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# DISCRETE-TIME CONTROL OF LINEAR MULTIVARIABLE SYSTEMS WITH EITHER SINGULAR OR ILL-CONDITIONED TRANSFER FUNCTION MATRICES

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**Abstract.** *The control of multivariable linear discrete-time, time-invariant systems whose transfer function matrices are either singular or ill-conditioned is considered. It is assumed that there are arbitrary unmeasurable but bounded disturbances, and the parameters of these systems may be somewhat unknown. The optimal controller is derived by using the pseudoinverse of the system transfer function matrix. The boundedness of all signals caused by this controller and also the robustness properties of the controller in the presence of parameter uncertainty are proved. Numerical examples are given to support the theoretical investigations.*

**Keywords:** bounded disturbance; discrete time; noninvertible matrix; multivariable system; optimality pseudo-inversion

## 1. Introduction

The problem of controlling multivariable systems in the presence of unmeasurable disturbances stated several decades ago in [10] remains actual up to now. This is an important problem from a theoretical and practical point of view.

Since the seventies, the internal model method become popular among other methods dealing with an improvement of the control system by exploiting the different types of plant and disturbances models.

Based on this principle, multivariable regulator problem was first approached in [4].

A reformulation of this problem utilizing the geometric approach tools is reported in the book [11].

A perspective modification of the internal model control principle is the so-called model inverse approach.

The perfect output control performance is an important multivariable control problem closely related to inverse systems.

The problem of inversion of linear time-invariant multivariable systems has attracted the attention of several researches [3, 6, 8].

During last years, a significant progress in this research area has been achieved in [7].

Most of these works except [7] dealt with continuous-time multivariable systems.

To the best of our knowledge, an inverse model approach to regulating discrete-time systems described by the first-order difference equations was first advanced in [5].

Unfortunately, the inverse model approach is not appropriate when the transfer function matrix is singular, because the one becomes noninvertible.

There are different methods to deal with possible noninvertibility in [6].

It turned out that the so-called pseudoinverse model can be exploited in closed loop to cope with the singularity of multivariable system.

Such an approach is based on the pseudoinverse transfer function matrix which is the generalized inverse matrix [1, 9].

In summary, the main contribution of this paper is the utilization of the pseudoinverse model concept as a tool for dealing with the control of some multivariable systems whose transfer function matrix are singular or ill-conditioned, and the proofs of the boundedness, optimality and robustness results.

Therein, the bridge between iterative algorithms for solving singular linear equations [9] and similar control algorithms for controlling these systems is provided.

## 2. Problem statement

The plant to be controlled is a linear multivariable time-invariant, discrete-time system described by

$$y_n = Bu_{n-1} + v_n, \qquad (1)$$

where $B$ represent a $N \times N$ transfer function matrix;

$$y_n = [y_n^{(1)}, \ldots, y_n^{(N)}]^{\mathrm{T}},$$

$$u_n = [u_n^{(1)}, \ldots, u_n^{(N)}]^{\mathrm{T}},$$

$$v_n = [v_n^{(1)}, \ldots, v_n^{(N)}]^{\mathrm{T}}$$

are the $N$-dimensional output vector, control input vector and unmeasurable external disturbance vector, respectively, $n = 1, 2, \ldots$ denotes the discrete time.

The following basic assumptions are made.

A1. $B$ is the singular matrix, i.e.,

$$\det B = 0, \quad (2)$$

or ill-conditioned matrix implying

$$\text{cond } B = \|B\|\|B^{-1}\| >> 1, \quad (3)$$

where cond $B$ denotes the conditionality number of $B$.

A2. The components $v_n^{(i)}$ of $v_n$ are upper bounded in modulus by some $\varepsilon_i s$ for all $n$

$$\left|v_n^{(i)}\right| \le \varepsilon_i \quad (i = 1,\ldots, N) \quad (4)$$

meaning that the norm of $v_n$ satisfies

$$\|v_n\| \le \varepsilon \quad (5)$$

with $\varepsilon = (\varepsilon_1^2 + \cdots + \varepsilon_N^2)^{1/2}$.

A3. The control sequence

$$\{u_n\} = u_0, u_1, u_2, \ldots$$

may be constrained $U$ according to

$$\|u_n\| \le U < \infty. \quad (6)$$

It is assumed that the matrix $B$ may be exactly unknown, in principle, whereas rank $B$ needs to be known.

*Comment.* Equation (1) may describe a process control with the sampling period $T_0$ larger than the transient time [5].

Let

$$y^0 = [y^{0(1)},\ldots, y^{0(N)}]^T \quad (y^0 \equiv \text{const}).$$

denote a given reference set-point vector for $y_n$ satisfying

$$|y^{0(1)}| + \ldots + |y^{0(N)}| \ne 0.$$

Define the output error vector $e_n$ as

$$e_n := y^0 - y_n \quad (7)$$

and introduce the semi-norm $\|e\|_{ss}$ of $\{\|e_n\|\}$ given by

$$\|e\|_{ss} := \limsup_{n\to\infty} \|e_n\| \quad (8)$$

to evaluate the ultimate (asymptotical) behavior of the control system.

The problem is to design a linear controller minimizing the upper bound on $\|e\|_{ss}$ defined in (8) for any finite $y^0$ from the $N$-dimensional Euclidean space $\mathrm{IR}^N$ provided that the resulting control system is Bounded-input Bounded-state (BIBS) stable.

Moreover, the robustness properties of such a controller need to be examined.

## 3. Preliminaries

It is known that if $B$ is non-singular, that is $\det B \ne 0$, then the feedback control

$$u_n = u_{n-1} + Ae_n \quad (9)$$

together with (7) solves immediately the problem stated above by setting

$$A = B^{-1}, \quad (10)$$

where $B^{-1}$ denotes the inverse matrix of $B$.

Namely, the choice of $A$ in the form (10) gives that the control objective

$$\|e\|_{ss} \to \inf_{\{A\}} \quad (11)$$

in which $\{A\}$ represents a set of admissible $A$s is achieved for arbitrary

$$y^0 \in \mathrm{IR}^N.$$

To implement the control law (9), we need the inverse model described by the equation

$$\Delta u_n = Ae_n \quad (12)$$

with $A$ given by (10) and also the discrete integrator whose output is

$$u_n = \sum_{k=1}^{n} \Delta u_k. \quad (13)$$

In view of (12), (13), the controller contained in the closed loop plays the role of an I-type multivariable controller whose matrix gain is $A$ (Fig. 1).

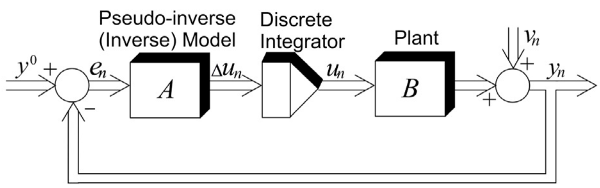


**Fig. 1.** Configuration of the closed-loop control system (1), (7), (9)

The controller (7), (9), (10) causes that the each – $i$-th current error

$$e_n^{(i)} := y^{0(i)} - y^{(i)}$$

will follow the corresponding change of $i$-th disturbance $v_n^{(i)}$ defined by

$$\Delta v_n^{(i)} := v_n^{(i)} - v_{n-1}^{(i)}$$

during the one sampling period $T_0$ with the accuracy of its sign:

$$e_n^{(i)} \equiv -\Delta v_n^{(i)}. \quad (14)$$

Thereby, the control law

$$u_n = u_{n-1} + B^{-1}e_n \quad (15)$$

obtained simply by substituting (10) into (9) is optimal.

Due to (4), (5), (14) it gives

$$\limsup_{n\to\infty} \left|e_n^{(i)}\right| \le 2\varepsilon_i$$

leading to

$$\limsup_{n\to\infty} \|e_n\| \le 2\varepsilon. \quad (16)$$

The equations (1), (15) together with (7) yield

$$u_n = B^{-1}y^0 - B^{-1}v_n.$$

from which, using (5), we obtain

$$\|u_n\| \le \|B^{-1}\|\left(\|y^0\| + \varepsilon\right) < \infty \quad (n = 1, 2, \ldots). \quad (17)$$

This and (16) imply that the vector

$$w_n = [u_n^{\mathrm{T}}, y_n^{\mathrm{T}}]^{\mathrm{T}}$$

remains bounded in the norm for all integer $n \in [1, \infty)$, meaning $\{w_n\} \in \ell_\infty$, where $\ell_\infty$ denotes the space of all bounded sequences.

Now, let us assume that the transfer function matrix $B$ in (1) is singular.

Then the control (9) with $A$ given by the inversion (10) is impossible.

In this case, a certain condition given below is necessary to avoid the instability of the closed-loop control system (1), (7), (9).

Before going to present this condition, the kernel and the image of an arbitrary matrix $H$ [9] denoted by $\ker H$ and $\operatorname{im} H$, respectively, are introduced.

Theorem 1. Let Assumption A1 be valid.

Suppose the set-point vector $y^0$ satisfies

$$y^0 \notin \operatorname{im} B$$

and the disturbances $v_n$ are absent.

Then the singularity of $A$ implying

$$\det A = 0 \quad (18)$$

is necessary to achieve an equilibrium state

$$w_\infty = \lim_{n\to\infty} w_n \in \ell_\infty$$

of the feedback control system (1), (7), (9).

*Proof.* Let $v_n \equiv 0$. Then by the definition of the image of the matrix [9, p. 10] from (1) we get $y_n \in \operatorname{im} B$ for all $u_{n-1}$s.

Due to (2), $\operatorname{im} B$ is a linear subspace in $\mathrm{IR}^N$ whose dimension satisfies $\dim(\operatorname{im} B) = \operatorname{rank} B < N$ [9, sect. 6.25].

In view of (17), the output error $e_n$ determined by (7) together with (1) as $e_n = y^0 - Bu_{n-1}$ will be nonzero vector for any $n$, i.e.,

$$\|e_n\| \ne 0 \quad \forall n. \quad (19)$$

From (1) it follows that an equilibrium state $w_\infty$ can be achieved if $\lim_{n\to\infty}\|u_n\| < \infty$ will be guaranteed.

By virtue of (12), (13), this condition requires

$$\|\Delta u_n\| = \|Ae_n\| \to 0 \text{ as } n \to \infty \quad (20)$$

(recall that $\Delta u_n = u_n - u_{n-1}$).

On the other hand, in order to satisfy the requirement (21), from the definition of the kernel of the matrix [1] it can be concluded that the ultimated vector $e_\infty := \lim_{n\to\infty} e_n$ must lie on $\ker A \subseteq \mathrm{IR}^N$.

Now, assume that $A$ is non-singular.

In this case we have $\ker A = \{0\}$ [1], where $\{0\}$ denotes the origin of $\mathrm{IR}^N$

However, according to (19), $e_\infty$ cannot lie on $\ker A$. Therefore, the assumption that $\det A \ne 0$ made above does not hold.

This fact establishes the validity of Proposition.

*Remark*. Due to Proposition, if $\det A \ne 0$ then $\|w_n\|$ goes to infinity as $n$ tends to $\infty$.

Nevertheless, this feature of the closed-loop control system (1), (7), (9) with a non-singular $A$ does not exclude a situation when the error vector $e_n$ may be bounded in norm for all time $n$, as shown in Fig. 2.

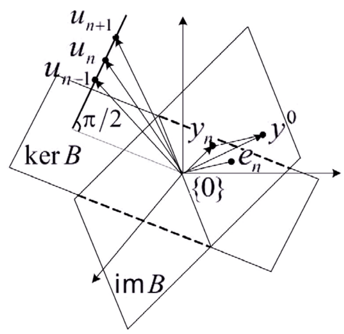


**Fig. 2.** Geometrical interpretation of Remark

In this special case, $y_n \underset{n\to\infty}{\to} y_\infty$ with $\|y_\infty\| < \infty$ whereas $u_n$s which are determined by (13) together with (12) will lie on the line orthogonal to $\ker B$ and become unbounded in norm (Fig. 2):

$$\lim_{n\to\infty}\|u_n\| = \infty$$

Thus,

$$y_\infty := \lim_{n\to\infty} y_n = \lim_{n\to\infty} Bu_{n-1}$$

remains bounded in norm, while $\{u_n\}$ is theoretically the unbounded control sequence: $\{u_n\} \notin \ell_\infty$.

Now, let the transfer function matrix $B$ be singular according to (2).

Then, the choice of $A$ in the form (10) becomes impossible.

At first sight, in this case, instead of (15), the control law

$$u_n = u_{n-1} + B^+ e_n \quad (21)$$

can immediately be derived from (9) via replacing $B^{-1}$ by the so-called pseudoinverse matrix $B^+$ according to

$$A = B^+. \quad (22)$$

Recall [1] that $B^+$ is specified as follows:

$$B^+ := \lim_{\delta\to 0}(B^T B + \delta^2 I)^{-1} B^T$$

where $I$ denotes the identity $N \times N$ matrix.

Thereby, the structure of the closed-loop control system containing the pseudo-inverse model together with the discrete integrator will be the same as in the presence of inverse model (Fig. 1).

To confirm the fact above established, a numerical example is given.

Example 1. Suppose

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \\ 5 & 10 & 15 \end{pmatrix}.$$

In this case, $\det B = 0$.

Let $A = \operatorname{diag}(0.02,\ 0.05,\ 0.03)$.

With

$$y^0 = [5, 12, 17]^{\mathrm{T}}$$

and initial

$$u_0 = [1, 2, 3]^{\mathrm{T}}$$

the closed-loop control system (1), (7), (9) was simulated.

The behavior of this system is shown in Fig. 3.

We see that the norm of control input $\|u_n\|$ increases whereas the output $y_n$ becomes constant in its norm.

Since the singularity of the matrix $B$ causes $\det B^+ = 0$, the requirement (18) given in Proposition can always be satisfied under condition (22).

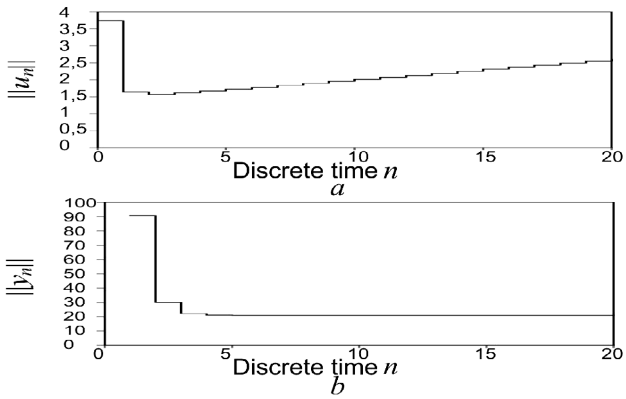


**Fig. 3.** Simulation results of Example 1:
*a* – the norm of control input $u_n$;
*b* – the norm of output $y_n$

Nevertheless, both the boundedness of all signals in the closed-loop control system (1), (7), (21) provided that $\{v_n\} \in \ell_\infty$ and the optimality of the controller are not obvious as yet and need to be argued.

Substantiations of these properties and also of its robustness are the main results.

## 4. Control of system with singular transfer function matrix

Suppose for the time being that the disturbances are removed and $B$ satisfying (2) is exactly known.

With these assumptions, the properties of the closed-loop system (1), (7), (21) are explored in the following theorem.

Theorem 2. Subject to Assumption A1 and provided that $v_n \equiv 0$ and $y^0$ represents an arbitrary vector from $\mathrm{IR}^N$, then the controller (7), (21) leads to stabilizing $w_n$ so that:

(i) the error vector $e_n$ remains constant equal to

$$e_n = (I - BB^+)y^0 \quad \forall n = 2, 3, \ldots \quad (23)$$

irrespective of initial $u_0$;

(ii) starting from the first time instant, $u_n$ becomes the constant vector depending on $u_0$ and $y^0$ in accordance with

$$u_n = (I - B^+ B)u_0 + B^+ y^0 \quad \forall n = 1, 2, \ldots. \quad (24)$$

*Proof.* Due to space limitation, the proof is omitted.

*Corollary.* In the special case, where

$$y^0 \in \operatorname{im} B \quad (25)$$

and there are no disturbances, the output error $e_n$ will be zero vector

$$0_N := [\underbrace{0,\dots,0}_{N}]^{\mathrm{T}}$$

for all integer $n \in [2, \infty)$:

$$e_n \equiv 0_N. \tag{26}$$

*Proof.* By Theorem of [9] any vector $y^0 \in \mathrm{IR}^N$ can be presented as

$$y^0 = \mathrm{pr}_{\mathrm{im}\,B} y^0 + \mathrm{pr}_{\ker B^T} y^0 \tag{27}$$

in which $\mathrm{pr}_{\mathrm{im}\,B} y^0$ and $\mathrm{pr}_{\ker B^T} y^0$ denote the projections of $y^0$ onto $\mathrm{im}\,B$ and $\ker B^T$, respectively.

In view of (25), from (27) we obtain

$$\mathrm{pr}_{\ker B^T} y^0 = 0_N. \tag{28}$$

Further, by Corollary 3.5 of [9], it can be written

$$(I - BB^+) y^0 = \mathrm{pr}_{\ker B^T} y^0. \tag{29}$$

Taking into account (29), and the property (23) together with (28) result in (26).

This completes the proof.

The geometrical interpretation of Theorem 2 is given in Fig. 4.

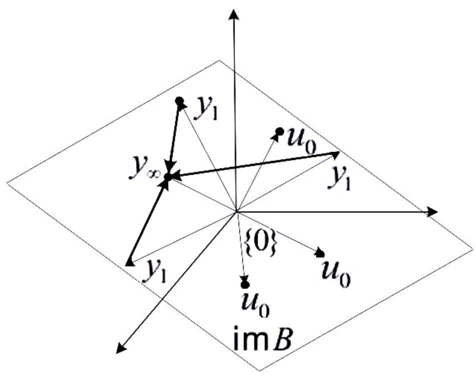


**Fig. 4.** The meaning of properties (i) and (ii)

Let $v_n$ be present. In this case, the following basic result holds.

Theorem 3. Under Assumptions A1 and A2, the closed-loop system (1), (4), (21) has the properties:

(a) the ultimate output error vector $e_\infty$ and also the input control vector $u_\infty$ are bounded in norm by

$$\|e_\infty\| \le \|Q_e\| \left( \|y^0\| + \varepsilon \right) + 2\varepsilon, \tag{30}$$

$$\|u_\infty\| \le \|Q_u\| \|u_0\| + \|B^+\| \left( \|y^0\| + \varepsilon \right), \tag{31}$$

where

$$Q_e := I - BB^+, \tag{32}$$

$$Q_u := I - B^+ B; \tag{33}$$

(b) the controller (22) is optimal in the sense of the requirement (11).

*Proof.* (a) Following the same steps as in proving Theorem 1 it can be written

$$e_n = Q_e (y^0 - v_{n-1}) - \Delta v_n, \tag{34}$$

$$u_n = Q_u u_0 + B^+ y^0 - B^+ v_n, \tag{35}$$

where $Q_e$ an $Q_u$ are the matrices given by (32) and (33), respectively, and the notation

$$\Delta v_n := v_n - v_{n-1} \tag{36}$$

is introduced.

Taking into account (5), by the triangle inequality from (36) we get $\|v_n\| \le 2\varepsilon$.

This together with (34), (35) and (10) leads to (30), (31).

(b) Equation (1) together with (4) yields

$$e_{n+1} = e_n - B\Delta u_n + \Delta v_{n+1}. \tag{37}$$

Using again the triangle inequality, from (38) we have

$$\|e_{n+1}\| \le \|e_n - B\Delta u_n\| + \|\Delta v_{n+1}\|. \tag{38}$$

Consider the general linear control of the form (9) rewritten as

$$\Delta u_n := A e_n. \tag{39}$$

By Theorem 3.4 of [1], the condition

$$\Delta u_n := B^+ e_n \tag{40}$$

applied to the first term of the right-hand side of (38) gives

$$\| e_n - B \underbrace{B^+ e_n}_{\Delta u_n} \| = \inf_{\Delta u \in \mathrm{IR}^N} \|e_n - B\Delta u\| \tag{41}$$

with $A \in \mathrm{IR}^{N \times N}$ and $e_n \in \mathrm{IR}^N$.

Defining an arbitrary vector $\Delta u$ as $\Delta u := A e_n$ from (41) we obtain

$$\|e_n - B\Delta u_n\| = \inf_A \| e_n - B \underbrace{A e_n}_{\Delta u_n} \| \tag{42}$$

provided that $\Delta u_n$ is given by (40).

Since the variables $\|e_n - B\Delta u_n\|$ and $\|\Delta v_n\| \le 2\varepsilon$ are independent, (42) means that the control (40) minimizes the upper bound on $\|e_{n+1}\|$ given by (38) for any nonnegative integer $n$.

Hence, this control is optimal on the set of controls having the form (39) for all admissible $A$s because (11) is satisfied.

The fact thus established completes the proof.

Example 2. Consider the system (1) with $B$, $u_0$ and $y^0$ as in Example 1. Let $\{v_n^{(1)}\}$ and $\{v_n^{(2)}\}$ be the pseudorandom variables taken from the range $[-1, 1]$.

Choose $A$ by (22) in which

$$B^+ = (1/588)\,B^{\mathrm{T}}.$$

The behavior of the closed-loop control system (1), (7), (21) is presented in Fig. 5.

From Figs 5, *d, e, f* we observe the outputs depicted by solid lines are close to the set-points depicted by dashed lines.

This shows that the performance of the closed-loop system with the controller containing the pseudoinverse matrix is successful.

Since $B$ may not exactly be known, the choice of $A$ in the form (22) becomes impossible.

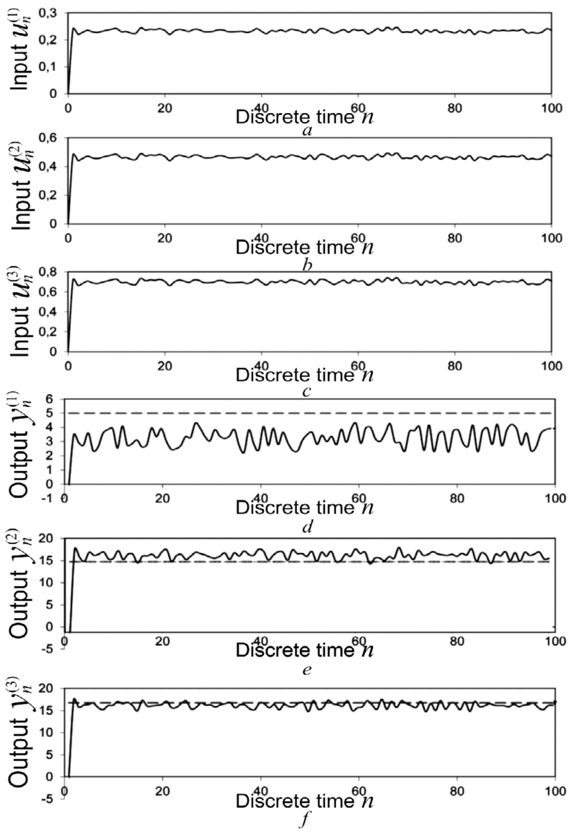


**Fig. 5.** Simulation results of Example 2:
*a* – first control input;
*b* – second control input;
*c* – third control input;
*d* – first control output;
*e* – second control output;
*f* – third control output

In this case, $B^+$ needs to be replaced by a suitable estimate $\tilde{B}^+$ close in some sense to $B$.

However the following question arises:

Is the controller

$$u_n = u_{n-1} + \tilde{B}^+ e_n \qquad (43)$$

robust in the presence of difference between $\tilde{B}^+$ and $B^+$?

The answer is given in Theorem 4 below.

The crucial step in deriving the robustness properties is based on utilization of the following lemma which is a reformulation of results that can be found in the handbook [9].

*Lemma.* Let $Q$ be a matrix of the form (32) or (33) satisfying the conditions:

(i) $\rho^*(Q) = \max_{\substack{\lambda\in\sigma(Q)\\ \lambda\neq 1}} |\lambda(Q)| < 1;$

(ii) $\operatorname{rank}(I-Q) = \operatorname{rank}(I-Q)^2,$

where the notification $\sigma(Q)$ of the matrix spectrum is introduced.

Then

(a) there exists a limit matrix

$$Q_\infty := \lim_{n\to\infty} Q^n;$$

(b) the series

$$\left(I - \tilde{B}^+B\right)\tilde{B}^+ + \left(I - \tilde{B}^+B\right)^2\tilde{B}^+ + \left(I - \tilde{B}^+B\right)^3\tilde{B}^+ + \ldots$$

converges.

With this lemma, the following result can be shown to be valid.

Theorem 4. Under Assumptions A1-A3, there is a set of $\tilde{B}^+$ including $\tilde{B}^+ = B^{\mathrm{T}}$ such the controller (43) guarantees the boundedness of $\{w_n\}$.

*Proof.* Due to space limitation, the proof is omitted.

Example 3. Consider the system with $B$ as in Example 1 and 2. Choose $\tilde{B}^+$ as

$$\tilde{B}^+ = (1/588)\begin{pmatrix} 1 & 4 & 4,8 \\ 2 & 8 & 9,6 \\ 3,02 & 12,08 & 14,496 \end{pmatrix}.$$

The performance of the closed-loop system containing the controller (43) with no disturbance and in the presence of disturbances as in Example 2 is depicted in Figs 6, 7, respectively.

Fig. 6, *b* demonstrates just the feature of this controller given by (24).

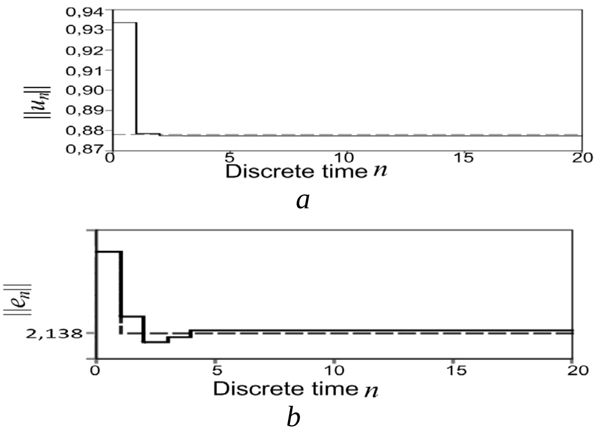


**Fig. 6.** Simulation results of Example 3 with no disturbance:
*a* – the norm of control input in the case $A=\tilde{B}^+$ (solid line) and in the case $A=B^+$ (dashed line);
*b* – the norm of error $A=\tilde{B}^+$ (solid line) and in the case $A=B^+$ (dashed line)

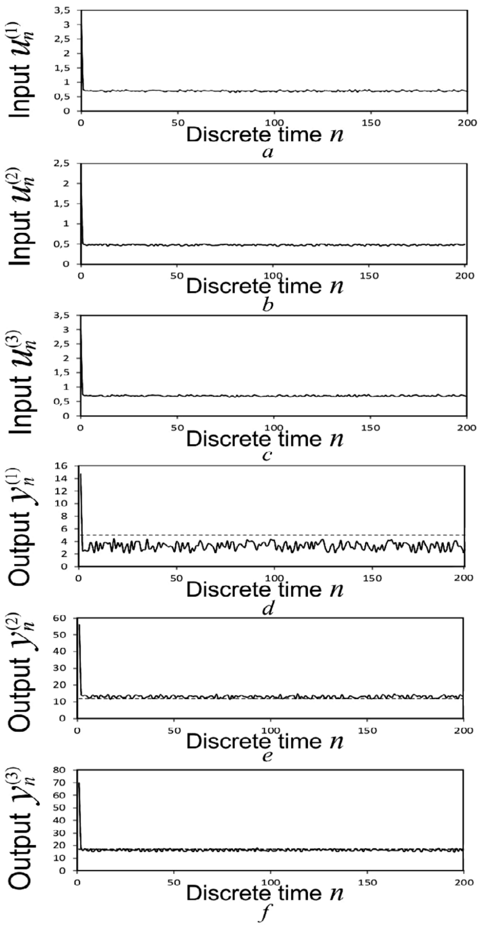


**Fig. 7.** Simulation results of Example 3 in the presents disturbance:
*a* – first control input;
*b* – second control input;
*c* – third control input;
*d* – first control output;
*e* – second control output;
*f* – third control output

## 5. Control of system with ill-conditioned transfer function matrix

Consider the case where

$$\operatorname{rank} B = N \tag{44}$$

but the conditionality number $\operatorname{cond} B$ satisfies (3) meaning that the norm of the inverse matrix $B^{-1}$ is sufficiently large number. In this case, the following result can be shown to be valid.

Theorem 5 [2]. Let (44) be satisfy.

Suppose assumptions A1 in which (3) takes place and A2 hold.

Define

$$\tilde{B}_* := \arg \min_{\{\underline{B}\}:\,\det \underline{B}=0} \left\|\tilde{B}-B\right\| \tag{45}$$

as the singular matrix which is the closest to $B$.

Let

$$\left\|B_*^+\Delta\right\| < 1, \tag{46}$$

where

$$\Delta := B-\tilde{B}_*. \tag{47}$$

Then the controller

$$u_n = u_n + \tilde{B}_*^+ e_n$$

guarantees

$$\limsup_{n\to\infty}\left\|u_n\right\| \le \left\|I-\tilde{B}_*^+\tilde{B}_*\right\|\left\|u_0\right\| + \frac{\left\|\tilde{B}_*^+\right\|}{1-\left\|\tilde{B}_*^+\Delta\right\|}\left(\left\|y^0\right\|+\varepsilon\right) < \infty.$$

*Corollary*. Let $B^{-1}$ be a non-singular matrix satisfying

$$\left\|B^{-1}\right\| > \left\|\tilde{B}_*^+\right\|$$

with $\tilde{B}_*$ determined by (44).

Under Assumptions A1 and A2 the system (1), (4), (45) will be dissipative.

The proof is based on utilizing the result taken from [11, Lemma 7.2].

The result thus established allows now to derive the controller satisfying the constrain (46) together with (47) of the form

$$u_n = \begin{cases} u_{n-1} + B^{-1}e_n & \text{if } \left\|B^{-1}\right\| \le 2\left\|\tilde{B}_*^+\right\| \\ u_{n-1} + \tilde{B}_*^+ e_n & \text{otherwise} \end{cases} \tag{48}$$

putting $u_0 = [0,\ldots,0]^T$.

Due to Corollary and Theorems 5, the controller (48) may be considered as a suboptimal controller whose actions $\{u_n\}$ satisfy (6) with

$$U = 2\left\|\tilde{B}_*^+\right\|\left(\left\|y^0\right\|+\varepsilon\right).$$

This controller is applicable to deal with the ill-conditioned transfer function matrix *B*.

## 6. Conclusions

It was established that the inverse matrix approach can be used to optimize the discrete-time control of linear multivariable systems whose transfer function matrices are either singular or ill-conditioned.

## References

[1] *Albert, A.* Regression and the Moore-Penrose Pseudoinverse. New York, Academic Press. 1972. 210 p.

[2] *Azarskov, V.N.; Skurikhin, V.I.; Solovchuk, K.Yu.; Zhiteckii, L.S.* Optimal and suboptimal control of static multivariable plants based on generalized inverse matrix approach. Proceedings 20th International Conference on Automatic Control "Automatics-2013". 25-27 September 2013. Mykolaiv, Ukraine. 2013. P. 67-68.

[3] *Dorato, P.* On the inverse of linear dynamical systems. IEEE Trans. Syst. Sc. and Cyber. 1969. Vol. 5, N 1. P. 43-48.

[4] *Francis, B.A.* The linear multivariable regulator problem. SIAM J. Control Optimiz. 1977. Vol 15, N 3. P. 486–505.

[5] *Lee, T.; Adams, G.; Gaines, W.* Computer Process Control: Modeling and Optimization. New York, Wiley. 1968. 437 p.

[6] *Lovass-Nagy, V.; Miller, J.R.; Powers, L.D.* On the application of matrix generalized inversion to the construction of inverse systems. Int. Journal of Control. 1976. Vol. 24, N 5. P. 733–739.

[7] *Lyubchik, L.M.* Inverse model control and subinvariance in linear discrete multivariable systems. Proceedings of the 3rd European Control Conference. Roma, Italy. 1995. Vol. 4, Part 2. P. 3651–3659.

[8] *Seraji, H.* Minimal inversion, command tracking and disturbance decoupling in multivariable systems. Int. J. Control. 1989. Vol. 49, N 6. P. 2093–2191.

[9] *Voevodin, V.V.; Kuznetsov, Yu.A.* Matrices and Computations. Moscow, Nauka. 1984. 320 p. (in Russian)

[10] *Wolovich, W.A.* Linear Multivariable Systems. New York, Springer. 1974. 582 p.

[11] *Wonham, W.M.* Linear Multivariable Control. A Geometrical Approach. New York, Springer. 1985. 613 p.

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**В.М. Азарсков[1], Л.С. Житецький[2], К.Ю. Соловчук[3]**. **Дискретне керування лінійними багатовимірними системами з виродженою або погано обумовленою матрицями передавальних функцій**
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Розглянуто керування багатовимірними лінійними дискретними стаціонарними системами, матриці передавальних функцій яких або вироджені, або погано обумовлені. Показано, що є довільні невимірювальні, але обмежені збурення, а параметри цих систем можуть бути частково невідомими. Оптимальний регулятор побудовано з використанням псевдоінверсії матриці передавальних функцій системи. Доведено обмеженість усіх сигналів, породжуваних цим регулятором, а також властивості робастності регулятора за наявності параметричної невизначеності. Для підтвердження теоретичних досліджень наведено числові приклади.

**Ключові слова**: багатовимірна система; дискретний час; матриця, яка не може бути оберненою; обмежене збурення; оптимальність; псевдоінверсія.

**В.Н. Азарсков[1], Л.С. Житецкий[2], К.Ю. Соловчук[3]**. **Дискретное управление линейными многомерными системами с вырожденными или плохо обусловленными матрицами передаточных функций**
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Рассмотрено управление многомерными дискретными стационарными системами, матрицы передаточных функций которых или вырождены, или плохо обусловлены. Показано, что есть произвольные неизмеряемые, но ограниченные возмущения, а параметры этих систем могут быть частично неизвестными. Оптимальный регулятор построен с использованием псевдоинверсии матрицы передаточных функций системы. Доказана ограниченность всех сигналов, порожденных этим регулятором. Приведены свойства робастности регулятора при наличии параметрической неопределенности. Для подтверждения теоретических исследований даны численные примеры.

**Ключевые слова**: дискретное время; многомерная система; необращаемая матрица; ограниченное возмущение; оптимальность; псевдоинверсия.

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