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HEAT EXCHANGE MODEL IN THE SPHERICAL AREA

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Purpose. In astrophysics and geophysics one of the important problems is the problem of the temperature field modelling in rotating with the angular velocity of the sphere when the direction of the heat flow intensity is orthogonal to the axis of the rotation. In this paper a mathematical model of the temperature field in spherical area with complex conditions of heat exchange with the environment is considered. The solution of the nonlinear initial boundary value problem is reduced to the solution of the nonlinear integral equation of Fredholm type respect to spatial coordinates and Volterra with the kernel in the form of the Green's function on the time coordinate. **Methodology**. The algorithm of numerical and analytical solution of the initial boundary value problem for the determination of the temperature field of a spherical area with complicated boundary conditions is proposed. **Results.** The initial boundary value problem for the heat equation is converted into the nonlinear integral equation of Hammerstein type and the corresponding quadrature for determining periodical quasi-stationary temperature field by using integral transformations.

Key words: mathematical model, integral equation of Fredholm, Volterra, equation of Hammerstein, quadrature.

МОДЕЛЬ ТЕПЛООБМІНУ У СФЕРИЧНІЙ ОБЛАСТІ

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В астрофізиці та геофізиці однією із важливих задач є задача моделювання температурного поля у кулі, що обертається зі сталою кутовою швидкістю при умові, що тепловий потік перпендикулярний осі обертання. У роботі розглянуто математичну модель температурного поля у сферичній області зі складними умовами теплообміну з оточуючим середовищем. Розв'язок нелінійної початково-крайової задачі зведено до розв'язання нелінійного інтегрального рівняння типу Фредгольма по просторовим координатам і типу Вольтерра по часовій координаті з ядром у вигляді функції Гріна. Запропоновано алгоритм чисельно-аналітичного розв'язку задачі визначення температурного поля у сферичній області зі складними граничними умовами. Шляхом інтегральних перетворень початково-крайова задачу для рівняння теплопровідності зведено до нелінійного інтегрального рівняння типу Гаммерштейна і відповідну квадратуру для визначення періодичного квазістаціонарного температурного поля.

Ключові слова: математична модель, інтегральні рівняння Фредгольма і Вольтерра, рівняння Гаммерштейна, квадратурні формули.

PROBLEM STATEMENT. The research of the heat transfer process is usually associated with carrying out of natural experiments on the temperature and thermal parameters measurements of the surface and inner areas of the object. Installing sensors on the surface thereof is often impossible due to the fact that the analyzed object (for example, an asteroid, a planet, a star) is located at a considerable distance.

Therefore, information about the temperature field is obtained from the limited set of the observation points located inside or on the surface of the investigated object [1]. Taking into account the observed data, based on a mathematical model, by solving the direct and inverse heat conduction problem, we can determine the temperature distribution and parameters of the thermal process [2–8].

One of the main methods of solving of the boundary problems for the heat equation in the areas of canonical form is the method of integral equations, which means replacing the boundary value problem for a differential equation to an integral equation on the surface area. Solutions of the equations are mainly numerical methods [6–14]. The equation solution by means of integrating «smoothes out», *i.e.*, the accuracy of the solution increases. The possibility to solve problems for the areas of any desired shape appears because the problem with the satisfaction of the boundary conditions does not occur.

One of the important problems in astrophysics and geophysics is the problem of the temperature field $U = U(r,\theta,\phi,t)$ modeling in rotating at a constant angular velocity ω of the sphere $r \le R$ when the direction of the constant heat flow intensity q is orthogonal to the rotation axis. From the mathematical point of view, most of the space objects may be considered as the spheres, i.e. bodies limited by sphere [4, 7, 9, 13–15].

PURPOSE. This paper is aimed at constructing of a mathematical model of temperature field of the spherical area with complex conditions of heat exchange with the environment.

EXPERIMENTAL PART AND RESULTS OBTAINED. Considering the temperature field of a planet or an asteroid, which is irradiated by the heat flow from the stars, let us assume that the heat exchange takes place according to Newton's law on the irradiated surface, and according to Newton and Stefan-Boltzmann law on not irradiated surface, then, determining the temperature $U(r,\theta,\phi,t)$ of the sphere, we obtain the following nonlinear initial boundary value problem in the area

$$\Omega \times t = \{0 < r < R, 0 < \theta < \pi, 0 < \phi < 2\pi, t > 0\}$$

$$
\begin{aligned}
&\Delta U-\frac{1}{a^2}U_t=0, && 0<r<R,\quad 0<\theta<\pi,\quad 0<\phi<2\pi,\quad t>0\\
&U(r,\theta,\phi,0)=U_0, && 0<r<R,\quad 0<\theta<\pi,\quad 0<\phi<2\pi\\
&\frac{\partial U}{\partial r}+h_2U\,|_{r=R}=\begin{cases} h_2U_c+\dfrac{q}{\lambda}\sin\theta\sin\phi & 0<\theta<\pi,\ \omega\cdot t<\phi<\omega\cdot t+\pi\\ (h_2-h_1)U+h_1U_c+\kappa\left(U_c^4-U^4\right) & 0<\theta<\pi,\ \omega\cdot t+\pi<\phi<\omega\cdot t+2\pi\end{cases}\\
&U(r,\theta,\phi,t)=U(r,\theta,\phi+2\pi,t),
\end{aligned}
\tag{1}
$$

where Δ – Laplace operator in the spherical coordinates, $a^2=\dfrac{\lambda}{c\rho}$, $h_i=\dfrac{\alpha_i}{\lambda}$, $\kappa=\dfrac{\xi\sigma}{\lambda}$, λ – thermal conductivity, c – heat capacity, ρ – density, α_i – heat transfer coefficient, σ – Stefan-Boltzmann constant, ξ – emissivity of the sphere surface, U_c – cooling medium temperature, R – sphere the radius.

We will search the solution $U(r,\theta,\phi,t)$ in the spherical coordinate system. The solution of (1) we obtain by using the Fredholm type integral equations in the angular coordinates $0<\phi<2\pi$ and $0<\theta<\pi$ and Volterra in the time coordinate for $t>0$ [2–4, 6]. To do this, multiply the equation (1) to $V\,r^2\sin\theta\,dr\,d\theta\,d\phi\,dt$ and integrate with respect to r

$$
\int_0^{\tau+0}\int_{\omega t}^{\omega t+2\pi}\int_0^{\pi}\int_0^{R}V\left(\Delta U-\frac{1}{a^2}U_t\right)r^2\sin\theta dr\,d\theta\,d\phi\,dt=
$$
$$
=\int_0^{\tau+0}\int_{\omega t}^{\omega t+2\pi}\int_0^{\pi}\int_0^{R}V\,\Delta U r^2\sin\theta\,dr\,d\theta\,d\phi\,dt-\frac{1}{a^2}\int_0^{\tau+0}\int_{\omega t}^{\omega t+2\pi}\int_0^{\pi}\int_0^{R}V\,U_t r^2\sin\theta\,dr\,d\theta\,d\phi\,dt.
$$

Taking into account the difference

$$
\int_0^{\tau+0}\int_{\omega t}^{\omega t+2\pi}\int_0^{\pi}\int_0^{R}\left[V\left(\Delta U-\frac{1}{a^2}U_t\right)-U\left(\Delta V+\frac{1}{a^2}V_t\right)\right]r^2\sin\theta\,dr\,d\theta\,d\phi\,dt=
$$

$$
=\int_0^{\tau+0}\int_{\omega t}^{\omega t+2\pi}\int_0^{\pi}\left(V\frac{\partial U}{\partial r}-U\frac{\partial V}{\partial r}\right)\Bigg|_0^R R^2\sin\theta\,d\theta\,d\phi\,dt-\frac{1}{a^2}\int_0^{\tau+0}\int_{\omega t}^{\omega t+2\rho}\int_0^{\pi}\int_0^{R}(U\,V)_t'\,r^2\sin\theta\,dr\,d\theta\,d\phi\,dt.
$$

we get the required second Green formula, where V is a harmonic function

$$
\begin{aligned}
&\int_0^{\tau+0}\int_{\omega t}^{\omega t+2\pi}\int_0^{\pi}\int_0^{R}\left[V\left(\Delta U-\frac{1}{a^2}U_t\right)-U\left(\Delta V+\frac{1}{a^2}V_t\right)\right]r^2\sin\theta\,dr\,d\theta\,d\phi\,dt=\\
&=\int_0^{\tau+0}\int_{\omega t}^{\omega t+2\pi}\int_0^{\pi}\left[V\left(\frac{\partial U}{\partial r}-h_2U\right)-U\left(\frac{\partial V}{\partial r}-h_2V\right)\right]_{r=R}R^2\sin\theta\,d\theta\,d\phi\,dt-\\
&-\frac{1}{a^2}\int_0^{2\pi}d\phi\int_0^{\pi}\int_0^{R}(U\,V)\Big|_0^{\tau+0}r^2\sin\theta\,dr\,d\theta.
\end{aligned}
\tag{2}
$$

Besides Green's formula we introduce the Green function for solving the problem $G=G(r,\theta,\phi,t)$. It can be obtained by solving the dual problem to the problem (1).

$$
\begin{aligned}
&\Delta_{r,\theta,\phi}G+\frac{1}{a^2}G_t=-\delta(r-\rho)\delta(\theta-\xi)\delta(\phi-\psi)\delta(t-\tau),\\
&0<r,\rho<R,\ 0<\theta<\pi,\ 0<\phi<2\pi,\quad t,\tau>0\\
&G=0,\quad t>\tau;\ G<\infty,\ r=0\\
&G_r+h_2G\big|_{r=R}=0,\qquad G\big|_{\phi+2\pi}=G\big|_{\phi},
\end{aligned}
\tag{3}
$$

where $\delta(r-\rho)$, $\delta(\theta-\varsigma)$, $\delta(\phi-\psi)$ – Dirac delta functions [2,3]. Assuming in (2) that $V=G$ and taking into account (1) and (3), we obtain the desired integral equation with respect to $W(\theta,\phi,t)$, when

$$
0<\theta<\pi,\ 0<\phi<2\pi,\ t>0
$$

$$\int_0^{\tau+0}\int_{\omega t}^{\omega t+2\pi}\int_0^{\pi}\int_0^{R} G\cdot 0+U\,\delta(r-\rho)\delta(\theta-\zeta)\delta(\phi-\psi)\delta(t-\tau)r^2\sin\theta\,dr\,d\theta\,d\phi\,dt=$$
$$=\int_0^{\tau+0}\int_{\omega t}^{\omega t+2\pi}\int_0^{\pi}\left[G\left(h_2U_c+\frac{q}{\lambda}\sin\theta\right)\right]\Big|_{r=R}R^2\sin\theta\,d\theta\,d\phi\,dt+$$
$$+\int_0^{\tau+0}\int_{\omega t}^{\xi t+2\pi}\int_0^{\pi}\left[G\left((h_2-h_1)U+h_1U_c+\kappa\left(U_c^4-U^4\right)\right)\right]\Big|_{r=R}R^2\sin\theta\,d\theta\,d\phi\,dt+$$
$$+\frac{1}{a^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}U_0G(r,\rho;\theta,\zeta;\phi-\psi;0-\tau)r^2\sin\theta\,dr\,d\theta\,d\phi.$$

The last equation can be written as

$$U(\rho;\zeta;\psi;\tau)=U_l(\rho;\zeta;\psi;\tau)-$$
$$-\int_0^{\tau}\int_{\omega t}^{\omega t+2\pi}\int_0^{\pi}G(R,\rho;\theta-\zeta;\phi-\psi;t-\tau)\left[(h_1-h_2)W+\kappa W^4\right]R^2\sin\theta\,d\theta\,d\phi\,dt,\qquad(4)$$

where

$$U_l(\rho,\zeta,\psi,\tau)=\frac{U_0}{a^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}G(r,\rho;\theta-\zeta;\phi-\psi;0-\tau)r^2\sin\theta\,dr\,d\theta\,d\phi+$$
$$+\int_0^{\tau}\int_{\omega t}^{\omega t+\pi}\int_0^{\pi}G(R,\rho;\theta-\zeta;\phi-\psi;t-\tau)\left(h_2U_c+\frac{q}{\lambda}\sin\theta\right)R_2\sin\theta\,d\theta\,d\phi\,dt+\qquad(5)$$
$$+\left(h_1U_c+\kappa U_c^4\right)\int_0^{\tau}\int_{\omega t+\pi}^{\omega t+2\pi}\int_0^{\pi}G(R,\rho;\theta-\zeta;\phi-\psi;t-\tau)R^2\sin\theta\,d\theta\,d\phi\,dt.$$

Denoting

$$U(R,\theta,\phi,t)=W(\theta,\phi,t);\;G(\rho;\theta-\zeta;\phi-\psi;t-\tau)=R^2G(R,\rho;\theta-\zeta;\phi-\psi;t-\tau);$$
$$G(\theta-\zeta;\phi-\psi;t-\tau)=G(R;\theta-\zeta;\phi-\psi;t-\tau);\;\left(h_1-h_2+\kappa W^3\right)W=\Phi\left[W(\theta,\phi,t)\right],$$

we obtain the temperature distribution on the sphere surface in the form of

$$W(\zeta,\psi,\tau)=W_l(\xi;\psi;\tau)-\int_0^{\tau}\int_{\omega t+\pi}^{\omega t+2\pi}\int_0^{\pi}G(\theta-\zeta;\phi-\psi;t-\tau)\Phi\left[W(\theta,\phi,t)\right]\sin\theta\,d\theta\,d\phi\,dt.(6)$$

From the obtained value $W(\zeta;\phi;\tau)$ from (6) using numerical methods (4) temperature can be found at any point of the sphere. The solution (3) is the Green's function represented as

$$G(r,\rho;\theta,\zeta;\phi-\psi;t-\tau)=\sum_{j=0}^{\infty}g_{jc}(r,\theta,t)\cos j\phi+g_{js}(r,\theta,t)\sin j\phi.$$

It satisfies the periodicity condition $G|_{\phi+2\pi}=G|_{\phi}$, at that j – an integer, $j=\overline{0,\infty}$. Substituting the assumed form of the solution to the equation of problem (3), we define functions

$$g_{jc}(r,\theta,t)=\frac{1}{\pi}\overline{g}_{jc}(r,\theta,t)\cos j\psi,\qquad g_{js}(r,\theta,t)=\frac{1}{\pi}\overline{g}_{js}(r,\theta,t)\sin j\psi,$$
$$\sum_{j=0}^{\infty}\left(\Delta_{r,\theta}-\frac{j^2}{a^2}\frac{\partial}{\partial t}\right)\left(g_{jc}(r,\theta,t)\cos j\phi+g_{js}(r,\theta,t)\sin j\phi\right)=-\delta(r-\rho)\delta(\theta-\zeta)\delta(\phi-\psi)\delta(t-\tau).$$

We multiply both sides of this equation by $\cos k\phi$ and integrate with respect to ϕ in the interval $(0,2\pi)$. Then we do the same, pre-multiplying the equation by $\sin k\phi$. As a result of transformation, we obtain $\overline{g}_{jc}(r,\theta,t)=\overline{g}_{js}(r,\theta,t)=\overline{g}_{j}(r,\theta,t)$. The Green's function G takes the form

$$G=\frac{1}{\pi}\sum_{j=0}^{\infty}\cos j(\phi-\psi)\,\overline{g}_j(r,\theta,t).\qquad(7)$$

The expression $\overline{g}_j(r,\theta,t)$ derived from the solution of equation.

$$\left(\Delta_{r,\theta}-\frac{j^2}{r^2\sin^2\theta}+\frac{1}{a^2}\frac{\partial}{\partial t}\right)\overline{g}_j(r,\theta,t)=-\delta(r-\rho)\delta(\theta-\zeta)\delta(t-\tau).\quad(8)$$

We substitute in (8) with $\overline{g}_j(r,\theta,t)=\sum_{n=0}^{\infty}P_{nj}(\cos\theta)g_{nj}(r,t)$ and using Legendre's equation [3, 6], we obtain:

$$\sum_{n=0}^{\infty}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{n(n+1)}{r^2}+\frac{1}{a^2}\frac{\partial}{\partial t}\right]P_{nj}(\cos\theta)g_{nj}(r,t)=-\delta(r-\rho)\delta(\theta-\zeta)\delta(t-\tau)\quad(9)$$

where $n(n+1)$ – the eigenvalues, $P_{nj}(\cos\theta)$ – the Legendre associated functions. We multiply both sides of this equation by $P_{kj}(\cos\theta)\sin\theta$ and integrate with respect to θ in the interval $(0,\pi)$

$$\int_0^{\pi}P_{kj}(\cos\theta)\sin\theta\sum_{n=0}^{\infty}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{n(n+1)}{r^2}+\frac{1}{a^2}\frac{\partial}{\partial t}\right)P_{nj}(\cos\theta)g_{nj}(r,t)d\theta=$$
$$=-\delta(r-\rho)\delta(t-\tau)\int_0^{\pi}P_{kj}(\cos\theta)\sin\theta\,\delta(\theta-\zeta)d\theta.$$

Taking into account that

$$\int_0^{\pi}P_{nj}(\cos\theta)P_{kj}(\cos\theta)\sin\theta\,d\theta=\begin{cases}0 & n\neq k\\ \dfrac{2}{(2n+1)}\dfrac{(n+j)!}{(n-j)!} & n=k,\end{cases}$$

we get

$$\sum_{n=0}^{\infty}\frac{2}{2n+1}\frac{(n+j)!}{(n-j)!}\left(\frac{1}{r^2}\frac{\partial}{\partial}\left(r^2\frac{\partial}{\partial r}\right)-\frac{n(n+1)}{r^2}+\frac{1}{a^2}\frac{\partial}{\partial t}\right)\times$$
$$\times g_{nj}(r,t)=-\delta(r-\rho)\delta(t-\tau)P_{nj}(\cos\zeta)$$

Standing for

$$g_{nj}(r,t)=\overline{g}_n(r,t)\frac{(2n+1)(n-j)!P_{nj}(\cos\zeta)}{2(n+j)!},$$

after some manipulations we get Bessel equation

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{n(n+1)}{r^2}+\frac{1}{a^2}\frac{\partial}{\partial t}\right)\overline{g}_n(r,t)=-\delta(r-\rho)\delta(t-\tau),\quad(10)$$

which has the form [4, 8]

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dH}{dr}\right)-\frac{n(n+1)}{r^2}H=-k^2H\,.$$

Function $H=A\frac{1}{\sqrt{r}}J_{n+\frac{1}{2}}(kr)+B\frac{1}{\sqrt{r}}Y_{n+\frac{1}{2}}(kr)$ satisfies this equation. When $r=0$ $G<\infty$, $B=0$. From the boundary condition (3) $G_r+h_2G\big|_{r=R}=0,$ – we get

$$G_r+h_2G=-\frac{1}{2}Ar^{-\frac{3}{2}}J_{n+\frac{1}{2}}(kr)+\frac{A}{\sqrt{r}}kJ'_{n+\frac{1}{2}}(kr)+\frac{h_2}{\sqrt{r}}J_{n+\frac{1}{2}}(kr)=0,$$

or

$$kR\,J'_{n+\frac{1}{2}}(kR)+\left(h_2R-\frac{1}{2}\right)J_{n+\frac{1}{2}}(kR)=0,\quad(11)$$

where $\mu_{n1},\mu_{n2},...$ – positive roots of the transcendental equation (11), $k^2=\left(\frac{\mu_{nm}}{R}\right)^2$ $m=1,2,...,$ $m=\overline{1,\infty}$ – the eigenvalues of Bessel's equation. Taking into account (11), Eq. (10) can be written as

$$\left(-\left(\frac{\mu_{nm}}{R}\right)^2+\frac{1}{a^2}\frac{\partial}{\partial t}\right)\frac{J_{n+\frac{1}{2}}\left(\frac{\mu_{nm}}{R}r\right)}{\sqrt{r}}\overline{g}_n(r,t)=-\delta(r-\rho)\delta(t-\tau),\quad(12)$$

where

$$\overline{g_n}(r,t)=\sum_{m=1}^{\infty}\frac{J_{n+\frac{1}{2}}\left(\frac{\mu_{nm}}{R}r\right)}{\sqrt{r}}g_{nm}(t)\,.$$

Multiplying both sides of the equation (12) on $\frac{r^2}{\sqrt{r}}J_{n+\frac{1}{2}}\left(\frac{\mu_{nk}}{R}r\right)$ and integrating with respect to r in the segment $[0,R]$, we obtain

$$\left(-\left(\frac{\mu_{nm}}{R}\right)^2+\frac{1}{a^2}\frac{\partial}{\partial t}\right)g_{nm}(t)\frac{R^2}{2}\left(1+\frac{(h_2R-n-1)(h_2R+n)}{\mu_{nm}^2}\right)J^2_{n+\frac{1}{2}}(\mu_{nm})=-\delta(t-\tau)\frac{1}{\rho}J_{n+\frac{1}{2}}\left(\frac{\mu_{nm}}{R}\rho\right).$$

Standing for

$$g_{nm}(t)=\bar{g}_{nm}(t)\frac{\frac{1}{\sqrt{\rho}}J_{n+\frac{1}{2}}\left(\frac{\mu_{nm}}{R}\rho\right)}{\frac{R^2}{2}\left(1+\frac{(h_2R-n-1)(h_2R+n)}{\mu_{nm}^2}\right)J^2_{n+\frac{1}{2}}(\mu_{nm})}$$

we obtain the homogeneous equation

$$\left(-\left(\frac{\mu_{nm}}{R}\right)^2+\frac{1}{a^2}\frac{\partial}{\partial t}\right)\bar{g}_{nm}(t)=-\delta(t-\tau),\quad \bar{g}_{nm}(t)=0,\quad t>\tau.$$

Its solution has the form $g_{nm}(t)=Ce^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 t}$, after transformations we obtain

$$\bar{g}_{nm}(t-\tau)=\eta(\tau-t)a^2e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2(t-\tau)}.$$

Taking into consideration the aforesaid, we obtain an expression for the Green function in the for

$$G(r,\rho;\theta,\zeta;\phi-\psi;t-\tau)=$$

$$=\eta(\tau-t)\frac{a^2}{\pi R^2}\sum_{j=0}^{\infty}{}'\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{J_{n+\frac{1}{2}}\left(\frac{\mu_{nm}}{R}r\right)J_{n+\frac{1}{2}}\left(\frac{\mu_{nm}}{R}\rho\right)P_{nj}(\cos\theta)P_{nj}(\cos\zeta)\cos j(\phi-\psi)}{\frac{\sqrt{r\rho}}{2n+1}\frac{(n+j)!}{(n-j)!}\left(1+\frac{(h_2R-n-1)(h_2R+n)}{\mu_{nm}}\right)J^2_{n+\frac{1}{2}}(\mu_{nm})}e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2(t-\tau)}\qquad(13)$$

The symbol «/» means the halving of all members of series at $j=0$. Substituting (13) into (5) we obtain an expression for $U_l(\rho,\zeta,\psi,\tau)$ in the form

$$U_l(\rho,\zeta,\psi,\tau)=\sum_{j=0}^{j=1}U_l^j(\rho,\zeta,\psi,\tau)+\sum_{j=2k+2}^{j=2k+3}U_l^j(\rho,\zeta,\psi,\tau).\qquad(14)$$

We denote by $T=\frac{2\pi}{\omega}$ the duration of one turn of the sphere and consider the time interval $(N+q+1)T$ for the turn $(N+q)T\le\tau\le(N+q+1)T$. If $N\to\infty$ $\lim_{\tau\to\infty}e^{-a^2\left(\frac{\mu_{nm}}{R}\right)^2\tau}=0$, then influence of the initial temperature disappears. At a certain value $N\to\infty$ periodic steady state occurs, $U(\rho,\zeta,\psi,\tau)=U(\rho,\zeta,\psi,\tau')$, where

$$\tau=(N+q)T+\tau'\qquad 0\le\tau'\le T.$$

When $\tau\to 0$ the linear part of Eq. (4) is (11).

Now we calculate the nonlinear part. Consider

$$\int_0^{\tau}\int_{\omega t+\pi}^{\omega t+2\pi}\int_0^{\pi}G(R,\rho;\theta,\zeta;\phi,\psi;t-\tau)\left[(h_1-h_2)W+\kappa W^4\right]R^2\sin\theta\, d\theta\, d\phi\, dt=\frac{a^2}{\pi\sqrt{R}}\times$$

$$\times\sum_{j=0}^{\infty}{}'\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{J_{n+\frac{1}{2}}\left(\frac{\mu_{nm}}{R}\rho\right)P_{nj}(\cos\zeta)\times\int_0^{\pi}P_{nj}(\cos\theta)\sin\theta\, d\theta}{\frac{\sqrt{\rho}}{2n+1}\frac{(n+j)!}{(n-j)!}\left(1+\frac{(h_2R-n-1)(h_2R+n)}{\mu_{nm}^2}\right)J_{n+\frac{1}{2}}(\mu_{nm})}\times$$

$$\times\int_0^{\tau}e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2(t-\tau)}\left(\int_{\omega t+\pi}^{\omega t+2\pi}\cos j(\phi-\psi)\left[(h_1-h_2)W+\kappa W^4\right]d\phi\right)dt.$$

Denote by $F(t)=\int_{\omega t+\pi}^{\omega t+2\pi}\cos j(\phi-\psi)\left[(h_1-h_2)W+\kappa W^4\right]d\phi$ and consider the integral with respect to $\tau=(N+q)T+\tau'\quad n\to\infty.$

$$\int_0^{\tau} F(t) e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 (t-\tau)} dt = \int_0^{NT} + \int_{NT}^{(N+q)T} + \int_{(N+q)T}^{(N+q)T+\tau'} F(t) e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 (t-\tau)} dt,$$

$$\lim_{N\to\infty} \int_0^{NT} F(t) e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 (t-\tau)} dt = \lim_{N\to\infty} \int_0^{NT} F(t) e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 \left(t-(N-t)T-\tau'\right)} dt =$$

$$= \lim_{N\to\infty} e^{-a^2\left(\frac{\mu_{nm}}{R}\right)^2 (N+q)T} \int_0^{NT} F(t) e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 \left(t-\tau'\right)} dt = 0,$$

when $q \sim N \to \infty$.

Consider the integral assuming that $W(\zeta,\psi,\tau)$ is periodic function of the variable τ with a period T

$$\int_{NT}^{(N+q)T} F(t) e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 (t-\tau)} dt = \int_{NT}^{(N+q)T} F(t) e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 \left(t-(N+q)T-\tau'\right)} dt =$$

$$= \sum_{i=1}^{q} \int_{(N+i-1)T}^{(N+i)T} F(t) e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 \left(t-(N+q)T-\tau'\right)} dt =$$

$$= \frac{1}{e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 T} - 1} \int_0^{\tau'} F(t') e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 \left(t'-\tau'\right)} dt' + e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 T} \int_{\tau'}^{T} F(t') e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 \left(t'-\tau'\right)} dt,$$

as

$$\sum_{i=1}^{q} e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 (i-q)T} = \left(1 - e^{-a^2\left(\frac{\mu_{nm}}{R}\right)^2 T}\right)^{-1}.$$

Eqs. (4) and (6) are transformed into the nonlinear integral equation of Hammerstein type [2, 5] and the corresponding quadrature for determining of periodical quasi-stationary temperature field, after such a limiting transition.

$$U\left(\rho,\zeta,\psi,\tau'\right) = U_l\left(\rho,\zeta,\psi,\tau'\right)$$

$$-\frac{a^2}{\pi\sqrt{R}} \sum_{j=0}^{\infty}{}' \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{J_{n+\frac{1}{2}}\left(\frac{\mu_{nm}}{R}\rho\right) P_{nj}(\cos\zeta) \int_0^{\pi} P_{nj}(\cos)\sin\theta\, d\theta}{\frac{\sqrt{\rho}}{2n+1} \cdot \frac{(n+j)!}{(n-j)!} \left(1 + \frac{(h_2 R - n - 1)(h_2 R + n)}{\mu_{nm}^2}\right) J_{n+\frac{1}{2}}(\mu_{nm})} \times$$

$$\times \left(\left(\frac{1}{e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 T} - 1} + 1 \right) \int_0^{\tau'} F(t') e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 \left(t'-\tau'\right)} dt' + \frac{1}{e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 T} - 1} \int_{\tau'}^{T} F(t') e^{a^2\left(\frac{\mu_{nm}}{R}\right)^2 \left(t'-\tau'\right)} dt' \right).$$

CONCLUSIONS. The algorithm of numerical and analytical solution of the initial boundary value problem for the determination of the temperature field of a spherical area with complicated boundary conditions is proposed. The initial boundary value problem for the heat equation is converted into a nonlinear integral equation of Hammerstein type and a corresponding quadrature for determining of periodical quasi-stationary temperature field by using integral transformations.

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МОДЕЛЬ ТЕПЛООБМЕНА В СФЕРИЧЕСКОЙ ОБЛАСТИ

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В астрофизике и геофизике одной из важных задач есть задача моделирования температурного поля во вращающемся с угловой скоростью шаре, когда направление теплового потока ортогонально оси вращения. В работе рассмотрена математическая модель температурного поля в сферической области со сложными условиями теплообмена с окружающей средой. Решение нелинейной начально-краевой задачи сведено к решению нелинейного интегрального уравнения типа Фредгольма по пространственным координатам и типа Вольтерра по временной координате с ядром в виде функции Грина. Предложен алгоритм численно – аналитического решения начально-краевой задачи определения температурного поля сферической области со сложными граничными условиями. В результате интегральных преобразований начально-краевая задача для уравнений теплопроводности преобразована в нелинейное интегральное уравнение типа Гаммерштейна и соответствующую квадратуру для определения периодического квазистационарного температурного поля.

Ключевые слова: математическая модель, интегральные уравнения Фредгольма, Вольтерра, уравнение Гаммерштейна, квадратурные формулы.

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