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THE THERMOELASTIC CONTACT PROBLEM FOR A RECTANGLE

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In this paper the vertical vibrations of a punch lying on an elastic isotropic rectangle under harmonic force $P = P_0 e^{-iwt}$ and the temperature field defined in [1] is considered.

In view of the statement of the problem the boundary conditions are

$$v(x,\ 0,\ t) = v_0 e^{-iwt}, \qquad if \quad x \in \Delta_1, \tag{1}$$

$$\sigma_y(x,\ 0,\ t) = 0, \qquad if \quad x \in \Delta_2, \tag{2}$$

$$\tau_{xy}(x,\ 0,\ t) = 0, \qquad if \quad x \in \Delta_1 \cup \Delta_2, \tag{3}$$

$$u(x,\ h,\ t) = v(x,\ h,\ t)0, \qquad if \quad x \in \Delta_1 \cup \Delta_2, \tag{4}$$

$$u(\pm\pi,\ y,\ t) = \tau_{xy}(\pm\pi,\ y,\ t) = 0, \qquad if \quad y \in \Delta_3, \tag{5}$$

where $\Delta_1 = [-a,\ a]$, $\Delta_2 = [-\pi,\ \pi] \setminus \Delta_1$, $\Delta_3 = [0,\ h]$.

v_0 is the unknown amplitude of the vibration.

The potentials Φ and Ψ can be thought of in the form [2]

$$\Phi(x,\ y,\ t) = \Phi^*(x,\ y)e^{-iwt}, \qquad \Psi(x,\ y,\ t) = \Psi^*(x,\ y)e^{-iwt},$$

where $\nabla^2\Phi^* + \alpha^2\Phi^* = mT^*, \quad \nabla^2\Psi^* + \beta^2\Psi^* = 0, \quad \alpha = \dfrac{w}{c_1}, \quad \beta = \dfrac{w}{c_2}.$

Boundary conditions (1) and (2) can be completed as follows:

$$v^*(x,\ 0) = v_-(x) + \psi_+(x), \qquad \sigma_y^*(x,\ 0) = \psi_-(x) \tag{6}$$

where $v(x,\ y,\ t) = v^*(x,\ y)e^{-iwt}, \quad \sigma_y(x,\ y,\ t) = \sigma_y^*(x,\ y)e^{-iwt},$

and

$$\psi_+(x)=\begin{cases}0, & x\in\Delta_1\\ \text{undertermined}, & x\in\Delta_2\end{cases}\qquad \psi_-(x)=\begin{cases}\text{undertermined}, & x\in\Delta_1\\ 0, & x\in\Delta_2\end{cases}$$

Working in the same way as in [3] we arrive at the following discrete Riemann problem

$$\Psi_{n+}=M_n^k\Psi_{n-}+F_n^k-V_{n-}\quad (n=0,\pm 1,\pm 2,\ldots), \tag{7}$$

where

$$M_n^h=\frac{S_n^h}{P_n^h},\quad F_n^h=\frac{T_n^h}{P_n^h}\frac{dQ_n(h)}{dy},\ T_n^h=\frac{T_{1n}^h}{T_{2n}^h},$$

$$T_{1n}^h=k_{2n}\left(P_n^w\cosh k_{2n}h-2Gn^2\cosh k_{1n}h\right),$$

$$T_{2n}^h=in\left(2Gk_{1n}k_{2n}\sinh k_{1n}h-P_n^w\sinh k_{2n}h\right),$$

$$P_n^h=T_{1n}^h\left(P_n^w\cosh k_{1n}h-2Gn^2k_{2n}h\right)\left(\rho w^2T_{2n}^h\right)^{-1}-$$

$$-in\left(\rho w^2k_{1n}\right)^{-1}\left(P_n^w\sinh k_{1n}-2Gk_{1n}\sinh k_{2n}h\right),$$

$$S_n^h=T_{1n}k_{1n}\sinh k_{1n}h\left(P_n^wT_{2n}^h\right)^{-1}-in\cosh k_{1n}h\left(P_n^w\right)^{-1},$$

$$k_{1n}^2(\gamma)=n^2-\alpha^2-iw\gamma,\quad k_{2n}^2(\gamma)=n^2-\beta^2-iw\gamma\ \ (\gamma\downarrow 0).$$

In this case the following formulas were used

$$\Phi_n^*=A_n\cosh k_{1n}y+B_n\sinh k_{1n}+Q_n(y),\ \Psi_n^*=C_n\cosh k_{2n}y+D_n\sinh k_{2n}y,$$

$$Q_n(y)=\frac{m}{k_{1n}}\int_0^y T_n^*(t)\sinh k_{1n}(y-t)dt \tag{8}$$

Multiplying (7) by n, we rewrite it in such a form

$$n\Psi_{n+}=-A_{\alpha\beta}\,\mathrm{sgn}\left(n+\frac{1}{2}\right)\Psi_{n-}+\Gamma_n^{\alpha\beta}\Psi_{n-}-nV_{n-}+n\frac{T_n^h}{P_n^h}\frac{dQ_n(h)}{dy},$$

$$\Gamma_n^{\alpha\beta}=nM_n^h+A_{\alpha\beta}\,\mathrm{sgn}\left(n+\frac{1}{2}\right),\quad A_{\alpha\beta}=\frac{\rho w^2}{2G\left(2\rho w^2-G\left(\alpha^2+\beta^2\right)\right)}, \tag{9}$$

and also $\left|\Gamma_n^{\alpha\beta}\right| = O\left(\frac{1}{n^2}\right)$ $(n \to \infty)$.

Additionally, the condition

$$\sum_{n=-\infty}^{+\infty} \left(k_{1n} B_n - in C_n\right) = v_0, \text{ where } C_n = -\frac{2G\, in\, k_{1n}}{P_n^w} B_n, \tag{10}$$

where $B_n = \frac{1}{D_n^{\alpha\beta h}} \left(X_{1n} T_{2n}^h + X_{2n} T_{1n}^h\right)$,

$$X_{1n} = in \cosh k_{1n} h \Psi_{n-} + in P_n^w Q_n(h), \quad X_{2n} = k_{1n} \sinh k_{1n} h \Psi_{n-} + P_n^w \frac{dQ_n(h)}{dy},$$

$$D_n^{\alpha\beta h} = in T_{2n}^h \left(P_n^w \sinh k_{1n} h - 2G k_{1n} k_{2n} \sinh k_{2n} h\right) -$$
$$- k_{1n} T_{1n}^h \left(P_n^w \cosh k_{1n} h - 2G n^2 k_{2n} h\right)$$

$P_n^w = 2Gn^2 - \rho w^2$ determines the solution of problem (9) equivalent to that of (7).

On performing the inverse Fourier transform

$$W^{-1} \Psi_{n\pm} = \sum_{n=-\infty}^{+\infty} \Psi_{n\pm} e^{inx} = \psi_\pm(x) \tag{11}$$

and using the formula [4]

$$W^{-1} \operatorname{sgn}\left(n + \frac{1}{2}\right) \Psi_{n-} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\psi_-(t) e^{it}}{e^{it} - e^{ix}} \tag{12}$$

we reduce the discrete problem (9) to the singular integral equation

$$\frac{1}{2\pi} \int_{-a}^{a} \cot \frac{t - x}{2} \psi_-(t) dt = -2 \sum_{n=1}^{\infty} \Gamma_n^{\alpha\beta} \Psi_{n-} \sin nx + f_h'(x), \tag{13}$$

where $f_h'(x) = \sum_{n=-\infty}^{\infty} \frac{n T_n^h}{P_n^h} \frac{dQ_n(h)}{dy} e^{inx}$.

the Hilbert-type integral equation (13) can be inverted in the class of integrable functions with the result

$$\psi_-(x) = \frac{1}{A_{\alpha\beta} X(x)} \left(m_{\alpha\beta}(x) + 2\sum_{n=1}^{\infty} \Gamma_n^{\alpha\beta} \Psi_{n-} J_n(x) + a_0 \cos\frac{x}{2} \right), \quad (14)$$

where $X(x) = \sqrt{2(\cos x - \cos a)}$, $J_n(x) = \sum_{m=0}^{n} \mu_{n-m}(\cos a)\cos\left(m + \frac{1}{2}\right)x$,

$\mu_k(\cos a) = \frac{P_{k-2}(\cos a) - P_k(\cos a)}{2k-1}$ $(k = 2, 3, \ldots)$, $\mu_0(\cos a) = 1$,

$\mu_1(\cos a) = -\cos a$,

$m_{\alpha\beta}(x) = \frac{1}{2\pi} \int_{-a}^{a} \frac{X(t) f_h'(t)}{\sin\frac{t-x}{2}} dt$.

The application of the finite Fourier transform to (14) leads to the following infinite system of linear algebraic equations:

$$A_{\alpha\beta}\Psi_{n-} = M_n^{\alpha\beta} + 2\sum_{k=1}^{\infty} \Gamma_k^{\alpha\beta} \Psi_{k-} N_{kn} + a_0 R_n \quad (n = 1, 2, \ldots), \quad (15)$$

$$A_{\alpha\beta}\Psi_{0-} = 2\sum_{k=1}^{\infty} \Gamma_k^{\alpha\beta} \Psi_{k-} N_{k0} + M_0^{\alpha\beta} + a_0 R_0, \quad (16)$$

where

$$M_n^{\alpha\beta} = \frac{1}{2\pi} \int_{-a}^{a} \frac{m_{\alpha\beta}(x)\cos nx}{X(x)} dx,$$

$$N_{kn} = -\frac{1}{2}\frac{n+1}{n-k}\left(P_n(\cos a)P_{k-1}(\cos a) - P_{n+1}(\cos a)P_k(\cos a)\right) \quad (k \geq 1), \quad (17)$$

$$R_n = \frac{1}{4}\left(P_n(\cos a) + P_{n+1}(\cos a)\right),$$

$P_n(\cos a) = \frac{1}{\pi} \int_{-a}^{a} \frac{\cos\left(n + \frac{1}{2}\right)x \, dx}{\sqrt{2(\cos x - \cos a)}}$ are Legendre polynomials.

Since system (15), (16) can in general be solved approximately, namely using the method of truncation, we investigate this system.
To do this we estimate the sum

$$S_n = \left|A_{\alpha\beta}^{-1}\right| \sum_{k=1}^{\infty} \left|\Gamma_k^{\alpha\beta}\right| \left|N_{kn}\right| \ \ (n \geq 1) \tag{18}$$

since $\left|\Gamma_k^{\alpha\beta}\right| = \mathrm{O}\left(\frac{1}{k^2}\right) \ \ (k \to +\infty)$,

then $\left|\Gamma_k^{\alpha\beta}\right| < \frac{C_{\alpha\beta} + \varepsilon}{k^2}$ if $k > N$,

where $C_{\alpha\beta} = \frac{\left|2\alpha^2\left(\beta^2 - 1\right) - \left(\alpha^4 + \beta^4\right)\right|}{2\left(G\left(\alpha^2 + \beta^2\right) - 2\rho w^2\right)^2}$.

Hence $\left|A_{\alpha\beta}\right| S_n \leq \sum_{k=1}^{N}{}' \left|\Gamma_k^{\alpha\beta}\right| \left|N_{kn}\right| + \left(C_{\alpha\beta} + \varepsilon\right) \sum_{k=N+1}^{\infty} \frac{\left|N_{kn}\right|}{k^2} + \left|\Gamma_n^{\alpha\beta}\right| \left|N_{nn}\right|$,

where the prime means that $k \neq n$.
Now consider the sum

$$\left|A_{\alpha\beta}\right| S_n^{(1)} \leq \sum_{k=1}^{N}{}' \left|\Gamma_k^{\alpha\beta}\right| \left|N_{kn}\right| + \left(C_{\alpha\beta} + \varepsilon\right) \sum_{k=N+1}^{\infty} \frac{\left|N_{kn}\right|}{k^2}.$$

Let us denote

$\left|\Gamma^{\alpha\beta}\right| = \max\limits_{1 \leq k \leq N} \left(\Gamma_1^{\alpha\beta}, \Gamma_2^{\alpha\beta}, \ldots, \Gamma_N^{\alpha\beta}\right)$, and $\left|A_{\alpha\beta}\right| S_n^{(1)} \leq Q_{\alpha\beta} \sum_{k=1}^{\infty}{}' \frac{\left|N_{kn}\right|}{k^2}$, where $Q_{\alpha\beta} = \max\left(N^2 \left|\Gamma^{\alpha\beta}\right|, \ C_{\alpha\beta} + \varepsilon\right)$.

Applying formula (17) we have

$$\left|A_{\alpha\beta}\right| S_n^{(1)} \leq Q_{\alpha\beta} (n+1) \left|P_n(\cos a)\right| \sum_{k=1}^{\infty}{}' \frac{\left|P_k(\cos \alpha)\right|}{k^2 |n-k|}.$$

Taking into account the inequality [5]

$$\left|P_n(\cos a)\right| \leq \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{\sqrt{n \sin a}} \ \ \left(0 < a < \pi, \ n = 1, 2, \ldots\right) \tag{19}$$

we get

$$\left|A_{\alpha\beta}\right|S_n^{(1)} \le \frac{2Q_{\alpha\beta}}{\pi\sin a}\frac{n+1}{\sqrt{n}}\sum_{k=1}^{\infty}{}'\frac{1}{k^2\sqrt{k}\left|n-k\right|}. \qquad (20)$$

Observe that

$$\sum_{k=1}^{\infty}{}'\frac{1}{k^2\sqrt{k}\left|n-k\right|} < \int_1^{n-1}\frac{dx}{x\sqrt{x}(n-x)} + \int_{n+1}^{\infty}\frac{dx}{x\sqrt{x}(x-n)},$$

and calculating the integrals we obtain

$$\sum_{k=1}^{\infty}{}'\frac{1}{k^2\sqrt{k}\left|n-k\right|} < \frac{2}{n} + \frac{2}{n\sqrt{n}}\ln\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}.$$

Since [6]

$$\frac{1}{\sqrt{n}}\ln\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}-\sqrt{n}} \le 2 \quad (n=1,\ 2,\ \ldots)$$

then $\sum_{k=1}^{\infty}{}'\frac{1}{k^2\sqrt{k}\left|n-k\right|} < \frac{6}{n}$.

On estimating $N_{nn} = \lim_{\mu\to\nu} N_{\mu\nu}$ we get

$$\left|N_{nn}\right| \le \frac{\pi}{2}\left(\frac{2}{\pi\sin a}\right)^{1/2}\frac{n+1}{\sqrt{n}} \quad (n=1,\ 2,\ \ldots) \qquad (21)$$

Finally we can write

$$S_n \le \frac{16Q_{\alpha\beta}(n+1)}{\left|A_{\alpha\beta}\right|\pi\sin a\cdot n\sqrt{n}} \quad (n>0) \qquad (22)$$

Infinite system of linear equations (15) is totally regular if the following condition is fulfilled [7] $S_n < 1$ $(n=1,\ 2,\ \ldots)$ which leads to such values of $Q_{\alpha\beta}$ satisfying the inequality

$$Q_{\alpha\beta} < \frac{\left|A_{\alpha\beta}\right|\pi\sin a}{32}. \qquad (23)$$

For arbitrary values of $Q_{\alpha\beta}$ the infinite system of linear equations is totally quasiregular.

Using formula (14) and taking into account boundary condition (2) we get the expression for the contact stress

$$p(x,t) = -\sigma_y(x,0,t) = -\frac{e^{-iwt}}{A_{\alpha\beta}X(x)}\left(m_{\alpha\beta}(x) + 2\sum_{n=1}^{\infty}\Gamma_n^{\alpha\beta}\Psi_{n-}J_n(x) + a_0\cos\frac{x}{2}\right). \qquad (24)$$

The quantity a_0 included in (24) is still to be defined. In fact the equation of motion of the punch is [8]

$$M\frac{d^2v}{dt^2} = e^{-iwt}\left(P_0 - P_R\right) \tag{25}$$

where M is the mass of the punch P_0 is the amplitude of the force acting on the punch, and P_R is the reaction of the elastic rectangle:

$$P_R = -\int_{-a}^{a}\sigma_y^*(x,0)dx = \int_{-a}^{a}\psi_-(x)dx = -\frac{\pi}{A_{\alpha\beta}}a_0 .$$

Substituting this expression together with $v = v_0e^{-iwt}$ into (25) we get

$$-Mw^2v_0 = P_0 + \frac{2Ga_0\pi}{A_{\alpha\beta}} \tag{26}$$

Thus the amplitude v_0 and the quantity a_0 can be calculated from equations (10) and (26)/

The real values for which $v^*(x,0)\to\infty$, the resonance frequencies, are the real roots of the resonance equations

$$D_n^{\alpha\beta h} = 0 \ \left(n = 1,2,\ldots,N\right), \ \ 2Gn^2 - \rho w^2 = 0 \ \left(n = 1,2,\ldots,N\right). \tag{27}$$

Let $\overline{w} = \frac{wa_T}{c_2^2}$, $\tau = \frac{tc_2^3}{a_T}$, $\overline{x} = \frac{x}{a}$ are the dimensionless frequency, time and coordinate. $\overline{M} = \frac{M}{\rho a_T^2}$ is the dimensionless mass.

Suppose that $\overline{w} = 0.1$; $\overline{M} = 1$; $\nu = 0.3$; $N = 10$. in the table the values of the contact stress corresponding to different values of $\overline{x}$ are exhibited when $\tau = 2\pi$.

$\overline{x}$	0	0.1	0.2	0.3	0.4	0.5
$\frac{\tilde{p}(\overline{x},\ \tau)}{P_0}$	0.4294	0.4302	0.4324	0.4422	0.4682	0.4889
$\overline{x}$	0.6	0.7	0.8	0.9	0.95	0.99
$\frac{\tilde{p}(\overline{x},\ \tau)}{P_0}$	0.5174	0.5713	0.6515	0.8653	1.1388	2.3455

The unknown amplitude of the vibration of the punch is $v_0 = 0.1217P$. Note that the values of the contact stresses increase unboundedly at the vicinities of the end points of the contact interval.

Conclusion

On choosing the number N and using formula (24) we can get an approximate solution of the thermoelastic contact problem for a rectangle to find a contact stress up to any prescribed accuracy.

Summary

The problem is formulated into a singular integral equation of Hilbert type, its solution providing an expression for the physically important unbounded normal stress. The integral equation is converted into an infinite system of algebraic equation. The investigation of this system was carried out and the approximate solution was obtained.

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