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PROPAGATION AND DIFFRACTION OF TM WAVE ON PERIODIC IMPEDANCE GRATING ABOVE A FLAT SCREEN REFLECTOR

Boundary-value problems in integral equation form for the stationary wave equations lead to problems of diffraction on plane-parallel structures from the theory of electromagnetic waves. They have been studied in the monograph [1] and in papers [2] - [4], [9], [10]. Problems of modeling the interaction of electromagnetic (EM) radiation with pre-fractal [8] structures, among which are the limited diffraction gratings with the flat screen reflector [1], [9], have been actual to the present day.

The overall aim of this work is to build a discrete mathematical model of the diffraction problem of a plane monochromatic TM wave on periodic impedance pre-Cantor strips, placed above a flat screen reflector. The discrete mathematical model is based on hypersingular integral equations (HSIE) of the 1st kind and on the Fredholm integral equations of 2nd kind. It allows one to perform numerical simulations of this diffraction problem with the help of an efficient discrete singularities method (DSM).

Diffraction problem of TM mode

We consider the diffraction problem of an H-polarized plane wave on periodic impedance pre-Cantor strips, located above a flat screen reflector. In this case the components of electric and magnetic fields are $(H_x, 0, 0), (0, E_y, E_z)$. Propagation direction of a plane wave is given by the direction of wave vector $\vec{k}$, where $k = \varepsilon\mu\omega^2$ (Fig. 1).

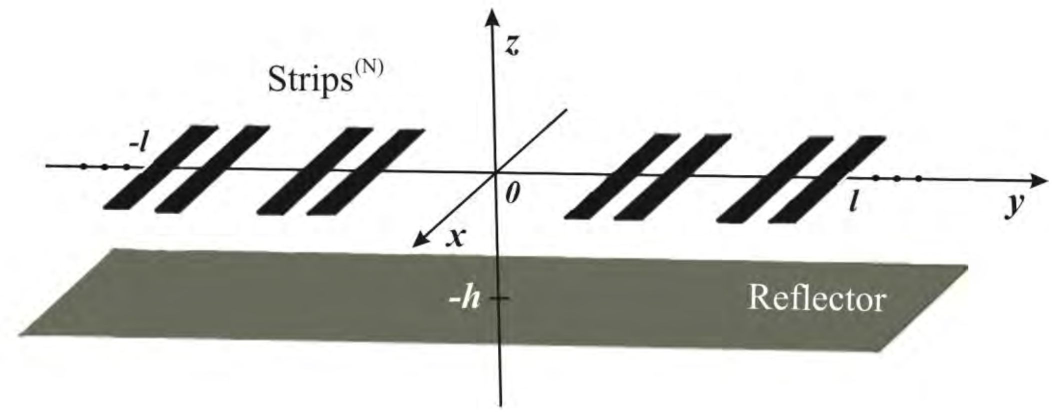

Fig. 1. Schematic of the considered diffraction structure.

$$Strips^{(N)} = \{(x, y, z) \in \Re^3, y \in St_{2l}^{(N)}, z = 0\},$$
$$St_{2l}^{(N)} = \bigcup_{r=-\infty}^{+\infty} \bigcup_{q=1}^{2^N} \left(a_q^N + 2l \cdot r, b_q^N + 2l \cdot r\right) \quad (1)$$

For convenience let us switch to dimensionless coordinates and 2π period:

$$\xi = \frac{\pi}{l} y, \zeta = \frac{\pi}{l} z, \kappa = \frac{l}{\pi} k, d = \frac{\pi}{l} h, \kappa = \frac{l}{\pi} k, \alpha_q^N = \frac{\pi}{l} a_q^N, \beta_q^N = \frac{\pi}{l} b_q^N,$$

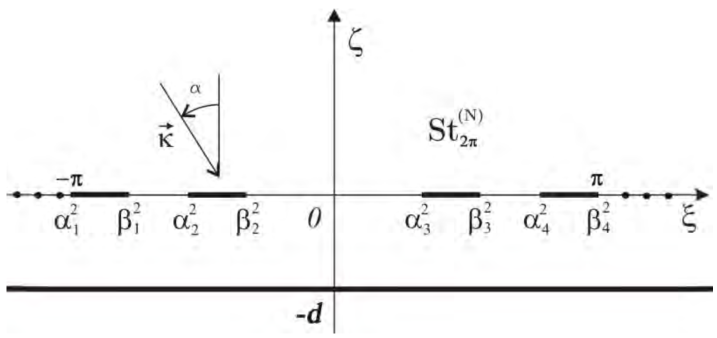

Fig. 2. Cross-section of the diffraction structure in $\zeta\xi$ plane.

$$St_{2\pi}^{(N)} = \bigcup_{q=1}^{2^N} \left(\alpha_q^N, \beta_q^N\right).$$

To solve a 2D diffraction problem we calculate the total field $H_\varsigma(\xi,\zeta)$ which satisfies Maxwell's equations, supplemented with Shchukin-Leontovich impedance boundary conditions. Besides, the total field must also satisfy the Sommerfeld radiation conditions and the Meixner edge condition.

The H-polarized plane wave falls from infinity at an angle α :

$$u_{inc}^N(\xi,\zeta) = H_\varsigma(\xi,\zeta) = e^{i\kappa(\xi\sin\alpha - \zeta\cos\alpha)}. \qquad (2)$$

The only non-zero component of magnetic field satisfies all the aforementioned conditions and also the two-dimensional Helmholtz equation off the metallic strips above flat screen reflector:

$$\frac{\partial^2}{\partial\xi^2} H_\varsigma(\xi,\zeta) + \frac{\partial^2}{\partial\zeta^2} H_\varsigma(\xi,\zeta) + \kappa H_\varsigma(\xi,\zeta) = 0, \quad -d < \zeta < 0. \qquad (3)$$

The total field $u^{(N)}(\xi,\zeta) = H_\varsigma(\xi,\zeta)$ is considered to be in the form of

$$u^{(N)}(\xi,\zeta) = \begin{cases} u_0^N(\xi,\zeta) + u_+^N(\xi,\zeta), & \zeta > 0, \\ u_0^N(\xi,\zeta) + u_-^N(\xi,\zeta), & -d < \zeta < 0, \end{cases} \qquad (4)$$

where $u_0^{(N)}(y,z)$ is a known solution to the Helmholtz equation. It represents the sum of incident and reflected waves in flat reflector. The functions $u_+^{(N)}(\xi,\zeta)$, $u_-^{(N)}(\xi,\zeta)$ are considered as Fourier series:

$$u_+^N(\xi,\zeta) = \sum_{s=-\infty}^{+\infty} C_{s+}^N e^{i\lambda_s\xi - \gamma(\lambda_s)\zeta}, \quad \zeta > 0, \qquad (5)$$

where $\gamma(\lambda_s) = \sqrt{\lambda_s^2 - \kappa^2}$, $\lambda_s = \frac{\pi s}{l}$, $s \in Z$. In this case the Helmholtz equation is satisfied for $u_+^N(\xi,\zeta)$, $\zeta > 0$. The radiation condition will be fulfilled if γ_s is given by $\mathrm{Re}(\gamma_s) \geq 0, \mathrm{Im}(\gamma_s) \leq 0$;

$$u_-^N(\xi,\zeta) = \sum_{s=-\infty}^{+\infty} C_{s-}^N Z(\lambda_s,\zeta) e^{i\lambda_s\xi}, \quad -d < \zeta < 0, \qquad (6)$$

where

$$Z(\lambda_s,\zeta) = \frac{Bsh(\gamma(\lambda_s)(\zeta+d)) + \gamma(\lambda_s)ch(\gamma(\lambda_s)(\zeta+d))}{Bch(\gamma(\lambda_s)d) + \gamma(\lambda_s)sh(\gamma(\lambda_s)d)},$$

In this case, the function $u_-^N(\xi,\zeta)$ (6) satisfies the Helmholtz equation and the following conditions for the function $Z(\lambda_s,\zeta)$, $-d < \zeta < 0$ are kept:

$$Z'_z(\lambda_s,0) = \gamma(\lambda_s), \qquad Z'_z(\lambda_s,-d) - BZ(\lambda_s,-d) = 0,$$

so that the boundary condition on the flat reflector is fulfilled.

In this paper the incident angle is considered as zero. As shown in the monograph [1] and paper [2] the boundary value problem considering all mentioned conditions is reduced to two systems of coupled integral equations:

$$\begin{cases} \sum\limits_{s=-\infty}^{+\infty}\left(C_{s+}^{N}-C_{s-}^{N}Z_s(0)\right)e^{is\xi}=0,\ \xi\in CSt_{2\pi}^{(N)}, \\ \sum\limits_{s=-\infty}^{+\infty}\left((\gamma_s+B)C_{s+}^{N}-(\gamma_s+BZ_s(0))C_{s-}^{N}\right)e^{is\xi}=2\dfrac{\partial u_0^N}{\partial\zeta}(\xi,0),\ \xi\in St_{2\pi}^{(N)}. \end{cases} \tag{7}$$

$$\begin{cases} \sum\limits_{s=-\infty}^{+\infty}\gamma_s\left(C_{s-}^{N}+C_{s+}^{N}\right)e^{is\xi}=0,\ \xi\in CSt_{2\pi}^{(N)}, \\ \sum\limits_{s=-\infty}^{+\infty}\left((\gamma_s+B)C_{s+}^{N}+(\gamma_s+BZ_s(0))C_{s-}^{N}\right)e^{is\xi}=-2Bu_0^N(\xi,0),\ \xi\in St_{2\pi}^{(N)}. \end{cases} \tag{8}$$

Let us introduce two new unknown functions $F_1^N(\eta), F_2^N(\eta)$ and representations of unknown coefficients C_{s+}^N, C_{s-}^N:

$$F_1^N(\eta)=\sum_{s=-\infty}^{\infty}\left(C_{s+}^{N}-C_{s-}^{N}Z_s(0)\right)e^{is\xi},\quad F_2^N(\eta)=\sum_{s=-\infty}^{+\infty}\gamma_s\left(C_{s-}^{N}+C_{s+}^{N}(s)\right)e^{is\xi}, \tag{9}$$

$$C_{s+}^{N}-C_{s-}^{N}Z_s(0)=\frac{1}{2\pi i}\int\limits_{St_{2\pi}^{(N)}}F_1^N(\eta)e^{-is\eta}d\eta,\quad C_{s-}^{N}+C_{s+}^{N}=\frac{1}{2\pi i\gamma_s}\int\limits_{St_{2\pi}^{(N)}}F_2^N(\eta)e^{-is\eta}d\eta,\ s\in Z. \tag{10}$$

Using parametric representations of hypersingular integral operators of periodic functions and integral operators with logarithmic kernels this problem is reduced [8] to hypersingular integral equation of the first kind on a set of intervals, and the Fredholm integral equation of the second kind:

$$\begin{gathered} \frac{1}{\pi}\int\limits_{St_{2\pi}^{(N)}}\frac{F_1^N(\eta)}{(\eta-\xi)^2}d\eta+\frac{C_1}{\pi}\int\limits_{St_{2\pi}^{(N)}}\ln|\eta-\xi|F_1^N(\eta)d\eta+\frac{1}{\pi}\int\limits_{St_{2\pi}^{(N)}}Q_1^N(\eta,\xi)F_1^N(\eta)d\eta+ \\ +\frac{1}{\pi}\int\limits_{St_{2\pi}^{(N)}}Q_2^N(\eta,\xi)F_2^N(\eta)d\eta=f_1^N(\xi),\ \xi\in St_{2\pi}^{(N)}, \\ F_2^N(\xi)+\frac{C_2}{\pi}\int\limits_{St_{2\pi}^{(N)}}\ln|\eta-\xi|F_2^N(\eta)d\eta+\frac{1}{\pi}\int\limits_{St_{2\pi}^{(N)}}Q_3^N(\eta,\xi)F_2^N(\eta)d\eta+ \\ +C_3F_1^N(\xi)+\frac{1}{\pi}\int\limits_{St_{2\pi}^{(N)}}Q_4^N(\eta,\xi)F_1^N(\eta)d\eta=f_2^N(\eta),\ \xi\in St_{2\pi}^{(N)}, \end{gathered} \tag{11}$$

where $f^{(N)}(\eta), g^{(N)}(\eta), K^{(N)}(\eta,\xi), Q^{(N)}(\eta,\xi), P^{(N)}(\eta,\xi), R^{(N)}(\eta,\xi)$ are known functions.

Discrete mathematical model of the hypersingular integral equations (HSIE)

To reduce the equation (11) on $St_{2\pi}^{(N)}$ to equations on a set of intervals $St_q^{(N)}=(\alpha_q^N,\beta_q^N)$, $q=\overline{1,2^N}$, we introduce restrictive conditions to functions:

$$F_{i,q}^{N}(\eta)=F_{i}^{N}(\eta)\Big|_{\eta\in St_{q}^{(N)}},\quad F_{i,p}^{N}(\xi)=F_{i}^{N}(\xi)\Big|_{\xi\in St_{q}^{(N)}},$$
$$f_{i,p}^{N}(\xi)=f_{i}^{N}(\xi)\Big|_{\xi\in St_{q}^{(N)}},\ i=1,2,\ p,q=\overline{1,2^{N}}. \tag{12}$$

The Meixner condition will be satisfied if unknown functions (8) are in the form of:

$$F_{i,q}^{N}(\eta)=v_{i,q}^{N}(\eta)\sqrt{\left(\beta_{q}^{N}-\eta\right)\left(\eta-\alpha_{q}^{N}\right)},\ q=\overline{1,2^{N}},i=1,2, \tag{13}$$

$$F_{i,p}^{N}(\xi)=v_{i,p}^{N}(\xi)\sqrt{\left(\beta_{p}^{N}-\xi\right)\left(\xi-\alpha_{p}^{N}\right)},\quad p=\overline{1,2^{N}},i=1,2, \tag{14}$$

where functions $v_{i,q}^{N}(\eta),v_{i,p}^{N}(\xi),i=1,2,\ p,q=\overline{1,2^{N}}$, are Hölder continuous.

By choosing a normalized interval (-1,1) the variables transform accordingly:

$$g_{q}^{(N)}:(-1,1)\mapsto(\alpha_{q}^{N},\beta_{q}^{N}):t\mapsto g_{q}^{(N)}(t)=\frac{\beta_{q}^{N}-\alpha_{q}^{N}}{2}t+\frac{\beta_{q}^{N}+\alpha_{q}^{N}}{2},$$
$$\eta=g_{q}^{(N)}(t),\ \xi=g_{q}^{(N)}(t_{0}),\ |t|<1,\ |t_{0}|<1,\ \eta\in St_{q}^{(N)},\ \xi\in St_{p}^{(N)}. \tag{15}$$

Then, for $|t|<1,|t_{0}|<1,\ p,q=\overline{1,2^{N}}$:

$$F_{i,q}^{N}(g_{q}^{(N)}(t))=v_{i,q}^{N}(t)\frac{\beta_{q}^{N}-\alpha_{q}^{N}}{2}\sqrt{1-t^{2}},\quad F_{i,p}^{N}(g_{p}^{(N)}(t_{0}))=v_{i,p}^{N}(t)\frac{\beta_{p}^{N}-\alpha_{p}^{N}}{2}\sqrt{1-t_{0}^{2}}. \tag{16}$$

Considering (12)-(16) and excluding logarithmic singularity at *p*=*q* from equations (11), we obtain a system of boundary hypersingular integral equations of the 1st kind and Fredholm integral equation of the 2nd kind on a normalized interval for $p,q=\overline{1,2^{N}}$:

$$\begin{aligned}
&\frac{1}{\pi}\int_{-1}^{1}\frac{v_{1,p}^{N}(g_{p}^{(N)}(t))}{(t-t_{0})^{2}}\sqrt{1-t^{2}}\,dt+\frac{C_{1}}{\pi}\int_{-1}^{1}\ln|t-t_{0}|v_{1,p}^{N}(g_{p}^{(N)}(t))\sqrt{1-t^{2}}\,dt+\\
&+\frac{1}{\pi}\int_{-1}^{1}P_{1}^{N}(t,t_{0})v_{1,p}^{N}(g_{p}^{(N)}(t))\sqrt{1-t^{2}}\,dt+\\
&+\frac{1}{\pi}\int_{-1}^{1}P_{2}^{N}(t,t_{0})v_{2,p}^{N}(g_{p}^{(N)}(t))\sqrt{1-t^{2}}\,dt=f_{1}^{N}(g_{p}^{(N)}(t)),\ |t_{0}|<1,\\
&\frac{\beta_{p}^{N}-\alpha_{p}^{N}}{2}v_{2,p}^{N}(g_{p}^{(N)}(t_{0}))+\frac{C_{2}}{\pi}\int_{-1}^{1}\ln|t-t_{0}|v_{2,p}^{N}(g_{p}^{(N)}(t))\sqrt{1-t^{2}}\,dt+\\
&+\frac{1}{\pi}\int_{-1}^{1}P_{3}^{N}(t,t_{0})v_{2,q}^{N}(g_{q}^{(N)}(t))\sqrt{1-t^{2}}\,dt+C_{3}\frac{\beta_{p}^{N}-\alpha_{p}^{N}}{2}v_{1,p}^{N}(g_{p}^{(N)}(t_{0})+\\
&+\frac{1}{\pi}\int_{-1}^{1}P_{4}^{N}(t,t_{0})v_{1,q}^{N}(g_{q}^{(N)}(t))\sqrt{1-t^{2}}\,dt=f_{2}^{N}(g_{p}^{(N)}(t_{0})),\ |t_{0}|<1
\end{aligned} \tag{16}$$

Discrete mathematical model has been developed with the help of an efficient numerical method DSM [1], [5], [6]. The unknown functions and smooth functions are interpolated by a Lagrange polynomial of (*n*-2) -th degree in the nodes which are the zeros of Chebyshev polynomials of the 2st kind. From (16) we have obtained a system of approximate solutions. In the next step, using quadrature formulas [7] for integrals with hypersingular and logarithmic singularity and integrals of smooth functions, we derive a system of linear algebraic equations (SLAE) for the values of unknown functions in the node points. Therefore we have obtained a SLAE for $p=\overline{1,2^{N}}$:

$$\sum_{q=1}^{2^N}\left(\sum_{\substack{j=1\\ j\neq k}}^{n-1} v_{1,p}^{N,(n-2)}(g_p^{(N)}(t_{0j}^n))\frac{(1-(-1)^{j+k})(1-(t_{0j}^n)^2)}{(t_{0k}^n-t_{0j}^n)^2}\frac{1}{n}-v_{1,p}^{N,(n-2)}(g_p^{(N)}(t_{0k}^n))\frac{n}{2}\right)-$$

$$-\frac{C_1}{n}\sum_{q=1}^{2^N}\sum_{j=1}^{n-1} v_{1,p}^{N,(n-2)}(g_p^{(N)}(t_{0j}^n))(1-(t_{0j}^n)^2)\left[\ln 2+2\sum_{k=1}^{n-1}\frac{T_k(t_{0j}^n)}{k}T_k(t_{0k}^n)+\frac{(-1)^j}{n}T_k(t_{0k}^n)\right]+$$

$$+\frac{1}{n}\sum_{q=1}^{2^N}\sum_{j=1}^{n-1} P_{1,q,p}^N(t_{0j}^n,t_{0k}^n)v_{1,q}^{N,(n-2)}(g_q^{(N)}(t_{0j}^n))(1-(t_{0j}^n)^2)+$$

$$+\frac{1}{n}\sum_{q=1}^{2^N}\sum_{j=1}^{n-1} P_{2,q,p}^N(t_{0j}^n,t_{0k}^n)v_{2,q}^{N,(n-2)}(g_q^{(N)}(t_{0j}^n))(1-(t_{0j}^n)^2)=f_1^{N,(n-2)}(g_p^{(N)}(t_{0k}^n)),$$

$$\frac{\beta_p^N-\alpha_p^N}{2}v_{2,p}^{N,(n-2)}(g_p^{(N)}(t_{0k}^n))-$$

$$-\frac{C_2}{n}\sum_{q=1}^{2^N}\sum_{j=1}^{n-1} v_{2,p}^{N,(n-2)}(g_p^{(N)}(t_{0j}^n))(1-(t_{0j}^n)^2)\left[\ln 2+2\sum_{k=1}^{n-1}\frac{T_k(t_{0j}^n)}{k}T_k(t_{0k}^n)+\frac{(-1)^j}{n}T_k(t_{0k}^n)\right]+$$

$$+\frac{1}{n}\sum_{q=1}^{2^N}\sum_{j=1}^{n-1} P_{3,q,p}^N(t_{0j}^n,t_{0k}^n)v_{2,q}^{N,(n-2)}(g_q^{(N)}(t_{0j}^n))(1-(t_{0j}^n)^2)+C_3\frac{\beta_p^N-\alpha_p^N}{2}v_{1,p}^{N,(n-2)}(g_p^{(N)}(t_{0k}^n))+$$

$$+\frac{1}{n}\sum_{q=1}^{2^N}\sum_{j=1}^{n-1} P_{4,q,p}^N(t_{0j}^n,t_{0k}^n)v_{1,q}^{N,(n-2)}(g_q^{(N)}(t_{0j}^n))(1-(t_{0j}^n)^2)=f_2^{N,(n-2)}(g_p^{(N)}(t_{0k}^n)).$$

Discrete mathematical model of the diffraction problem of H-polarized EM waves on the gratings, consisting of periodic impedance pre-Cantor strips with an impedance flat reflector underneath, has been developed with the help of an efficient DSM. As a next step we plan to perform numerical simulations and analyze the performance of this approach for both E- and H-polarization cases.

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