УДК 539.3

THE JUSTIFICATION OF THE TRUNCATION APPLIED TO INHOMOGENEOUS INTEGRAL EQUATIONS WITH CAUCHY'S KERNEL

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Let us consider the following Sturm-Liouville problem:

To find the values of γ and the functions u odd with respect to x, such that

$$\begin{aligned} &\Delta u + \gamma^2 u = 0 && \text{in } \Omega \subset R^2, \\ &u = 0 && \text{on } \Gamma_-, \\ &\frac{\partial u}{\partial n} = 0 && \text{on } \Gamma_+, \end{aligned} \tag{1}$$

where $\quad \Omega = (-\pi,\ \pi)\times(0,1),\ \ \Gamma_+ = \{(x,\ 0),\ |x| < c\}$ and $\Gamma_- = \Omega \setminus \Gamma_+.$ (2)

The restriction $u\big|_{y=0}$ is Hölder-continuous function at the points $(\pm c,\ 0)$ where the boundary conditions change while $\left.\dfrac{\partial u}{\partial y}\right|_{y=0}$ can at most be a piecewise continuous function. Following the technique and procedure of [1] closely, the solutions, odd with respect to x, of problem (1) are found to be

$$\begin{aligned} u(x, y; \gamma_i) = &\sum_{n>1}^{[\gamma_i]} 2\Phi_{n-}(\gamma_i) \frac{\sin\left(\sqrt{\gamma_i^2 - n^2}(1-y)\right)}{\sin\sqrt{\gamma_i^2 - n^2}} \sin n\,x + \\ &+ \sum_{n>[\gamma_i]} 2\Phi_{n-}(\gamma_i) \frac{\sinh\left(\sqrt{n^2 - \gamma_i^2}(1-y)\right)}{\sin\sqrt{n^2 - \gamma_i^2}} \sin n\,x, \end{aligned} \tag{3}$$

where γ_i stand for the eigenvalues and $\Phi_{n-}(\gamma_i)$, $n \in N$, and the complex Fourier components of the function $\varphi_-(x;\ \gamma_i) = u(x;\ 0;\ \gamma_i)$ vanishing on the intervals $c \le |x| \le \pi$ and compatible with the third condition of (1).

The γ_i's are simultaneously the eigenvalues of Cauchy-type integral equation

$$\frac{1}{\pi}\int_{-c}^{c}\frac{\varphi_-(t;\ \gamma)}{1-e^{i(x-t)}}\,dt = 2\sum_{n=1}^{\infty}\frac{1}{n}Q_n(\gamma)\Phi_{n-}(\gamma)\cos nx \qquad (4)$$

and the sets $\{\Phi_{n-}(\gamma_i),\ n \in N\}$ belong to the corresponding solutions. Here we have

$$Q_n(\gamma) = \begin{cases} n - \sqrt{\gamma^2 - n^2}\cot\sqrt{\gamma^2 - n^2} & \text{for } n < \gamma, \\ n - \sqrt{n^2 - \gamma^2}\coth\sqrt{\gamma^2 - n^2} & \text{for } n > \gamma; \end{cases} \qquad (5)$$

and

$$Q_n(\gamma) = o\left(\frac{1}{n}\right). \qquad (6)$$

Equivalently, it has been shown that the zeroes of the determinant of the homogeneous algebraic system

$$\Phi_{l-}(\gamma) = \sum_{n=1}^{\infty}\frac{1}{n}Q_n\left(N_{nl} + N_{-nl}\right)\Phi_{n-}(\gamma), \quad l \in N, \qquad (7)$$

define the eigenvalues of the problem and their corresponding solutions of the integral equation (4)/ the exact definitions of the coefficients $N_{\pm nl}$ are given in [2]. Here, it is sufficient to stress their asymptotic behavior

$$\frac{1}{n}N_{\pm nl} < O\left(\frac{1}{l}\right) \text{ and } \frac{1}{n}N_{\pm nl} < O\left(\frac{1}{n}\right) \qquad (8)$$

Denoting by $\gamma_i^{(j)}$ the i^{th} zero of the determinant truncated at order j, the components $\Phi_{n-}\left(\gamma_i^{(j)}\right)$, $n = 1, 2, \ldots, j$ can immediately be obtained by solving the truncated system while the next can be obtained according to the approximation

$$\Phi_{l-}\left(\gamma_i^{(j)}\right) = Q_1\left(\gamma_i^{(j)}\right)\left(N_{1l} + N_{-1l}\right) + \sum_{n=2}^{j}\frac{Q_n\left(\gamma_i^{(j)}\right)}{n}\left(N_{nl} + N_{-nl}\right)\Phi_{n-}\left(\gamma_i^{(j)}\right) \qquad (9)$$

$$l = j+1,\ j+2, \ldots$$

Here $\Phi_{1-}\left(\gamma_i^{(j)}\right)$ is already set equal one, an additional condition fixing the solution of the truncated homogeneous algebraic system (7).

According to the Rayleigh-Ritz technique, the eigenvalue γ_i of problem (1) is related to the corresponding eigenfunction by means of the equation [3]

$$\gamma^2 = \frac{\int\limits_{\Omega} |\Delta u|^2 dxdy}{\int\limits_{\Omega} u^2 dxdy}, \tag{10}$$

and it is possible to prove that

$$\gamma_i^2 = \lim_{j\to\infty} \frac{A\left(\gamma_i^{(j)}\right)}{B\left(\gamma_i^{(j)}\right)}, \tag{11}$$

where

$$\begin{aligned} A(\gamma) = & \sum_{n<\gamma} \left[\left(\frac{\gamma^2}{2} - n^2 \right) \frac{\cot\sqrt{\gamma^2 - n^2}}{\sqrt{\gamma^2 - n^2}} + \frac{\gamma^2}{2\sin^2\sqrt{\gamma^2 - n^2}} \right] \Phi_{n-}^2(\gamma) + \\ & + \sum_{n>\gamma} \left[\left(n^2 - \frac{\gamma^2}{2} \right) \frac{\coth\sqrt{n^2 - \gamma^2}}{\sqrt{n^2 - \gamma^2}} - \frac{\gamma^2}{2\sinh^2\sqrt{n^2 - \gamma^2}} \right] \Phi_{n-}^2(\gamma) \end{aligned} \tag{12}$$

and

$$\begin{aligned} B(\gamma) = & \frac{1}{2} \sum_{n<\gamma} \left[-\frac{\cot\sqrt{\gamma^2 - n^2}}{\sqrt{\gamma^2 - n^2}} + \frac{1}{2\sin^2\sqrt{\gamma^2 - n^2}} \right] \Phi_{n-}^2(\gamma) + \\ & + \frac{1}{2} \sum_{n>\gamma} \left[\frac{\coth\sqrt{n^2 - \gamma^2}}{\sqrt{n^2 - \gamma^2}} - \frac{1}{2\sinh^2\sqrt{n^2 - \gamma^2}} \right] \Phi_{n-}^2(\gamma) \end{aligned} \tag{13}$$

It is clear that the expansion of both $A\left(\gamma_i^{(j)}\right)$ and $B\left(\gamma_i^{(j)}\right)$ converge at any value of $j \in N$. Further, the usefulness of the truncation method on designating the eigenvalues is confirmed in as much as how rapidly the γ_i and $\gamma_i^{(j)}$ will coincide. In spite of the homogeneity of equation (4) or equivalently the algebraic system (7), the truncation of both at a certain eigenvalue γ_i can be justified. To this end, we shall first show that the homogeneous system (7) with the so normalized solution $\varphi_-(x;\gamma_i)$ that the first Fourier component $\Phi_{1-}(\gamma_i)$ equals 1 is equivalent to the inhomogeneous equation

$$K\varphi_- = f,$$

where $$K\varphi_- \equiv \frac{1}{\pi i}\int_{-c}^{c}\frac{\varphi_-(t;\,\gamma_i)}{1-e^{i(x-t)}}dt - 2\sum_{n=2}^{\infty}\frac{1}{n}Q_n(\gamma_i)\Phi_{n-}(\gamma_i)\cos nx \qquad (15)$$

and $f = 2Q_1(\gamma_i)\cos x$.

To verify, we note that equation (14) can be reduced to the inhomogeneous algebraic system

$$\Phi_{1-}(\gamma_i) - \sum_{n=2}^{\infty}\frac{Q_n(\gamma_i)}{n}(N_{n1}+N_{-n1})\Phi_{n-}(\gamma_i) = Q_1(\gamma_i)(N_{11}+N_{-11});$$

$$\Phi_{l-}(\gamma_i) - \sum_{n=2}^{\infty}\frac{Q_n(\gamma_i)}{n}(N_{nl}+N_{-nl})\Phi_{n-}(\gamma_i) = Q_1(\gamma)(N_{1l}+N_{-1l}), \qquad (17)$$

$$l \in N - \{1\}.$$

The equation obtained by the approximation of the equation (14) can be written in the form

$$\tilde{K}\tilde{\varphi}_- = f \qquad (18)$$

Since the Banach space $L_2[-c,\, c]$ is the domain and the range of both the operators K and $\tilde{K}$, the norm of the operator $K-\tilde{K}$ clearly satisfies

$$\left\|(K-\tilde{K})\varphi_-(x)\right\| = \int_{-c}^{c}\left|\frac{1}{\pi}\sum_{n=j+1}^{\infty}\frac{Q_n}{n}\int_{-c}^{c}\cos n(x-t)\varphi_-(t)dt\right|^2 dx <$$

$$< 4\frac{c^2}{\pi^2}\left(\sum_{n=j+1}^{\infty}\frac{Q_n}{n}\right)^2\|\varphi_-\|^2$$

which leads to the result

$$\left\|(K-\tilde{K})\right\| < 2\frac{c}{\pi}\sum_{n=j+1}^{\infty}\frac{Q_n}{n} \to 0 \qquad (19)$$

but since equation (14) has a unique bounded solution [4], it follows that the operator $\tilde{K}^{-1}(K-\tilde{K})$ is bounded and at sufficiently large j, its norm satisfies

$$\left\|\tilde{K}^{-1}(K-\tilde{K})\right\| < 1. \qquad (20)$$

Thus, it follows that [5] equation (14) or equivalently (17) in which $\gamma = \gamma_i$ has the unique solution

$$\varphi_-(\gamma_i) = \tilde{\varphi}(\gamma_i) + \left[I + \tilde{K}^{-1}(K-\tilde{K})\right]^{-1}\tilde{K}^{-1}\left(f - K\tilde{\varphi}(\gamma_i)\right), \qquad (21)$$

where I is the unit operator. Additionally, the error resulting from the truncation can be estimated

$$\|\varphi_{-}\tilde{\varphi}_{-}\| \leq \frac{\left\|\tilde{K}^{-1}\left(f - K\tilde{\varphi}_{-}\right)\right\|}{1 - \left\|\tilde{K}^{-1}\left(K - \tilde{K}\right)\right\|}. \tag{22}$$

The calculations were carried out right to the numerical results at several values of the parameter c. The coefficients $N_{\pm nl}$ in system (7) were replaced be their values according to the expressions given in [1]. The eigenvalues were obtained in two different ways: the first was directly achieved by equating the determinants of the truncated system to zero; these are the values $\gamma_i^{(j)}$ as defined above. The second way consists in substituting the values $\gamma_i^{(j)}$ as well as the corresponding solutions $\Phi_{n-}\left(\gamma_i^{(j)}\right)$ of system (7) in equation (11) together with (12) and (13). To get some idea, table 1 provides the first three eigenvalues of the problem in the cases $c = \frac{s}{\pi}$, $s = 2,3,4$. In this table, a value calculated through the Rayleigh-Ritz technique is always listed under its corresponding one. It is to be noted that $\gamma_i^{(j)}$ are always on the high side, becoming lower, converging on their corresponding values, and which are almost stable from the outset. These results were to be expected. Thus, the eigenvalues of our mixed problem can be exactly designated, the arguments of the above section can be applied to get the corresponding eigenfunctions as precise as desired.

c		$\frac{\pi}{3}$			$\frac{\pi}{2}$			$\frac{2\pi}{3}$	
$j \setminus i$	1	2	3	1	2	3	1	2	3
4	2.816 2.806	3.464 3.463	4.128 4.120	2.328 2.322	3.416 3.411	3.943 3.929	2.067 2.064	3.055 3.044	4.670 4.603
8	2.813 2.806	3.463 3.463	4.119 4.118	2.326 2.322	3.413 3.411	3.931 3.928	2.065 2.064	3.047 3.043	4.578 4.575
20	2.812 2.806	3.463 3.463	4.118 4.118	2.325 2.322	3.413 3.411	3.930 3.928	2.065 2.064	3.047 3.043	4.577 4.575

Table 1. The first three eigenvalues at $c = \frac{s}{6}\pi$, $s = 2,3,4$, respectively.

Conclusion

It has been shown that an eigenvalue of the truncated equation tends to the exact corresponding one on increasing the order of the truncation set on that equation. As for a truncate eigenfunction, it leads to the designation of the exact one in the limit where the corresponding eigenvalue is sufficiently precise.

Summary

This paper is devoted to studying the influence of the truncation applied to a homogeneous Cauchy-type singular integral equation on its eigenvalues and eigenfunctions. This equation represents the class to which mixed Sturm-Liouville problems of the Dirichlet-Neuman type are reducible. The study is illustrated through considering a concrete problem.

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