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SAMPLING THE CONTINUOUS TWO-SIDED NONCOOPERATIVE GAME ON UNIT HYPERCUBE AND MULTIDIMENSIONAL MATRIX RESHAPING FOR SOLVING THE CORRESPONDING BIMATRIX GAME

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There is suggested a method of obtaining the approximate NE-solution of continuous two-sided noncooperative games on unit multidimensional cube with uniform sampling along each of dimensions. For that there are presented requirements to sample the game kernels smoothly and relationships letting reshape multidimensional matrices into ordinary two-dimensional ones with maintaining one-to-one indexing between their elements. The corresponding bimatrix game NE-solution is checked for two types of consistency, whose being suggested requirements allow to predetermine how this solution changes by changing the sampling step minimally.

Key words: two-sided noncooperative games, unit hypercube, multidimensional matrix, NE-strategies, approximate solution, NE-solution consistency.

ДИСКРЕТИЗАЦІЯ КОНТИНУАЛЬНОЇ ДВОСТОРОННЬОЇ БЕЗКОАЛІЦІЙНОЇ ГРИ НА ОДИНИЧНОМУ ГІПЕРКУБІ І ПЕРЕТВОРЕННЯ БАГАТОВИМІРНОЇ МАТРИЦІ ДЛЯ РОЗВ'ЯЗУВАННЯ ВІДПОВІДНОЇ БІМАТРИЧНОЇ ГРИ

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Пропонується метод отримання наближеного розв'язку у смислі рівноваги за Нешем континуальних двосторонніх безкоаліційних ігор на одиничному багатовимірному кубі за допомогою рівномірної дискретизації уздовж кожного з вимірів. Ставляться вимоги до того, щоб гладко дискретизувати ядра гри, і співвідношення, що дозволяють здійснювати перетворення багатовимірних матриць у звичайні двовимірні зі збереженням взаємооднозначної індексації між їх елементами. Розв'язок у смислі рівноваги за Нешем відповідної біматричної гри перевіряється на два типи узгодженості, пропоновані умови яких дозволяють зважати на те, наскільки цей розв'язок змінюється за мінімальної зміни кроку дискретизації.

Ключові слова: двосторонні безкоаліційні ігри, одиничний гіперкуб, багатовимірна матриця, NE-стратегії (рівноважні стратегії за Нешем), наближений розв'язок, узгодженість NE-розв'язку (розв'язку у смислі рівноваги за Нешем).

PROBLEM STATEMENT. One of the principal purposes of game modeling is resolution of conflict events or processes, springing up permanently under natural disproportion of resources and demands. In economics, politics, military science, jurisprudence, social and ecologic processes, the conflict game model solution allows to distribute resources mainly according to Nash or Pareto equilibrium [1, 2]. Moreover, under specific hard uncertainties, two-sided game solutions may be applied to technological and technical processes, reducing their risks or losses on average [3].

Games in Euclidean finite-dimensional subspaces of non-single dimension. While stating the problem of noncooperative game modeling, there are sets of the players' pure strategies to be defined, whereupon the game kernels (the players' payoff functions) are defined on the Cartesian product of these sets. For a lot of events the player's pure strategy is an action, featured with a sequence of parameters. And often these parameters cannot be aggregated into a single value [4, 5]. So if one or more parameters belong to some intervals of their acceptable values, the player gets a continuous set of one's pure strategies. This set can be equivalent to a Euclidean finite-dimensional subspace of non-single dimension [6, 7]. And even two-sided noncooperative games on such subspaces' product are pretty difficult to be solved analytically [2, 8].

Solutions of two-sided noncooperative games on compacts. There is a few ways for obtaining the exact solution of two-sided noncooperative games. For finite games, they are based on methods of linear programming or linear inequalities, implying algorithms of Lemke — Howson [9, 10] or of Vorobyov [11] and Kuhn [12]. When the game is infinite, there is no any universal method of solving, but just narrowly specified technique, oriented on particular cases [13, 14]. One of those particularities works on compact games, having solutions at least in mixed strategies [15]. It is worthy to note that the procedure of defining the player's pure strategies set as a compact in Euclidean finite-dimensional subspace is similar to normalization, concerning a payoff function or components of the pure strategy. Without loss of generality, further normalization drives at two-sided noncooperative games on unit hypercubes of Euclidean finite-dimensional spaces, letting identify the game class and its solution way effectiveness, comparing to others. However, even two-sided noncooperative games on unit square in R^2 are solvable only for "good" (for instance, without discontinuities) payoff functions of players, not speaking about games on unit cube in R^3 or hypercubes [16, 17].

Goal and tasks. It is clear that there is no algorithmic approach to solving two-sided noncooperative games on hypercubes of Euclidean finite-dimensional spaces with continuous players' payoff functions [2, 10, 18]. Thus there is a goal to develop a way of solving them approximately via sampling the players' payoff functions. For attaining the goal there are three tasks. Primarily the conditions of acceptance of the sampled payoff functions must be declared. The players' sampled payoff functions are multidimensional matrices, which are to be reshaped into ordinary matrices (flat arrays) of the same format with maintaining one-to-one indexing of their elements. In an assumed solution of the bimatrix game there should be conditioned the consistency of the players' NE-strategies supports: it will allow to predetermine whether the bimatrix game solution changes vastly by changing minimally the sampling step. An enough stable game solution is going to be suggested as the approximate solution of the initial continuous two-sided noncooperative game.

EXPERIMENTAL PART AND RESULTS OBTAINED. Let's consider a two-sided noncooperative game

$$\left\langle U_M, U_N, K_1(\mathbf{X}, \mathbf{Y}), K_2(\mathbf{X}, \mathbf{Y}) \right\rangle \tag{1}$$

with the players' pure strategies sets

$$U_M = \bigtimes_{m=1}^{M} [0; 1] \subset \mathbb{R}^M \tag{2}$$

and

$$U_N = \bigtimes_{n=1}^{N} [0; 1] \subset \mathbb{R}^N, \tag{3}$$

and the players' payoff functions $K_1(\mathbf{X}, \mathbf{Y})$ and $K_2(\mathbf{X}, \mathbf{Y})$, defined on $(M+N)$-dimensional unit hypercube

$$U_M \times U_N = \left\{ \bigtimes_{m=1}^{M} [0; 1] \right\} \times \left\{ \bigtimes_{n=1}^{N} [0; 1] \right\} = \\ = \left\{ \bigtimes_{k=1}^{M+N} [0; 1] \right\} \subset \mathbb{R}^{M+N} \tag{4}$$

by

$$\mathbf{X} = [x_m]_{1\times M} \in \bigtimes_{m=1}^{M} [0; 1] = U_M \subset \mathbb{R}^M \text{ and}$$

$$\mathbf{Y} = [y_n]_{1\times N} \in \bigtimes_{n=1}^{N} [0; 1] = U_N \subset \mathbb{R}^N$$

$$\text{at } M \in \mathbb{N},\ N \in \mathbb{N}. \tag{5}$$

It is assumed that each of the functions $K_1(\mathbf{X}, \mathbf{Y})$ and $K_2(\mathbf{X}, \mathbf{Y})$ is differentiable with respect to any of variables $\left\{ \{x_m\}_{m=1}^{M}, \{y_n\}_{n=1}^{N} \right\}$. Also there exist mixed derivatives of each of those functions by any combination of variables, where every variable is included no more than just once.

Conditions of sampling the game kernels smoothly. Obviously, it's impossible to find all the solutions of the game (1) on the hypercube (4). Therefore consider uniform sampling along each of M dimensions in U_M and each of N dimensions in U_N with a constant step, where endpoints of unit segments are included into the sampling necessarily. If S is a number of intervals between the selected points in each of dimensions then the sampling step is S^{-1}, and $S \in \mathbb{N}$ for the vaguest sampling. In m-th dimension the first player instead of the segment $[0; 1]$ of values of m-th component of its pure strategy $\mathbf{X}$ now possesses the set of points

$$D_m^{\langle \mathbf{X} \rangle}(S) = \left\{ x_m^{\langle s \rangle} \right\}_{s=1}^{S+1} \subset [0; 1]$$

$$\text{by } x_m^{\langle s \rangle} = \frac{s-1}{S} \quad \forall\, m = \overline{1, M}. \tag{6}$$

In n-th dimension the second player instead of the segment $[0; 1]$ of values of n-th component of its pure strategy $\mathbf{Y}$ now possesses the set of points

$$D_n^{\langle \mathbf{Y} \rangle}(S) = \left\{ y_n^{\langle s \rangle} \right\}_{s=1}^{S+1} \subset [0; 1]$$

$$\text{by } y_n^{\langle s \rangle} = \frac{s-1}{S} \quad \forall\, n = \overline{1, N}. \tag{7}$$

Number S shall clearly not be assigned arbitrarily, because the sampling mustn't erase specificities of the players' payoff functions. These specificities consist in local extremums and gradient over hypersurfaces $K_1(\mathbf{X}, \mathbf{Y})$ and $K_2(\mathbf{X}, \mathbf{Y})$. Formally, the head condition of the number S assignment is that $\forall\, s = \overline{1, S}$

$$\frac{\partial^{M+N} K_r(\mathbf{X}, \mathbf{Y})}{\partial x_1 \partial x_2 \ldots \partial x_M \partial y_1 \partial y_2 \ldots \partial y_N} \geqslant 0 \text{ or}$$

$$\frac{\partial^{M+N} K_r(\mathbf{X}, \mathbf{Y})}{\partial x_1 \partial x_2 \ldots \partial x_M \partial y_1 \partial y_2 \ldots \partial y_N} \leqslant 0$$

$$\forall\, x_m \in \left[x_m^{\langle s \rangle}; x_m^{\langle s+1 \rangle} \right] \text{ and}$$

$$\forall\, y_n \in \left[y_n^{\langle s \rangle}; y_n^{\langle s+1 \rangle} \right],\ r \in \{1, 2\}. \tag{8}$$

Requirements (8) mean that if extremums of the players' payoff functions exist off the boundary of the hypercube (4), then they must be reached at points, having only components

$$\left\{ \left\{ \left\{ x_m^{\langle s \rangle} \right\}_{s=2}^{S} \right\}_{m=1}^{M}, \left\{ \left\{ y_n^{\langle s \rangle} \right\}_{s=2}^{S} \right\}_{n=1}^{N} \right\}.$$

Of course, this is too severe condition that can be hardly satisfied, unless the game kernels are artificially configured before. So, an alternative condition of sampling the game kernels smoothly is that $\forall\, s = \overline{1, S}$

$$\left| \frac{\partial^{M+N} K_r(\mathbf{X}, \mathbf{Y})}{\partial x_1 \partial x_2 \ldots \partial x_M \partial y_1 \partial y_2 \ldots \partial y_N} \right| \leqslant \alpha \quad \forall\, x_m \in \left[x_m^{\langle s \rangle}; x_m^{\langle s+1 \rangle} \right]$$

$$\text{and } \forall\, y_n \in \left[y_n^{\langle s \rangle}; y_n^{\langle s+1 \rangle} \right],\ r \in \{1, 2\}, \tag{9}$$

where parameter $\alpha > 0$ is a tolerable unsteadiness of the players' payoff functions. Requirements (9) shall substitute requirements (8) almost in every two-sided noncooperative game with the pre-assigned parameter α. It is pre-assigned on some practical reasonings, concerning the value

$$v_\alpha = \max_{r \in \{1, 2\}} \max_{\mathbf{X} \in U_M} \max_{\mathbf{Y} \in U_N} K_r(\mathbf{X}, \mathbf{Y}) - \\ - \min_{r \in \{1, 2\}} \min_{\mathbf{X} \in U_M} \min_{\mathbf{Y} \in U_N} K_r(\mathbf{X}, \mathbf{Y}). \tag{10}$$

Roughly speaking, the players' payoff functions are sampled sufficiently smooth in points (6) and (7) if, say,

$$\alpha \leqslant 0.01 \cdot v_\alpha \text{ or } \alpha \leqslant 0.001 \cdot v_\alpha \tag{11}$$

for (9). It is more convenient if values of the game kernels are normalized, when $v_\alpha = 1$. Then it is expedient to accept $\alpha \in \{0.001, 0.005, 0.01\}$ conventionally. Nevertheless, the parameter α may be lowered as it depends upon whether the game approximate solution is stable enough, what is going to be spoken about later on.

Reshaping the multidimensional matrix into two-dimensional matrix. After having sampled the hypersurfaces $K_1(\mathbf{X}, \mathbf{Y})$ and $K_2(\mathbf{X}, \mathbf{Y})$ there are $(M+N)$-dimensional matrices $\mathbf{P}_1 = \left[p_J^{\langle 1 \rangle} \right]_{\mathscr{F}}$ and $\mathbf{P}_2 = \left[p_J^{\langle 2 \rangle} \right]_{\mathscr{F}}$ of the format $\mathscr{F} = \bigtimes_{k=1}^{M+N} (S+1)$ instead of them, whose elements

$$p_J^{\langle r \rangle} = K_r(\mathbf{X}, \mathbf{Y}) \text{ by } x_m = S^{-1}(j_m - 1) \quad \forall\, m = \overline{1, M}$$

$$\text{and } y_n = S^{-1}(j_{M+n} - 1) \quad \forall\, n = \overline{1, N} \tag{12}$$

have their indices

$$J = \{j_k\}_{k=1}^{M+N}, \quad j_k \in \left\{\overline{1, S+1}\right\} \quad \forall\, k = \overline{1, M+N}. \tag{13}$$

Inasmuch as first M indices in the sequence (13) of the element (12) of matrix $\mathbf{P}_r = \left[p_J^{\langle r \rangle} \right]_{\mathscr{F}}$ correspond to components of the first player pure strategy, and the last N ones correspond to components of the second player pure strategy, then $(M+N)$-dimensional matrix $\mathbf{P}_r = \left[p_J^{\langle r \rangle} \right]_{\mathscr{F}}$ can be reshaped into ordinary flat matrix

$$\mathbf{G}_r = \left[g_{uw}^{\langle r \rangle} \right]_{(S+1)^M \times (S+1)^N} \tag{14}$$

of the format $(S+1)^M \times (S+1)^N$ with elements $g_{uw}^{\langle r \rangle} = p_J^{\langle r \rangle}$ by $r \in \{1, 2\}$, whose indices are

$$u = \sum_{m=1}^{M} (S+1)^{m-1} \cdot \left(j_{M-m+1} - \operatorname{sign}(m-1) \right) \tag{15}$$

and

$$w = \sum_{n=1}^{N} (S+1)^{n-1} \cdot \left(j_{M+N-n+1} - \operatorname{sign}(n-1) \right). \tag{16}$$

Surely, some pure strategy $z_u^{\langle \mathbf{X} \rangle}(S)$ by its number (15) of the first player is unrolled back to the M-dimensional point of hypercube (2) through indices $\{j_m\}_{m=1}^{M}$:

$$j_M = \psi(u, S+1) + (S+1)\left(1 - \operatorname{sign}\left[\psi(u, S+1)\right]\right),$$

$$j_{M-m} = 1 + \psi\left(\frac{u - j_M - \sum_{m_1=1}^{m-1} (S+1)^{m_1} \left(j_{M-m_1} - 1 \right)}{(S+1)^m}, S+1 \right)$$

$$\forall\, m = \overline{1, M-1}, \tag{17}$$

where function $\psi(a, b)$ returns the fractional part of the ratio $\frac{a}{b}$. And some pure strategy $z_w^{\langle \mathbf{Y} \rangle}(S)$ by its number (16) of the second player is unrolled back to the N-dimensional point of hypercube (3) through indices $\{j_{M+n}\}_{n=1}^{N}$:

$$j_{M+N} = \psi(w, S+1) + (S+1)\left(1 - \operatorname{sign}\left[\psi(w, S+1)\right]\right),$$

$$j_{M+N-n} = 1 + \psi\left(\frac{w - j_{M+N} - \sum_{n_1=1}^{n-1} (S+1)^{n_1} \left(j_{M+N-n_1} - 1 \right)}{(S+1)^n}, S+1 \right)$$

$$\forall\, n = \overline{1, N-1}. \tag{18}$$

Hence, instead of the continuous game (1) by requirements (9) for its kernels on hypercube (4) there can be considered the approximation of this game in the form of bimatrix $(S+1)^M \times (S+1)^N$-game

$$\left\langle \left\{ z_u^{\langle \mathbf{X} \rangle}(S) \right\}_{u=1}^{(S+1)^M}, \left\{ z_w^{\langle \mathbf{Y} \rangle}(S) \right\}_{w=1}^{(S+1)^N}, \mathbf{G}_1, \mathbf{G}_2 \right\rangle. \tag{19}$$

The games (1) and (19) are connected due to the following. In the game (19) pure strategy $z_u^{\langle \mathbf{X} \rangle}(S)$ of the first player corresponds to its strategy $\mathbf{X}$ in the initial game (1) with components $\left\{ x_m = S^{-1}(j_m - 1) \right\}_{m=1}^{M}$, and pure strategy $z_w^{\langle \mathbf{Y} \rangle}(S)$ of the second player corresponds to its strategy $\mathbf{Y}$ in the initial game (1) with components $\left\{ y_n = S^{-1}(j_{M+n} - 1) \right\}_{n=1}^{N}$.

Consistency of supports of the players' NE-strategies. Denote by

$$\left\{ \left\{ p_*\left(z_u^{\langle \mathbf{X} \rangle}(S) \right) \right\}_{u=1}^{(S+1)^M}, \left\{ q_*\left(z_w^{\langle \mathbf{Y} \rangle}(S) \right) \right\}_{w=1}^{(S+1)^N} \right\} \tag{20}$$

an NE-solution of the game (19), where $p_*\left(z_u^{\langle \mathbf{X} \rangle}(S) \right)$ and $q_*\left(z_w^{\langle \mathbf{Y} \rangle}(S) \right)$ are probabilities of applying the strategies $z_u^{\langle \mathbf{X} \rangle}(S)$ and $z_w^{\langle \mathbf{Y} \rangle}(S)$ correspondingly. And let the support of NE-strategy of the first player in the game (19) be the set

$$\left\{ z_{u_i(S)}^{\langle \mathbf{X} \rangle}(S) \right\}_{i=1}^{U_*(S)}, \quad \left\{ u_i(S) \right\}_{i=1}^{U_*(S)} \subset \left\{ \overline{1, (S+1)^M} \right\}, \tag{21}$$

and the support of NE-strategy of the second player in this game be the set

$$\left\{ z_{w_l(S)}^{\langle \mathbf{Y} \rangle}(S) \right\}_{l=1}^{W_*(S)}, \quad \left\{ w_l(S) \right\}_{l=1}^{W_*(S)} \subset \left\{ \overline{1, (S+1)^N} \right\}. \tag{22}$$

Then, returning to the support definition, players within their NE-strategies in the solution (20) possess sets

$$\left\{ p_*\left(z_{u_i(S)}^{\langle \mathbf{X} \rangle}(S) \right) \right\}_{i=1}^{U_*(S)} \tag{23}$$

and

$$\left\{ q_*\left(z_{w_l(S)}^{\langle \mathbf{Y} \rangle}(S) \right) \right\}_{l=1}^{W_*(S)} \tag{24}$$

of nonzero probabilities. Then let $v_{NE}^{\langle 1 \rangle}(S)$ and $v_{NE}^{\langle 2 \rangle}(S)$ be payoffs of the first and second players correspondingly in the game (19):

$$v_{NE}^{\langle r\rangle}(S)=\sum_{u=1}^{(S+1)^M}\sum_{w=1}^{(S+1)^N} g_{uw}^{\langle r\rangle}\cdot p_*\left(z_u^{\langle \mathbf{X}\rangle}(S)\right)\cdot q_*\left(z_w^{\langle \mathbf{Y}\rangle}(S)\right)=$$
$$=\sum_{i=1}^{U_*(S)}\sum_{l=1}^{W_*(S)} g_{u_i(S)w_l(S)}^{\langle r\rangle}\cdot p_*\left(z_{u_i(S)}^{\langle \mathbf{X}\rangle}(S)\right)\cdot q_*\left(z_{w_l(S)}^{\langle \mathbf{Y}\rangle}(S)\right),$$
$$r\in\{1,\,2\}. \qquad (25)$$

It is apparent that there can be selected such a number $S\in\mathbb{N}$, for which the payoffs (25) of the game (19) will be significantly different from the payoffs in the game (1), taken by NE-situation, whose approximation is (20). Another difference is that NE-situations in the game (19) will have structure, being hardly comparable to the corresponding structure of NE-situations in the game (1). So, for acceptance of the game (19) solution as an approximate solution of the game (1) there is needed a sufficient closeness of the players' NE-strategies, being obtained by different versions of the number S. Surely, it is impracticable to sweep the players' NE-strategies in a wide range of S. Therefore first of all it should be conditioned the sufficient closeness of the one player's NE-strategies when the number S is changed minimally.

The spoken closeness may be called the consistency of the players' NE-strategies supports. For its definition consider a piecewise linear hypersurfaces $h_1(u,\,S)$ and $h_2(w,\,S)$, where the hypersurface $h_1(u,\,S)$ vertices are in points

$$\left\{\left\{S^{-1}(j_m-1)\right\}_{m=1}^{M},\ p_*\left(z_u^{\langle \mathbf{X}\rangle}(S)\right)\right\} \qquad (26)$$

in $\mathbb{R}^{M+1}$, and the hypersurface $h_2(w,\,S)$ vertices are in points

$$\left\{\left\{S^{-1}(j_{M+n}-1)\right\}_{n=1}^{N},\ q_*\left(z_w^{\langle \mathbf{Y}\rangle}(S)\right)\right\} \qquad (27)$$

in $\mathbb{R}^{N+1}$. Additionally will mark out the nonzero vertices among (26) and (27) as points of hypercubes U_M and U_N, matching the sets (21) and (22). For this, having unrolled $\{u_i(S)\}_{i=1}^{U_*(S)}$ and $\{w_l(S)\}_{l=1}^{W_*(S)}$ by (17) and (18) back to multidimensional indices, the index $u_i(S)$ is matched to the point

$$\mathbf{X}_i(S)=\left[x_m^{\langle i\rangle}(S)\right]_{1\times M}=\left[S^{-1}\left(j_m^{\langle i\rangle}(S)-1\right)\right]_{1\times M}\in U_M$$
$$\forall\ i=\overline{1,\,U_*(S)}$$

and the index $w_l(S)$ is matched to the point

$$\mathbf{Y}_l(S)=\left[y_n^{\langle l\rangle}(S)\right]_{1\times N}=\left[S^{-1}\left(j_{M+n}^{\langle l\rangle}(S)-1\right)\right]_{1\times N}\in U_N$$
$$\forall\ l=\overline{1,\,W_*(S)}.$$

Furthermore, let the points $\{\mathbf{X}_i(S)\}_{i=1}^{U_*(S)}$ and $\{\mathbf{Y}_l(S)\}_{l=1}^{W_*(S)}$ be sorted into sets

$$\left\{\overline{\mathbf{X}}_i(S)\right\}_{i=1}^{U_*(S)}=$$
$$=\left\{\left[S^{-1}\left(\overline{j}_m^{\langle i\rangle}(S)-1\right)\right]_{1\times M}\right\}_{i=1}^{U_*(S)}=\left\{\mathbf{X}_i(S)\right\}_{i=1}^{U_*(S)}$$

and

$$\left\{\overline{\mathbf{Y}}_l(S)\right\}_{l=1}^{W_*(S)}=$$
$$=\left\{\left[S^{-1}\left(\overline{j}_{M+n}^{\langle l\rangle}(S)-1\right)\right]_{1\times N}\right\}_{l=1}^{W_*(S)}=\left\{\mathbf{Y}_l(S)\right\}_{l=1}^{W_*(S)}$$

that the value

$$\min_{i_1\in\left\{\overline{i+1,\,U_*(S)}\right\}}\sqrt{\sum_{m=1}^{M}\left(\overline{j}_m^{\langle i\rangle}(S)-\overline{j}_m^{\langle i_1\rangle}(S)\right)^2} \qquad (28)$$

is reached at $i_1=i+1$ for each $i=\overline{1,\,U_*(S)-1}$, and the value

$$\min_{l_1\in\left\{\overline{l+1,\,W_*(S)}\right\}}\sqrt{\sum_{n=1}^{N}\left(\overline{j}_{M+n}^{\langle l\rangle}(S)-\overline{j}_{M+n}^{\langle l_1\rangle}(S)\right)^2} \qquad (29)$$

is reached at $l_1=l+1$ for each $l=\overline{1,\,W_*(S)-1}$. Thus here is the definition of the most primitive consistency for the approximate solution of the game (1).

Definition 1. Solution (20) of the game (19) is called weakly consistent for being the game (1) approximate solution if

$$U_*(S+1)\geqslant U_*(S),\ \ W_*(S+1)\geqslant W_*(S), \qquad (30)$$

$$\left|v_{NE}^{\langle r\rangle}(S)-v_{NE}^{\langle r\rangle}(S+1)\right|\leqslant\left|v_{NE}^{\langle r\rangle}(S-1)-v_{NE}^{\langle r\rangle}(S)\right|$$
$$\text{by}\ \ r\in\{1,\,2\}, \qquad (31)$$

$$\max_{i\in\left\{\overline{1,\,U_*(S+1)-1}\right\}}\left\{\sqrt{\sum_{m=1}^{M}\left(\overline{j}_m^{\langle i\rangle}(S+1)-\overline{j}_m^{\langle i+1\rangle}(S+1)\right)^2}\right\}\leqslant$$
$$\leqslant\max_{i\in\left\{\overline{1,\,U_*(S)-1}\right\}}\left\{\sqrt{\sum_{m=1}^{M}\left(\overline{j}_m^{\langle i\rangle}(S)-\overline{j}_m^{\langle i+1\rangle}(S)\right)^2}\right\},$$
$$\max_{l\in\left\{\overline{1,\,W_*(S+1)-1}\right\}}\left\{\sqrt{\sum_{n=1}^{N}\left(\overline{j}_{M+n}^{\langle l\rangle}(S+1)-\overline{j}_{M+n}^{\langle l+1\rangle}(S+1)\right)^2}\right\}\leqslant$$
$$\leqslant\max_{l\in\left\{\overline{1,\,W_*(S)-1}\right\}}\left\{\sqrt{\sum_{n=1}^{N}\left(\overline{j}_{M+n}^{\langle l\rangle}(S)-\overline{j}_{M+n}^{\langle l+1\rangle}(S)\right)^2}\right\}, \qquad (32)$$

$$\max_{U_M}\left|h_1(u,\,S)-h_1(u,\,S+1)\right|\leqslant$$
$$\leqslant\max_{U_M}\left|h_1(u,\,S-1)-h_1(u,\,S)\right|,$$
$$\max_{U_N}\left|h_2(w,\,S)-h_2(w,\,S+1)\right|\leqslant$$
$$\leqslant\max_{U_N}\left|h_2(w,\,S-1)-h_2(w,\,S)\right|, \qquad (33)$$

$$\left\|h_1(u,\,S)-h_1(u,\,S+1)\right\|\leqslant$$
$$\leqslant\left\|h_1(u,\,S-1)-h_1(u,\,S)\right\|\ \text{in}\ \mathbb{L}_2(U_M),$$
$$\left\|h_2(w,\,S)-h_2(w,\,S+1)\right\|\leqslant$$
$$\leqslant\left\|h_2(w,\,S-1)-h_2(w,\,S)\right\|\ \text{in}\ \mathbb{L}_2(U_N). \qquad (34)$$

According to (30) — (34) the meaning of the weak consistency for each of the players' NE-strategies in the solution (20) includes five claims. Namely, with the minimal decrement of the sampling step the cardinality of the support of NE-strategy will not decrease due to (30). Also due to (31) the players' payoffs in the new bimatrix game will change no more than if the sampling step was increased minimally, where those payoffs are taken only by NE-situation, whose approximation is (20). And due to (32) with the minimal decrement of the sampling step there is no decrement in density of points with nonzero probabilities of their selection on the corresponding hypercube. Moreover, due to (33) and (34) with the minimal decrement of the sampling step the piecewise linear hypersurfaces $h_1(u, S)$ and $h_2(w, S)$ approximating the supports of the players' NE-strategies, will change no more than if the sampling step was increased minimally.

The primitiveness of Definition 1 can be obviated with adding conditions of the minimal increment of the sampling step at (30) and (32). Thus the "weakness" disappears.

Definition 2. Solution (20) of the game (19) is called consistent for being the game (1) approximate solution if

$$U_*(S) \geqslant U_*(S-1),\ W_*(S) \geqslant W_*(S-1), \quad (35)$$

$$\max_{i \in \{1, U_*(S)-1\}} \left\{ \sqrt{\sum_{m=1}^{M} \left(\overline{j}_m^{\langle i \rangle}(S) - \overline{j}_m^{\langle i+1 \rangle}(S) \right)^2} \right\} \leqslant$$

$$\leqslant \max_{i \in \{1, U_*(S-1)-1\}} \left\{ \sqrt{\sum_{m=1}^{M} \left(\overline{j}_m^{\langle i \rangle}(S-1) - \overline{j}_m^{\langle i+1 \rangle}(S-1) \right)^2} \right\},$$

$$\max_{l \in \{1, W_*(S)-1\}} \left\{ \sqrt{\sum_{n=1}^{N} \left(\overline{j}_{M+n}^{\langle l \rangle}(S) - \overline{j}_{M+n}^{\langle l+1 \rangle}(S) \right)^2} \right\} \leqslant$$

$$\leqslant \max_{l \in \{1, W_*(S-1)-1\}} \left\{ \sqrt{\sum_{n=1}^{N} \left(\overline{j}_{M+n}^{\langle l \rangle}(S-1) - \overline{j}_{M+n}^{\langle l+1 \rangle}(S-1) \right)^2} \right\} \quad (36)$$

along with (30) — (34).

The consistency is apparently a particular case of weak consistency, reinforced with (35) and (36). The weak consistency reinforcement implies that, firstly, according to (35) with the minimal increment of the sampling step the cardinality of the support of NE-strategy will not increase. Secondly, due to (36) with the minimal increment of the sampling step the density of points with nonzero probabilities of their selection on the corresponding hypercube mustn't increase. Speaking generally, the player's NE-strategy support is consistent, if it is weakly consistent and in the minimal vicinity of the sampling step the NE-strategy support cardinality changes monotonously, as well as the density of points with nonzero probabilities of their selection on the corresponding hypercube. Nonetheless, one should remember that any consistency of the player's NE-strategy support does not mean necessary any consistency of the other player's NE-strategy support.

Certainly, consistency of the game (19) solution (20) does not always refer to unambiguous eligibility of the approximate solution of the game (1). Once again, one should remember that the players' genuine payoffs in the game (1), taken by NE-situation, whose approximation is (20), stay still unknown. And requirements (31) imply just that discrepancies in the players' payoffs would change monotonously (but not to increase with the minimal decrement of the sampling step), although the limits $\lim_{S\to\infty} v_{NE}^{\langle 1 \rangle}(S)$ and $\lim_{S\to\infty} v_{NE}^{\langle 2 \rangle}(S)$ existence is non-asserted. Besides there may be many other NE-situations, giving diverse payoffs for players. Nevertheless, after requirements (30) — (34) are satisfied there appears a preference of applying the approximate solution (20) of the game (1). And if those requirements are supplemented with (35) and (36) then it only reinforces the preference, where monotonicity in the approximate solution (20) becomes "wider".

CONCLUSIONS. The suggested sampling for solving approximately the continuous two-sided noncooperative game (1) on unit hypercube (4) is fulfilled in three stages. Primarily the conditions of sampling the game kernels smoothly in (9) are checked out. This is helped with the value (10) and one of the conditions (11). Then, having got the points in sets (6) and (7) for the assigned number S, $(M+N)$-dimensional matrices $\mathbf{P}_1 = \left[p_J^{\langle 1 \rangle} \right]_{\mathscr{F}}$ and $\mathbf{P}_2 = \left[p_J^{\langle 2 \rangle} \right]_{\mathscr{F}}$ with their elements (12) and indices (13) are reshaped into ordinary flat matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ in (14), whose indices are (15) and (16). At the third stage the game (19) solution (20) is checked out for its consistency. If the solution (20) appears nonconsistent even weakly, then the number of intervals S between the selected points should be increased.

The defined consistency is a particular case of weak consistency, wherein controlling the conditions (36) will take increasingly more time on hypercubes of greater dimensions and hundreds-order S. This is because of the additional sorting of elements of the sets $\left\{ \mathbf{X}_i(S-1) \right\}_{i=1}^{U_*(S-1)}$ and $\left\{ \mathbf{Y}_l(S-1) \right\}_{l=1}^{W_*(S-1)}$ into the sets $\left\{ \overline{\mathbf{X}}_i(S-1) \right\}_{i=1}^{U_*(S-1)}$ and $\left\{ \overline{\mathbf{Y}}_l(S-1) \right\}_{l=1}^{W_*(S-1)}$ within solving the problems alike in (28) and (29). Consequently, "multidimensional" games with very scrupulous approximation are better to check out for weak consistency, what is much faster. All the more so since consistency of the one player's NE-strategy doesn't guarantee even weak consistency of the other player's NE-strategy support. And, generally, consistency isn't necessarily followed with that limits $\lim_{S\to\infty} h_1(u, S)$ and $\lim_{S\to\infty} h_2(w, S)$ exist (anyhow, this unfortunately is not proved yet) and how they are close (in sense of the corresponding functional spaces metrics) to the genuine NE-strategies in the game (1).

Hence, solving the game (1) approximately needs at least the weakly consistent NE-solution. And the appropriate approximation lets solve even games without NE-situations [19, 20]. But shall one player use its (weakly) consistent NE-strategy if the other player's NE-strategy isn't consistent (or just weakly consistent)? And is it possible to determine (weak) consistency of the other player's NE-strategy if the one's has been determined? These questions are motives for further work on sampling the continuous games. Furthermore, the way with generalization in sampling hypercubes U_M and U_N non-uniformly is going to be stated, where also sampling on open and semi-open hypercubes will be discussed.

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ДИСКРЕТИЗАЦИЯ КОНТИНУАЛЬНОЙ ДВУХСТОРОННЕЙ БЕСКОАЛИЦИОННОЙ ИГРЫ НА ЕДИНИЧНОМ ГИПЕРКУБЕ И ПРЕОБРАЗОВАНИЕ МНОГОМЕРНОЙ МАТРИЦЫ ДЛЯ РЕШЕНИЯ СООТВЕТСТВУЮЩЕЙ БИМАТРИЧНОЙ ИГРЫ

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Предлагается метод получения приближённого решения в смысле равновесия по Нешу континуальных двухсторонних бескоалиционных игр на единичном многомерном кубе с помощью равномерной дискретизации вдоль каждого из измерений. Предъявляются требования к тому, чтобы гладко дискретизировать ядра игры, и соотношения, позволяющие осуществлять преобразование многомерных матриц в обычные двухмерные с сохранением взаимнооднозначной индексации между их элементами. Решение в смысле равновесия по Нешу соответствующей биматричной игры проверяется на два типа согласованности, предлагаемые условия которых позволяют судить о том, насколько это решение изменяется при минимальном изменении шага дискретизации.

Ключевые слова: двухсторонние бескоалиционные игры, единичный гиперкуб, многомерная матрица, NE-стратегии (равновесные стратегии по Нешу), приближённое решение, согласованность NE-решения (решения в смысле равновесия по Нешу).

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