^{5/3}.$$

The radius of the contact area, as well as pressure at its center, depend on the values for Ateb-sine as well, because, according to (2) and (5):

$$a(t)=\left(\beta\frac{Q_1+Q_2}{3AJ_2}\right)^{2/5}x_c^{2/3}\left[\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0 t}{x_c}\right)\right]^{2/3};$$

$$p(0,t)=\frac{2,5}{\pi}\left(\frac{3AJ_2}{Q_1+Q_2}\right)^{4/5}\beta^{1/5}x_c^{1/3}\left[\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0 t}{x_c}\right)\right]^{1/3}. \tag{11}$$

Maximum values are acquired by the characteristics of an impact interaction at $t=t_c$ or when:

$$\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0 t}{x_c}\right)=1. \tag{12}$$

Maxima are represented by concise formulae:

$$\max x(t)=x_c=\left(\frac{4M\upsilon_0^2}{3\beta}\right)^{3/8};\ \max P(t)=P_c=\beta x_c^{5/3};$$

$$\max a(t)=a_c=0,67650\left(\frac{x_c}{A}\right)^{2/3};$$

$$\max p(0,t)=p_c=2,5\frac{P_c}{\pi a_c^2}. \tag{13}$$

The root of equation (12) is related to integral:

$$\frac{\upsilon_0 t_c}{x_c}=I=\int_0^1\frac{d\xi}{\sqrt{1-\xi^{8/3}}},$$

which is expressed through a gamma function tabulated according to formula [18]:

$$I=\frac{3}{8}\sqrt{\pi}\frac{\Gamma(3/8)}{\Gamma(7/8)}.$$

Given $\Gamma(3/8)\approx 2,370437,\ \Gamma(7/8)\approx 1,089653,$ we find: $I\approx 1,445927.$

The result is the obtained formula for the calculation of compression process duration:

$$t_c=1,445927\frac{x_c}{\upsilon_0}. \tag{14}$$

To simplify the numerical realization of solution derived, we give a table of Ateb-sine whose values, with an acceptable accuracy, can be found through a linear interpolation of data from Table 1.

Table 1

Values for Ateb-sine (brackets: approximated)

10η	$10\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\eta\right)$	10η	$10\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\eta\right)$
0.0	0.00 (0.00)	7.5	7.04 (7.02)
0.5	0.50 (0.50)	8.0	7.42 (7.42)
1.0	1.00 (1.00)	8.5	7.78 (7.78)
1.5	1.50 (1.50)	9.0	8.12 (8.12)
2.0	2.00 (2.00)	9.5	8.43 (8.43)
2.5	2.49 (2.50)	10.0	8.72 (8.72)
3.0	2.98 (3.00)	10.5	8.98 (8.98)
3.5	3.47 (3.49)	11.0	9.22 (9.22)
4.0	3.95 (3.97)	11.5	9.43 (9.43)
4.5	4.43 (4.44)	12.0	9.60 (9.60)
5.0	4.89 (4.89)	12.5	9.75 (9.75)
5.5	5.35 (5.34)	13.0	9.86 (9.86)
6.0	5.79 (5.78)	13.5	9.94 (9.94)
6.5	6.22 (6.20)	14.0	9.99 (9.99)
7.0	6.64 (6.62)	$10I$	10.00 (10.00)

Here in parentheses we give approximate values, which are derived via approximation:

$$\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\eta\right)\approx\begin{cases}\eta & \text{at } 0\le\eta\le 0{,}2,\\ 0{,}1996+1{,}0277(\eta-0{,}2)-0{,}2064(\eta-0{,}2)^2 & \text{at } 0{,}2<\eta<0{,}8,\\ 1-1{,}2\sin^2\left[\left(\sqrt{5}/3\right)\cdot(I-\eta)\right] & \text{at } 0{,}8\le\eta\le I.\end{cases}$$

This approximation is a special case of the more general approximation of Ateb-sine, proposed in [24]. Its error is less than one percent.

Construction of the solution to the equation of impact at decompression of bodies.

Next, consider the process of the dynamic decompression of bodies that occurs over interval $t\in(t_c;t_y)$. To this end, we build a solution to the differential equation (7) under initial conditions:

$$x(t_c)=x_c,$$

$$\dot{x}(t_c)=0.$$

Upon double integration, we obtain:

$$x(t)=x_c\cdot\mathrm{Ca}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0}{x_c}(t-t_c)\right), \qquad (15)$$

where

$$\mathrm{Ca}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0}{x_c}(t-t_c)\right)$$

is the periodic Ateb-cosine [21, 22].

A change in the force of impact over time is governed by law:

$$P(t)=\beta x_c^{5/3}\left[\mathrm{Ca}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0}{x_c}(t-t_c)\right)\right]^{5/3}.$$

To calculate a time-dependent change in the radius of the contact area and pressure at its center, we obtain formulae:

$$a(t)=a_c\cdot Ca\left[\frac{5}{3};1;\frac{4}{3}\frac{\upsilon_0}{x_c}(t-t_c)\right]^{2/3};$$

$$p(0,t)=p_c\cdot Ca\left[\frac{5}{3};1;\frac{4}{3}\frac{\upsilon_0}{x_c}(t-t_c)\right]^{1/3}.$$

Thus, the decompression process of bodies is described by Ateb-cosine and its powers.

The impact process ends at $t=t_y$, which is a root of equation:

$$\mathrm{Ca}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0}{x_c}(t_y-t_c)\right)=0,$$

which has a solution:

$$\frac{\upsilon_0}{x_c}(t_y-t_c)=I.$$

Hence, we obtain the formula to calculate the impact duration:

$$t_y=t_c+\frac{x_c}{\upsilon_0}I=2t_c=2{,}891854\frac{x_c}{\upsilon_0}.$$

In order to simplify calculation of the decompression process, we give the values for Ateb-cosine in Table 2.

Table 2

Values for Ateb-cosine (brackets: approximated)

10ζ	$\mathrm{Ca}\left(\frac{5}{3},1,\frac{4}{3}\zeta\right)$	10ζ	$\mathrm{Ca}\left(\frac{5}{3},1,\frac{4}{3}\zeta\right)$
0.0	10.00 (10.00)	7.5	6.61 (6.62)
0.5	9.98 (9.98)	8.0	6.19 (6.17)
1.0	9.93 (9.93)	8.5	5.76 (5.74)
1.5	9.85 (9.85)	9.0	5.31 (5.30)
2.0	9.74 (9.74)	9.5	4.86 (4.86)
2.5	9.59 (9.59)	10.0	4.39 (4.40)
3.0	9.41 (9.41)	10.5	3.91 (3.93)
3.5	9.20 (9.20)	11.0	3.43 (3.45)
4.0	8.96 (8.96)	11.5	2.94 (2.96)
4.5	8.70 (8.70)	12.0	2.45 (2.46)
5.0	8.41 (8.41)	12.5	1.96 (1.96)
5.5	8.09 (8.09)	13.0	1.46 (1.46)
6.0	7.75 (7.76)	13.5	0.96 (0.96)
6.5	7.39 (7.40)	14.0	0.46 (0.46)
7.0	7.01 (7.02)	$10I$	0.00 (0.00)

In brackets, Table 2 gives the approximated values for Ateb-cosine, obtained through approximation:

$$Ca\left(\frac{5}{3},1,\frac{4}{3}\zeta\right)\approx\begin{cases}1-1{,}2\sin^2\left[\left(\sqrt{5}/3\right)\cdot\zeta\right] & \text{at } 0\le\zeta\le 0{,}75,\\ 0{,}1996+1{,}0277(1{,}2459-\zeta)-0{,}2064(1{,}2459-\zeta)^2 & \text{at } 0{,}75<\zeta<1{,}25,\\ \zeta & \text{at } 1{,}25\le\zeta\le I.\end{cases}$$

Table 2 shows that an error of the recorded analytical approximation is less than one percent. This indicates the adequacy of the represented formulae.

5. Results of calculations of the impact parameters, their comparative analysis

By using the derived formulae, we shall compute the characteristics of an impact at: M=0.7 kg; υ_0=3 m/s; A=5 m$^{-1/2}$; $E_1=2\cdot10^{11}$ Pa; μ_1=0.25. Material of the half-space is rubber for which [25]: $E_2=7.5\cdot10^6$ Pa, μ_2=0.5. For these initial data $\beta=2.77632\cdot10^6$ N·m$^{-5/3}$. From formulae (13) and (14), we obtain: x_c=0.008517 m; P_c=986.2176 N; a_c=0.009649 m; p_c=8429081.797 Pa. Duration of the compression and impact are, respectively: t_c=0.004105 s; t_y=0.008210 s. Values for $x(t)$, calculated based on formulae (10), (15), at different moments are given in Table 3.

Values for Ateb functions were computed by the method of linear interpolation of data from Tables 1, 2. For comparison, Table 3 also gives values for $x(t)$, obtained by a computer-based integration of equation (6) in the programming environment "Maple" [26]. There is a good consistency of results from calculations by two procedures, confirming the probability of analytical solutions to the nonlinear problem.

Table 3

Values for $x(t)/x_c$ calculated by two procedures

t/t_c	$\frac{\upsilon_0 t}{x_c}$	$x(t)/x_c$, formulae (10, 15)	$x(t)/x_c$, numerical integration
0.25	0.3615	0.3580	0.3582
0.50	0.7230	0.6824	0.6826
0.75	1.0845	0.9146	0.9149
1.00	1.4459	1.0000	1.0000
1.25	1.8074	0.9145	0.9149
1.50	2.1689	0.6826	0.6826
1.75	2.5303	0.3580	0.3582

The diagrams of change in $x(t)$ and $P(t)$ over time t, obtained from calculations, are shown in the dimensionless coordinates in Fig. 3. Over a compression interval, the diagram of $x(t)$ is convex, and the diagram for $P(t)$ changes concavity to convexity.

Fig. 4 shows the estimated diagrams of change in $a(t)$ and p(t) over time. There is a rapid growth in pressure $p(t)$ in the center of the contact area of bodies.

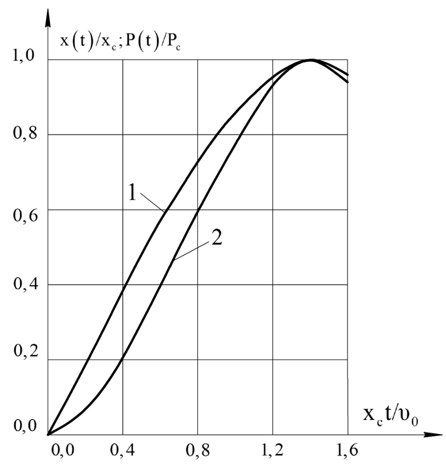

Fig. 3. Diagrams of time-dependent change: 1 – x(t)/x$_c$; 2 – P(t)/P$_c$

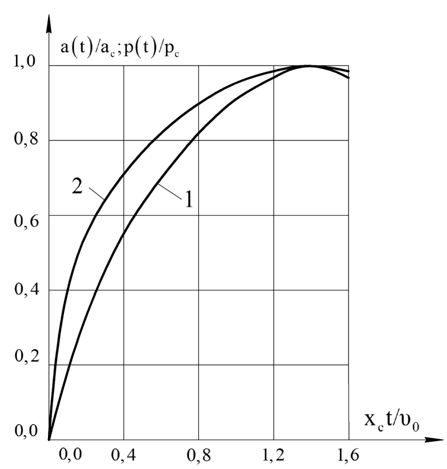

Fig. 4. Diagrams of time-dependent change: 1 – $a(t)/a_c$; 2 – $p(t)/p_c$

The course of the impact process is affected by the geometry of the hitting body. This is confirmed by results from the calculation of an impact characteristics, given in Table 4 for different values of A. The estimations employed formulae (13), (14).

Table 4

Impact characteristics at different A

A, m$^{-1/2}$	$10^4\, x_c$, m	P_c, N	$10^4\, a_c$, m	$10^{-6}\, p_c$, Pa	$10^4\, t_c$, s
2	67.74	1240.24	152.6	4.23825	32.65
4	80.55	1042.76	107.9	7.12741	38.82
6	89.15	942.36	88.1	9.66174	42.97

An increase in A leads to an increase in x_c, pressure p_c, and the impact duration. However, it is accompanied by a decrease in P_c and a_c. Attention should be paid to that at a change in A the product $P_c{\cdot}t_c$ remains an approximately constant magnitude. To find out why it happens, we shall consider the calculation of impact pulse $S(P)$. At time interval $t\in[0;t_*]$, it is represented by integral:

$$S(P)=\int_0^{t_*}P(t)\,dt=P_c\int_0^{t_*}\left[\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0 t}{x_c}\right)\right]^{5/3}\mathrm{d}t.$$

Over the entire impact duration, when $t_*=t_y$, the pulse is equal to:

$$S(P)=2P_c\int_0^{t_c}\left[\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\frac{\upsilon_0 t}{x_c}\right)\right]^{5/3}\mathrm{d}t.$$

Compute this integral approximately from the formula of trapezoid [27]:

$$S(P)\approx\frac{1}{3}P_c t_c\left\{1+4\left[\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\frac{I}{2}\right)\right]^{5/3}\right\}.$$

Since

$$\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\frac{I}{2}\right)\approx 0{,}682,$$

then

$$\frac{1}{3}\left\{1+4\left[\mathrm{Sa}\left(\frac{5}{3},1,\frac{4}{3}\frac{I}{2}\right)\right]^{5/3}\right\}\approx 1{,}03788.$$

Therefore, strength of pulse $P(t)$, over an impact duration, is approximately:

$$S(P)\approx 1{,}03788\,P_c\cdot t_c. \tag{16}$$

By using the results from Table 4 and formula (16), we obtain, at A=2, 4, 6 m$^{-1/2}$, respectively, $S(P)$=4.2028; 4.2013; 4.2027 N×s. These numbers are little different from the exact value for pulse $S(P)=2M\upsilon_0$=4.2 N×s. Thus, the stable product $P_c{\cdot}t_c$ in Table 4 is a consequence of the fact that $S(P)=2M\upsilon_0$=const in the considered example.

Next, we determine the influence on the impact parameters exerted by the initial velocity of bodies' collision, which, in accordance with [25], can reach 5 m/s. To conduct calculations, we save the above-specified source data, where A=5 m$^{-1/2}$, β=2.77632·10^6 N·m$^{-5/3}$

$$x_c = \left(\frac{4M\upsilon_0^2}{3\beta}\right)^{3/8} = 0{,}6037365\upsilon_0^{3/4},$$

$P_c=\beta x_c^{5/3}$. Changing υ_0, we obtain the impact parameters, given in Table 5.

Table 5

Impact characteristics at different υ_0

υ_0, m/s	$x_c \cdot 10^4$, m	P_c, N	$10^4 a_c$, m	$10^{-6} p_c$, Pa	$10^4 t_c$, s
3	85.17	986.22	96.49	8.429	41.05
4	105.68	1413.03	111.42	9.058	38.20
5	124.94	1867.63	124.57	9.578	36.13

The calculations confirm that an increase in υ_0 leads to an increase in x_c, P_c, a_c, p_c and a decrease in t_c. At υ_0=5 m/s, force P_c is close to 1,900 N. Paper [25] noted that force P_c may exceed 1,960 N.

Information about the effect of a body mass on the impact characteristics is given in Table 6. In the calculations, we used previous numerical data by assigning υ_0=4 m/s.

Table 6

Impact characteristics at different *M*

M, kg	$x_c \cdot 10^4$, m	P_c, N	$10^4 a_c$, m	$10^{-6} p_c$, Pa	$10^4 t_c$, s
0.7	105.68	1413.03	111.42	9.058	38.20
0.8	111.11	1536.05	115.20	9.211	40.16
0.9	116.13	1653.39	118.65	9.346	41.98
1.0	120.81	1765.93	121.81	9.471	43.67

Here, an increase in the hitting body mass leads to an increase in both x_c, P_c, a_c, p_c and t_c, which was not the case when increasing υ_0.

The calculations show that a velocity change in υ_0 has a considerably larger effect on the impact characteristics than a change in *M*.

6. Discussion of results of studying the derived analytic solutions to an elastic impact of the body with a special point

The built analytical solutions to the impact are quite universal. They do not have analogs in the scientific literature because they relate to the development of the theory of a quasi-static impact of bodies that have a particular point at the contact surface.

The obtained solutions are not linked to specific values for the physical and geometrical parameters of bodies exposed to impact. They make it possible to calculate a change in the impact process parameters over time, as well as extreme values: the strength of an impact, the magnitude of bodies' compression, the radius of the contact area, and the maximum pressure at the center of this plane. An apparatus of the periodic Ateb functions has proven to be an effective means to represent the analytical solutions, owing to these special functions, tabulated in this work. Comparison of numerical results obtained by different methods confirms the reliability of the derived analytical solutions. The results of calculations correspond to the physical essence of the impact process (small duration in time, a large maximum of the impact force, etc.).

The results obtained could be used in hybrid theories where they synthesize the quasi-static and wave theories. They could be applied when calculating a response to the impact from both homogeneous and composite beams, plates, and shells.

The outlined theory applies to low impact velocities, not exceeding 5 m/s, so that the deformations of bodies are within the limits of the elasticity theory.

The considered theory applies to only one case of a particular point at the boundary surface of a body. However, there are cases with points with higher attributes (a vertex of the cone, etc.). Development of the impact theory for such bodies could be a direction for further research.

7. Conclusions

1. We have derived a formula for the calculation of a coefficient in the impact equation, which depends on the geometric characteristics and materials of bodies exposed to impact. It is easy to calculate when using the formula derived in this work. Calculations are given in the results computation.

2. We have built a solution to the equation of impact at compression of bodies, which is expressed by Ateb-sine, and makes it possible to calculate time-dependent changes in the impact strength, in the convergence of the centers of mass of bodies, the radius of the contact area and pressure at its center, by using the compiled table of this function. Ateb-sine has not been used in the theory of impact before.

3. We have built a solution to the equation of impact at decompression of bodies, which is expressed through Ateb-cosine. We have derived a formula for the calculation of an impact duration. The table of this function makes it easy to use the solution in calculation.

4. The calculations have been performed, which confirmed the adequacy of the built analytical solutions; the effect of various factors on the main impact characteristics has been investigated. It was established:

– increasing the geometrical parameter leads to an increase in the maximum of compression, impact strength, pressure and duration of the impact, but it is accompanied by a decrease in the maximum of impact strength and the radius of the contact area;

– a growth of the initial impact velocity is accompanied by the growth of all impact parameters, except for the impact duration, which decreases;

– a growth in the hitting body mass is accompanied by an increase in all impact parameters due to an increase in the kinetic energy of the hitting body;

– a change in the initial velocity exerts a more essential, compared to a change in mass, influence on the impact characteristics, which has been represented in tables.

References

1. Pysarenko H. S., Kvitka O. L., Umanskyi E. S. Opir materialiv. Kyiv: Vyshcha shkola, 2004. 655 p.
2. Ol'shanskii V. P., Ol'shanskii S. V. Calculation of the dynamic deflection of a beam on inelastic impact by the Cox and Saint-Venant theories // Strength of Materials. 2013. Vol. 45, Issue 3. P. 361–368. doi: https://doi.org/10.1007/s11223-013-9466-x
3. Gol'dsmit V. Udar. Teoriya i fizicheskie svoystva soudaryaemyh tel. Moscow: Stroyizdat, 1965. 447 p.

4. Kil'chevskiy N. A. Dinamicheskoe kontaktnoe szhatie tverdyh tel. Udar. Kyiv: Naukova dumka, 1976. 319 p.
5. Abrate S. Impact on Laminated Composites: Recent Advances // Applied Mechanics Reviews. 1994. Vol. 47, Issue 11. P. 517. doi: https://doi.org/10.1115/1.3111065
6. Jones N. Structural impact. Cambridge: Cambridge Univ. Press, 1989. 320 p.
7. Hertz H. Ueber die Berührung fester elastischer Körper // Journal für die reine und angewandte Mathematik (Crelle's Journal). 1882. Issue 92. P. 156–171. doi: https://doi.org/10.1515/crll.1882.92.156
8. Chun L., Lam K. Y. Dynamic response of fully-clamped laminated composite plates subjected to low-velocity impact of a mass // International Journal of Solids and Structures. 1998. Vol. 35, Issue 11. P. 963–979. doi: https://doi.org/10.1016/s0020-7683(96)00231-4
9. Smetankina N. V. Vibrations of laminated orthotropic shells with complex shape at impact loading // Vibratsiyi v tekhnitsi ta tekhnolohiyakh. 2016. Issue 2 (82). P. 77–84.
10. Finite element modeling of low-velocity impact on laminated composite plates and cylindrical shells / Khalili S. M. R., Soroush M., Davar A., Rahmani O. // Composite Structures. 2011. Vol. 93, Issue 5. P. 1363–1375. doi: https://doi.org/10.1016/j.compstruct.2010.10.003
11. Low velocity impact on CFRP plates with compressive preload: Test and modeling / Heimbs S., Heller S., Middendorf P., Hähnel F., Weiße J. // International Journal of Impact Engineering. 2009. Vol. 36, Issue 10-11. P. 1182–1193. doi: https://doi.org/10.1016/j.ijimpeng.2009.04.006
12. Li D. H., Liu Y., Zhang X. Low-velocity impact responses of the stiffened composite laminated plates based on the progressive failure model and the layerwise/solid-elements method // Composite Structures. 2014. Vol. 110. P. 249–275. doi: https://doi.org/10.1016/j.compstruct.2013.12.011
13. Faggiani A., Falzon B. G. Predicting low-velocity impact damage on a stiffened composite panel // Composites Part A: Applied Science and Manufacturing. 2010. Vol. 41, Issue 6. P. 737–749. doi: https://doi.org/10.1016/j.compositesa.2010.02.005
14. Habil H., Hayder H. The low-velocity impact response of laminated composite plates with holes // Journal of Multidisciplinary Engineering Science and Technology. 2015. Vol. 2, Issue 4. P. 726–733.
15. Shtaerman I. Ya. Kontaktnaya zadacha teorii uprugosti. Moscow-Leningrad: Gostekhizdat, 1949. 272 p.
16. Abramovic M., Stigan I. Spravochnik po special'nym funkciyam (s formulami, grafikami i matematicheskimi tablicami). Moscow: Nauka, 1979. 832 p.
17. Yanke E., Emde F., Lesh F. Special'nye funkcii. Moscow: Nauka, 1977. 344 p.
18. Gradshteyn I. S., Ryzhik I. M. Tablicy integralov, summ, ryadov i proizvedeniy. Moscow: Nauka, 1962. 1100 p.
19. Fu G. An Extension of Hertz's Theory in Contact Mechanics // Journal of Applied Mechanics. 2007. Vol. 74, Issue 2. P. 373. doi: https://doi.org/10.1115/1.2188017
20. Sokil B. I. Pro zastosuvannia Ateb-funktsiy dlia pobudovy rozviazkiv deiakykh rivnian, yaki opysuiut neliniyni kolyvannia odnovymirnykh seredovyshch // Dopovidi Natsionalnoi akademiyi nauk Ukrainy. 1997. Issue 1. P. 55–58.
21. Hrytsyk V. V., Nazarkevych M. A. Matematychni modeli alhorytmiv i realizatsiya Ateb-funktsiy // Dopovidi Natsionalnoi akademiyi nauk Ukrainy. 2007. Issue 12. P. 37–42.
22. Pukach P. Ya. Yakisni metody doslidzhennia neliniynykh kolyvalnykh system. Lviv: Lvivska politekhnika, 2014. 288 p.
23. Olshanskiy V. P., Olshanskiy S. V. Ateb-sine in the solution of Hertz's problem of impact // Visnyk Natsionalnoho tekhnichnoho universytetu "KhPI". Seriya: Matematychne modeliuvannia v tekhnitsi ta tekhnolohiyakh. 2018. Issue 3. P. 98–103.
24. Olshansky V. P., Olshansky S. V. About the motion of the oscillator with the degree of characteristic of elasticity // Vibratsiyi v tekhnitsi ta tekhnolohiyakh. 2017. Issue 3. P. 34–40.
25. Nadutiy V. P., Yagnyukov V. F., Yagnyukova I. V. Vzaimodeystvie kuskov materiala s futerovannym valkom vibracionnogo klassifikatora // Vibratsiyi v tekhnitsi ta tekhnolohiyakh. 2014. Issue 1. P. 94–99.
26. D'yakonov V. P. Maple 8 v matematike, fizike i obrazovanii. Moscow: Solon-Press, 2013. 656 p.
27. Bahvalov N. S., Zhidkov N. P., Kobel'nikov G. M. Chislennye metody. Moscow: Binom, 2001. 630 p.