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A PROBLEM OF IMMERSING THE MAXIMUM RADIUS CIRCLE IN THE POLYHEDRON

The article considers the solution of the problem to determine the maximum radius of the circle placed (immersed) in the polyhedral region bounded by straight lines. This problem can be of practical use for the initial optimal pattern cutting and subsequent manufacture of expensive parts or in serial production. In theoretical terms this problem is solved in a complex with the optimization problem of the minimum coverage.

Keywords. *Circle, maximum radius, polyhedron, linear optimization, objective function, set of constraints, simplex-method.*

У статті розглянуто розв'язок задачі обчислення максимального радіусу кола, зануреного у поліедральну область (обмежену прямими лініями). Це завдання може мати практичне використання для початкового оптимального розкрою і подальшого виготовлення дорогих деталей або в серійному виробництві. У теоретичному плані, таке завдання пов'язується з оптимізаційною проблемою про мінімальне покриття.

Ключові слова. *Коло, максимальний радіус, поліедр, лінійна оптимізація, цільова функція, система обмежень, симплекс-метод.*

В статье рассмотрено решение задачи вычисления максимального радиуса окружности, погруженного в полиэдральную область (ограниченную прямыми линиями). Эта задача может иметь практическое использование для первоначального оптимального раскроя и последующего изготовления дорогостоящих деталей или в серийном производстве. В теоретическом плане, такая задача связана с оптимизационной проблемой минимального покрытия.

Ключевые слова: *Окружность, максимальный радиус, полиэдр, линейная оптимизация, целевая функция, система ограничений, симплекс-метод.*

Analysis of recent researches and publications. The modern mathematical apparatus of the optimization theory in combination with the use of computer technology allows to solve nonlinear problems, but always there is appropriateness of linearization of complex nonlinear problems. Such simplification allows to use accurate classical methods of optimization solution as opposed to approximate one for nonlinear optimization [1; 2; 3; 4].

The purpose of the article. The research task was rigorous mathematical reduction (linearization) to the problem of linear optimization, nonlinear problem of the maximum radius circle immersion in the range of the polyhedron type convex region.

Statement of the base material. We will consider a two-dimensional case. Let us suppose there is a polyhedron Ω_I represented by a system of linear algebraic inequalities

$$\Omega_I : \begin{cases} a_{11}x_1 + a_{12}x_2 \le b_1, \\ a_{21}x_1 + a_{22}x_2 \le b_2, \\ \dots\dots\dots\dots\dots\dots, \\ a_{m1}x_1 + a_{m2}x_2 \le b_m, \end{cases}$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

It is necessary to place the maximum radius circle in Ω_I region. Let us interpret the problem graphically (fig.1).

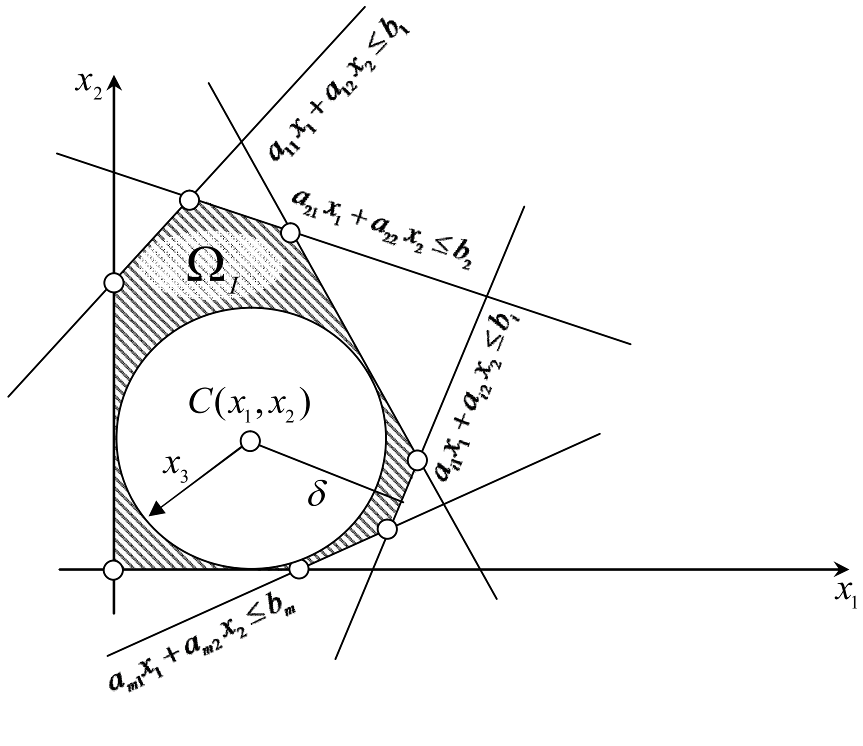

Fig. 1. Graphic interpretation of solving the circle problem

Let x_1, x_2 are coordinates of the circle centre, and x_3 is it's radius. The normal equation of the half-plane boundary represented as $a_{i1}x_1 + a_{i2}x_2 \le b_i$ is given by

$$\frac{-siqn(b_i)a_{i1}}{\sqrt{a_{i1}^2 + a_{i2}^2}}x_1 + \frac{-siqn(b_i)a_{i2}}{\sqrt{a_{i1}^2 + a_{i2}^2}}x_2 - p_i = 0 \cdot$$

It is known that the substitution of the point coordinates in the normal equation of a line is its δ deviation from the line. Since the circle center point $C(x_1, x_2)$ should be placed in the half-plane $a_{i1}x_1 + a_{i2}x_2 \le b_i$,its deviation

$$\delta = \frac{-siqn(b_i)a_{i1}}{\sqrt{a_{i1}^2 + a_{i2}^2}}x_1 + \frac{-siqn(b_i)a_{i2}}{\sqrt{a_{i1}^2 + a_{i2}^2}}x_2 - p_i$$

and the circle radius form the inequality

$$\delta + x_3 \le 0 .$$

There is a statement: the half-plane $a_{i1}x_1 + a_{i2}x_2 \le b_i$ is guaranteed to contain a circle with the point $C(x_1, x_2)$ as a centre and the radius x_3 in the case of the inequality satisfaction

$$a_{i1}x_1 + a_{i2}x_2 + x_3\sqrt{(a_{i1})^2 + (a_{i2})^2} \le b_i .$$

Taking this into account, the problem of the maximum radius circle which is located in the polyhedron is formulated as a problem of linear optimization

$$W_I = x_3 \to \max$$

$$\Omega_I : \begin{cases} a_{11}x_1 + a_{12}x_2 + x_3\sqrt{(a_{11})^2 + (a_{12})^2} \le b_1, \\ a_{21}x_1 + a_{22}x_2 + x_3\sqrt{(a_{21})^2 + (a_{22})^2} \le b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ a_{m1}x_1 + a_{m2}x_2 + x_3\sqrt{(a_{m1})^2 + (a_{m2})^2} \le b_m, \\ -x_1 + x_3 \le 0, \\ -x_2 + x_3 \le 0, \end{cases}$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$$

A model example.

The region is set by the system

$$\Omega_I : \begin{cases} -4x_1 + 3x_2 \le 12, \\ 5x_1 + 12x_2 \le 60, \\ 4x_1 - 3x_2 \le 24, \\ 3x_1 + 4x_2 \le 24, \end{cases}$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

It is necessary to find the center and radius coordinates of the circle that is the largest in area and located in the Ω_I region.

According to the proposed method let us to form the linear optimization problem that is the standard problem of linear optimization. The objective function and the set of constraints are given by

$$W_I = x_3 \to \max$$
$$\Omega_I : \begin{cases} -4x_1 + 3x_2 + 5x_3 \le 12, \\ 5x_1 + 12x_2 + 13x_3 \le 60, \\ 4x_1 - 3x_2 + 5x_3 \le 24, \\ 3x_1 + 4x_2 + 5x_3 \le 24, \\ -x_1 + x_3 \le 0, \\ -x_2 + x_3 \le 0, \end{cases}$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

In order to solve the problem by the simplex-method it needs to proceed to the canonical form of the problem [1; 2]

$$W_I = x_3 \to \max$$
$$\Omega_I : \begin{cases} -4x_1 + 3x_2 + 5x_3 + x_4 = 12, \\ 5x_1 + 12x_2 + 13x_3 + x_5 = 60, \\ 4x_1 - 3x_2 + 5x_3 + x_6 = 24, \\ 3x_1 + 4x_2 + 5x_3 + x_7 = 24, \\ -x_1 + x_3 + x_8 = 0, \\ -x_2 + x_3 + x_9 = 0, \end{cases}$$
$$x_i \ge 0,\ i = 1, \dots, 9.$$

Therefore we have an admissible pivot of beginning of the simplex-method $X_0 = [\,0, 0, 0, | 12, 60, 24, 24, 0, 0\,] \in \Omega_I$. The calculation is performed by the usual simplex-method (Table 1).

Table 1

Basis	C	B	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
			0	0	1	0	0	0	0	0	0
a_4	0	1	-4	3	5	1	0	0	0	0	0
a_5	0	60	5	12	13	0	1	0	0	0	0
a_6	0	2	4	-3	5	0	0	1	0	0	0
a_7	0	24	3	4	5	0	0	0	1	0	0
a_8	0	0	-1	0	1	0	0	0	0	1	0
a_9	0	0	0	-1	1	0	0	0	0	0	1
Δ_j	$W_I(X_0) = 0$		0	0	-1	0	0	0	0	0	0
a_4	0	12	1	3	0	1	0	0	0	-5	0
a_5	0	60	18	12	0	0	1	0	0	-13	0
a_6	0	24	9	-3	0	0	0	1	0	-5	0
a_7	0	2	8	4	0	0	0	0	1	-5	0
a_3	1	0	-1	0	1	0	0	0	0	1	0
a_9	0	0	1	-1	0	0	0	0	0	-1	1
Δ_j	$W_I(X_1) = 0$		-1	0	0	0	0	0	0	1	0
a_4	0	12	0	4	0	1	0	0	0	-4	-1
a_5	0	6	0	3	0	0	1	0	0	5	-18
a_6	0	24	0	6	0	0	0	1	0	4	-9
a_7	0	24	0	12	0	0	0	0	1	3	-8
a_3	1	0	0	-1	1	0	0	0	0	0	1
a_1	0	0	1	-1	0	0	0	0	0	-1	1
Δ_j	$W_I(X_2) = 0$		0	-1	0	0	0	0	0	0	1
a_4	0	4	0	0	0	1	- 2/15	0	0	- 14/3	7/5
a_2	0	2	0	1	0	0	1/30	0	0	1/6	- 3/5
a_6	0	1	0	0	0	0	- 1/5	1	0	3	- 27/5
a_7	0	0	0	0	0	0	- 2/5	0	1	1	- 4/5
a_3	1	2	0	0	1	0	1/30	0	0	1/6	2/5
a_1	0	2	1	0	0	0	1/30	0	0	- 5/6	2/5
Δ_j	$W_I(X_3) = 2$		0	0	0	0	1/30	0	0	1/6	2/5

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Стаття надійшла до редакції

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