УДК 72.03

GEOMETRY OF THE TRACERY OF THE GOTHIC WINDOWS

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Abstract. Window tracery takes a special place in the Gothic architecture. One can observe some problems with the conjugations on such windows – inscribed circle touches two arches of different diameters. Modern computer technology makes it easy to determine the points and radii of such conjugates, but the aim of the work is to understand the general pattern of the construction of such windows. There are three types of the most common Gothic windows: lowered arch, lancet arch and equilateral arch. Exactly the different combinations of such arches and the circumferences inscribed in them give a great variety of window tracery. Studying the geography and various configurations of such windows, one can pay attention to the fact that in most cases such windows are designed according to the same algorithm. This algorithm, according to the authors, ancient architects used window tracery in their designs. The essence of it is as follows: 1. Two rectangular triangles are made; 2. In accordance with their configuration, their legs and hypotenuse are calculated; 3. The radii of inscribed circles are determined with the Pythagorean theorem. The examples of authentic Gothic windows of different buildings were considered to confirm the proposed algorithm: the Cathedral in Salisbury, the church of St. Matthias in Budapest, the Cathedral in Poitiers, the arcade on the Westminster Abbey, the Heidelberg Castle. The authors also suggest using of this method in the restoration of old windows.

Keywords: tracery, lancet window, lowered arch, equilateral arch.

ГЕОМЕТРІЯ МАСВЕРКІВ ГОТИЧНИХ ВІКОН

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Анотація. Особливе місце в готичній архітектурі займають масверкові вікна. При детальному розгляді таких вікон виникають задачі на спряження – вписане коло дотикається двох дуг різного діаметру. Сучасні комп'ютерні технології дозволяють легко визначити точки і радіуси таких спряжень, але мета роботи полягала в тому, що б зрозуміти загальну закономірність конструкції таких вікон. Існує три типи найбільш поширених готичних вікон: ланцет, стрільчата стисла арка і стрільчата рівнобічна арка. Саме різні поєднання таких арок і вписані в них кола дають велике різноманіття масверків. Вивчаючи географію і різні конфігурації таких вікон, можна звернути увагу на те, що в більшості випадків такі вікна спроектовані за єдиним алгоритмом. Цей алгоритм на думку авторів, давні зодчі використовували в

своїх конструкціях масверкових вікон. Його суть в наступному: 1. Складаються два прямокутних трикутника. 2. Відповідно до їх конфігурації розраховуються їх катети і гіпотенузи. 3. За допомогою теореми Піфагора визначаються радіуси вписаних кіл. Для підтвердження запропонованого алгоритму були розглянуті приклади автентичних готичних масверків різних будівель: собору в Солсбері, церкви святого Матіаша в Будапешті, собору в Пуатьє, аркади в Вестмінтерському абатстві, замку в Гейдельберзі. Автори пропонують також використання цього методу при реставрації старовинних вікон.

Ключові слова: масверк, вікно ланцет, стрілчаста стисла арка, стрілчаста рівнобічна арка.

ГЕОМЕТРИЯ МАСВЕРКОВ ГОТИЧЕСКИХ ОКОН

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Аннотация. Особое место в готической архитектуре занимают масверковые окна. При детальном рассмотрении таких окон возникают задачи на сопряжения – вписанная окружность касается двух дуг разного диаметра. Современные компьютерные технологи позволяют легко определить точки и радиусы таких сопряжений, но цель работы состояла в том, что бы понять общую закономерность конструкции таких окон. Существует три типа наиболее распространённых готических окон: ланцет, стрельчатая сжатая арка и стрельчатая равносторонняя арка. Именно различные сочетания таких арок и вписанные в них окружности дают большое многообразие масверков. Изучая географию и различные конфигурации таких окон, можно обратить внимание на то, что в большинстве случаев такие окна спроектированы по единому алгоритму. Этот алгоритм, по мнению авторов, древние зодчие использовали в своих конструкциях масверков. Его суть в следующем: 1. Составляются два прямоугольных треугольника. 2. В соответствии с их конфигурацией рассчитываются их катеты и гипотенузы. 3. При помощи теоремы Пифагора определяются радиусы вписанных окружностей. Для подтверждения предложенного алгоритма были рассмотрены примеры аутентичных готичных окон различных зданий: собора в Солсбери, церкви святого Матиаша в Будапеште, собора в Пуатье, аркады в Вестминтерском аббатстве, замка в Гейдельберге. Авторы предлагают также использование этого метода при реставрации старинных окон.

Ключевые слова: масверк, окно ланцет, стрельчатая сжатая арка, стрельчатая равносторонняя арка.

Introduction. In Gothic architecture, large windows with tracery (decorative frame ornament with circular construction of all elements) fill almost all the walls of the temples. Decorated with colourful stained glass windows, they enhance the presence of the «Divine Light» in the churches and basilicas. The forms of gothic windows are diverse: lancet windows, window-roses, round window-medallions, quadrilateral, elliptical with pictures of historical episodes. The geometry of the tracery appears in the exact location of each element of the figure, centre and concentric partitions [1, 2]. Windows in the form of triangles, squares, stars, circles symbolize the unification of matter and spirit, heavenly and earthly, end and endless, eternal and mortal.

Analysis of recent research and publications. Several Gothic windows were considered in such articles [3, 4], where the radii of conjugate arcs and conjugation points were also determined.

Purpose and tasks. After considering thoroughly the geometry of the distribution of such windows in Western Europe and their various configurations, we can assume that the ancient architects engineered them with a single algorithm. The algorithm made it possible to construct easily various designs of Gothic windows and at the same time to be spread easily throughout the territory where Gothic temples were built.

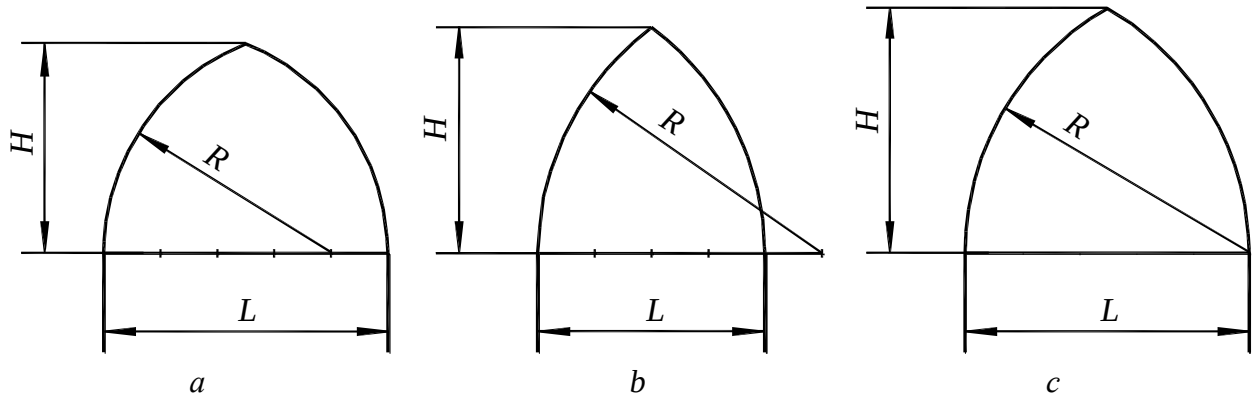
Presenting main material. A special role in the Gothic architecture is played by windows. One of such types of windows is a long narrow window that consists of two arches intersecting at the top (Fig. 1). Changing the arc length and position of its centre changes the proportions of windows. Between these arches a «rose» is always placed. «Rose» is a symbol that appeared in the Gothic after the Crusades of the Templars and it is essentially connected with them. Regarding that, let's consider three most popular shape types of gothic windows.

The first type is lowered arch which radius is equal to 0.8 of the span L, and the centre of the arches are placed inside the gap (Fig. 1, *a*). The height of the arc equals: $H = \frac{\sqrt{55}}{10} L \approx 0.74L$.

The second type is the lancet or lancet arch. In this case, the arc radius of the circle is equal to 1.25 of span L and its centres are situated outside the gap. We can add that the height of the arch-opening in this case is equal to its width $H = L$ (Fig. 1, *b*).

Third type is equilateral arch. In this case, the radius is equal to the span L and height $H = \frac{\sqrt{3}}{2} L \approx 0.87L$ (Fig. 1, *c*).

The variety of combinations of these arch types and «rose» make up the various types of windows that are described below.



a *b* *c*

Fig. 1 Configuration of Gothic arches:
a – lowered arch; *b* – lancet arch; *c* – equilateral arch

Regarding that, we propose to use an algorithm that describes the shape of a Gothic window (Fig. 2):

1. We build a triangle with vertices O_1, O_2 and O_3.
2. We determine the height of O_3K.
3. We determine two hypotenuses $O_1O_3 = R_1 - r$ and $O_2O_3 = R_2 + r$.
4. We determine the bases of the triangles O_1O_3K and O_2O_3K depending on the configuration of the window.
5. Based on Pythagorean Theorem we obtain $O_1O_3^2 - O_1K^2 = O_2O_3^2 - O_2K^2$.

Below, we present some examples of windows and tracery of various authentic buildings coming from XII-XIV centuries. Such a focus on the authenticity is regarded to the need of considering only the windows, which are the real examples of the Medieval Gothic. "Pseudogothic" windows are not considered, as they are only the imitation of the old style. For our examples, the windows with next configuration (as in Fig. 1) were chosen.

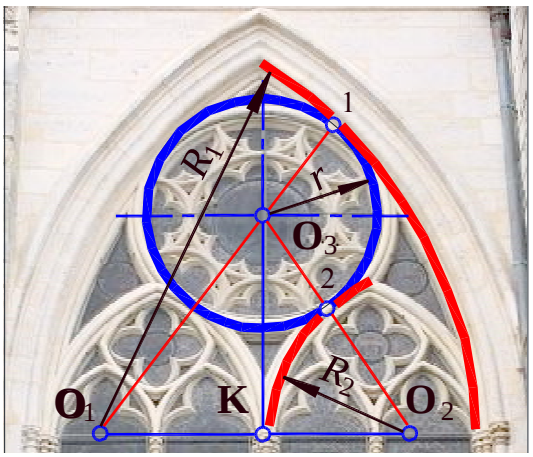


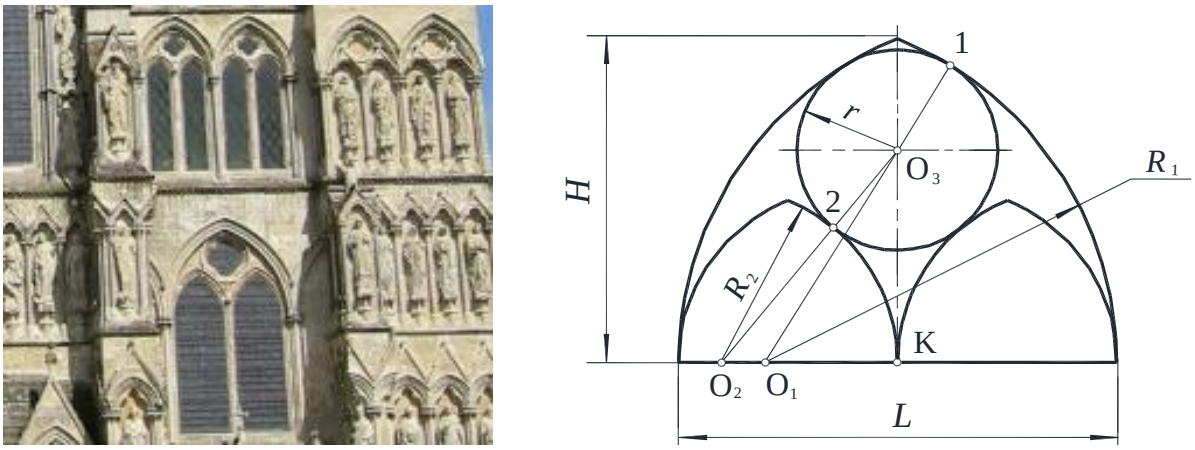
Fig. 2. Configuration of a Gothic window

The first example is the windows and bas-reliefs at the Cathedral of the Virgin Mary in Salisbury (1220-1258, England, Fig. 3, *a*) [5]. By fixing the size of the window, we can say that the outer and inner arches are constructed based on a lowered arch, so we can calculate the radii of both arcs, two hypotenuse and the base of the triangle (Fig. 3, *b*):

$$R_1 = \frac{4}{5}L,\ R_2 = \frac{4}{5}\cdot\frac{L}{2} = \frac{2}{5}L \tag{1}$$

$$O_1K = \frac{4}{5}L - \frac{1}{2}L = \frac{3}{10}L,\quad O_2K = R_2 = \frac{2}{5}L \tag{2}$$

$$O_1O_3 = R_1 - r = \frac{4}{5}L - r,\ \ O_2O_3 = R_2 + r = \frac{2}{5}L + r \tag{3}$$



a *b*

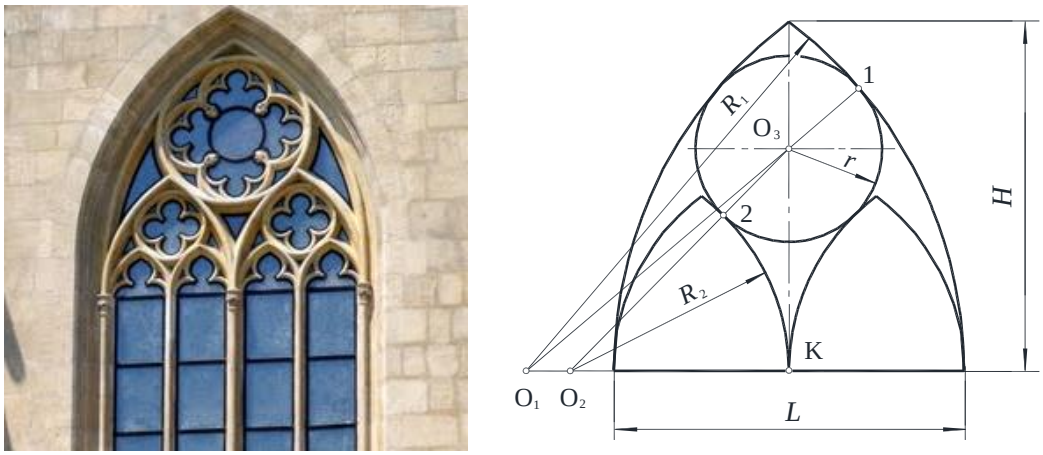
Fig. 3. Windows and bas-reliefs at Salisbury Cathedral and their schemes:
a – windows and bas-reliefs; *b* – the scheme of the window

We can form an equation: $O_1O_3^{\,2} - O_1K^{\,2} = O_2O_3^{\,2} - O_2K^{\,2}$, or:

$$\left(\frac{4}{5}L - r\right)^2 - \left(\frac{3}{10}L\right)^2 = \left(\frac{2}{5}L + r\right)^2 - \left(\frac{2}{5}L\right)^2 \tag{4}$$

where the inscribed circle radius equals: $r = \frac{11L}{48}$.

The second example is shown in Fig. 4, *a* – a window in the church of St. Matthias in Budapest (second half of the 14th century, Hungary). The outer and inner arches are constructed based on the lancet. It means that the height of the ogive is equal to its width (Fig. 4, *b*).



a *b*

Fig. 4. The window in the St. Matthias church in Budapest and its scheme:
a – the window; *b* – the scheme of the window

In this case:

$$R_1 = \frac{5}{4}L, \quad R_2 = \frac{5}{4}\cdot\frac{L}{2} = \frac{5}{8}L \tag{5}$$

$$O_1K = \frac{5}{4}L - \frac{1}{2}L = \frac{3}{4}L, \quad O_2K = R_2 = \frac{5}{8}L \tag{6}$$

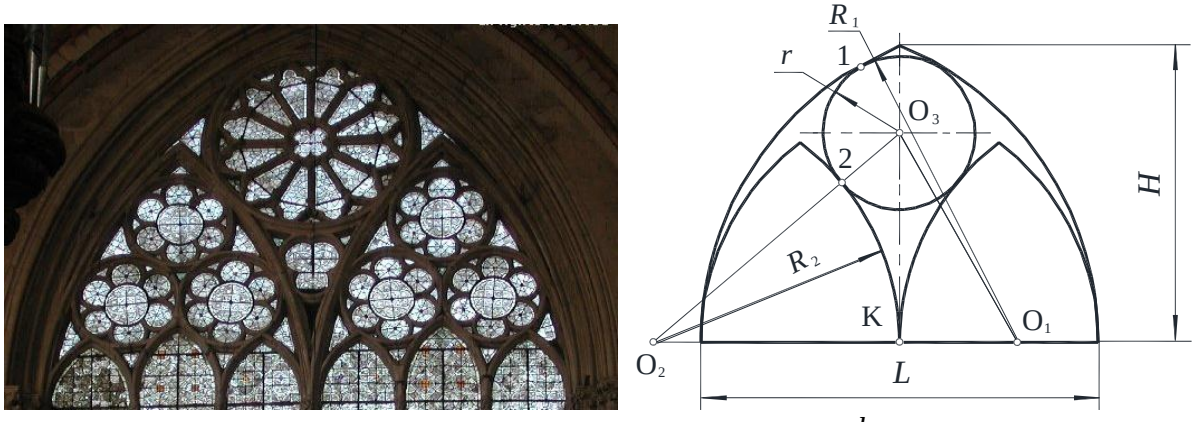
$$O_1O_3 = R_1 - r = \frac{5}{4}L - r, \quad O_2O_3 = R_2 + r = \frac{5}{8}L + r \tag{7}$$

We can form an equation:

$$\left(\frac{5}{4}L - r\right)^2 - \left(\frac{3}{4}L\right)^2 = \left(\frac{5}{8}L + r\right)^2 - \left(\frac{5}{8}L\right)^2, \tag{8}$$

where the inscribed circle radius equals: $r = \frac{4L}{15}$.

The third example is the window from St. Peter's Cathedral in Poitiers, (XII-XIII centuries, France, Fig. 5, *a*). Having made the measurements of the window, we can assert that the outer arches are constructed by the design of a lowered arch, while the interior is constructed with the lancet (Fig. 5, *b*).



a *b*

Fig. 5. The window on the Cathedral of Poitiers and its scheme:
a – the window; *b* – the scheme of the window

Regarding the shape of the window, we can define the following formulas:

$$R_1 = \frac{4L}{5}, \quad R_2 = \frac{L}{2} + \frac{L}{2}\cdot\frac{1}{4} = \frac{5L}{8} \tag{9}$$

$$O_1K = \frac{L}{2} - \frac{L}{5} = \frac{3L}{10}, \quad O_2K = R_2 = \frac{5L}{8} \tag{10}$$

$$O_1O_3 = R_1 - r = \frac{4L}{5} - r, \quad O_2O_3 = R_2 + r = \frac{5L}{8} + r \tag{11}$$

Subsequently, can form an equation:

$$\left(\frac{4L}{5} - r\right)^2 - \left(\frac{3L}{10}\right)^2 = \left(\frac{5L}{8} + r\right)^2 - \left(\frac{5L}{8}\right)^2 \tag{12}$$

where the inscribed circle radius equals: $r = \frac{11L}{57}$.

Below we present an example of an arcade on the outer wall of the Westminster Abbey, London, (1245-1745, England, Fig. 6, *a*) [5]. The construction of the arcade arch is made in the form of a lowered arch and consists of two semicircles and an inscribed circle (Fig. 6, *b*).

The arcade design has the following dimensions:

$$R_1 = \frac{4}{5}L, \; R_2 = \frac{1}{4}L \tag{13}$$

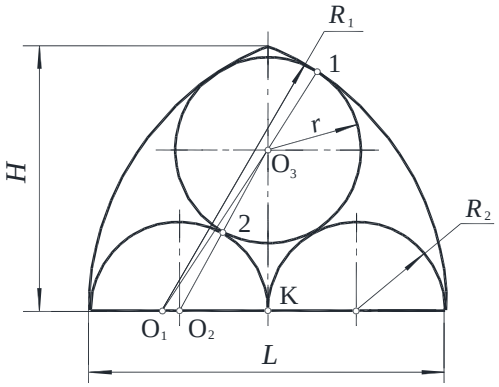
$$O_1K = \frac{4}{5}L - \frac{1}{2}L = \frac{3}{10}L, \quad O_2K = \frac{1}{4}L \tag{14}$$

$$O_1O_3 = R_1 - r = \frac{4}{5}L - r, \quad O_2O_3 = R_2 + r = \frac{1}{4}L + r \tag{15}$$

Thus, we can form the formula:

$$\left(\frac{4}{5}L - r\right)^2 - \left(\frac{3}{10}L\right)^2 = \left(\frac{1}{4}L + r\right)^2 - \left(\frac{1}{4}L\right)^2 \tag{16}$$

where the inscribed circle equals: $r = \frac{11L}{42}$.



a *b*

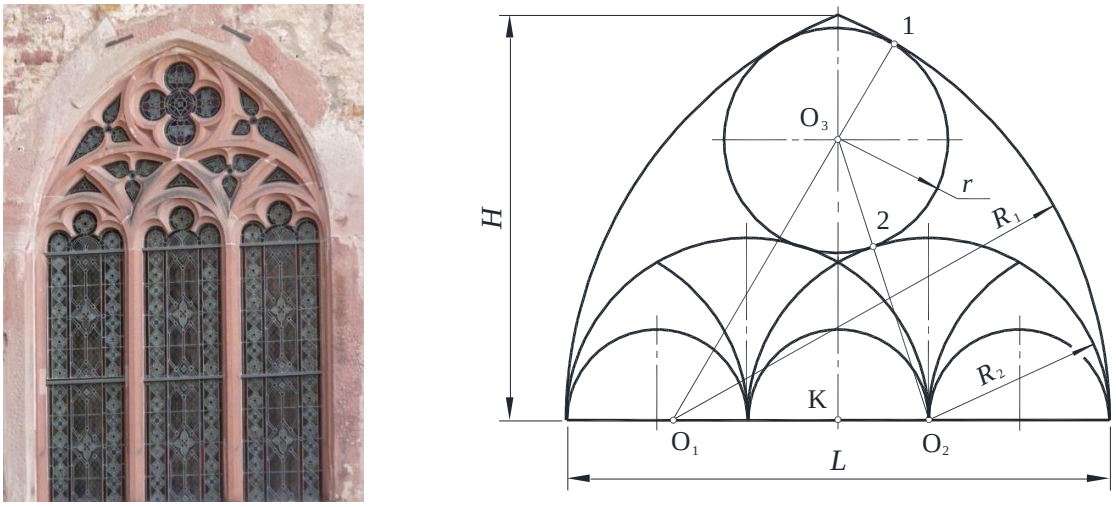
Fig. 6. The arcade on the Westminster Abbey and its scheme:
a – the arcade*;* *b* – the scheme of the arcade

In the following figure (Fig. 7, *a*) we present a window in the Heidelberg Castle (1225-1303, Germany). The given window contains two superimposed semicircles, three semicircles in one row, one circle inscribed, and the outer arches have the construction of lowered arch (Fig. 7, *b*).

$$R_1 = \frac{4L}{5}, \; R_2 = \frac{L}{3} \tag{17}$$

$$O_1K = \frac{L}{2} - \frac{L}{5} = \frac{3L}{10}, \quad O_2K = \frac{L}{6} \tag{18}$$

$$O_1O_3 = R_1 - r = \frac{4L}{5} - r, \; O_1O_2 = R_2 + r = \frac{L}{3} + r \tag{19}$$



a *b*

Fig. 7. The window in the Heidelberg Castle and its scheme:
a – the window; *b* – the scheme of the window

We form the equation:

$$\left(\frac{4L}{5}-r\right)^2-\left(\frac{3L}{10}\right)^2=\left(\frac{L}{3}+r\right)^2-\left(\frac{L}{6}\right)^2 \qquad (20)$$

where the inscribed circle is: $r=\frac{7L}{34}$.

Conclusions and perspectives of further research. With the help of the presented algorithm, we can easily calculate the size of all elements in different window configurations. We can assume that the ancient architects used such method in the design of tracery in Gothic architecture. We believe that it was this method that allowed medieval architects to create such a variety of forms and configurations of these gothic windows. The simplicity of this method allowed the rapid and widespread distribution of such windows throughout the Catholic Europe. In our opinion, the proposed algorithm can also be useful for the restoration of old tracery windows.

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Стаття надійшла 13.03.2018