Mathematical Subject Classification: 34A08, 34B15
UDC 517.9

A. V. Mykhailenko
Odesa National Economic University

FRACTIONAL BOUNDARY-VALUE PROBLEM

Михайленко А. В. Дробові крайові задачі. В роботі отримано достатні умови існування та єдиності розв'язку крайової задачі для нелінійного диференціального рівняння дробового порядку з похідною Рімана—Ліувіля.
Ключові слова: крайова задача, існування, єдиність, дробова похідна.

Михайленко А. В. Дробные краевые задачи. В работе получены достаточные условия существования и единственности решения краевой задачи для нелинейного дифференциального уравнения дробного порядка с производной Римана—Лиувилля.
Ключевые слова: краевая задача, существование, единственность, дробная производная.

Mykhailenko A. V. Fractional boundary-value problem. In this paper we establish sufficient conditions for the existence and uniqueness of solution of boundary-value problem for fractional differential equation with Riemann—Liouville derivative.
Key words: boundary-value problem, existence, uniqueness, fractional derivative.

Introduction. Differential equations of fractional order have numerous applications to problems in electrochemistry, biology, electromagnetics, control, viscoelasticity, etc. [11,3,7,4] Treatises of many autors are dedicated to the research of initial value problems [10,11]. Boundary-value problem for fractional differential equations have been considered in [1,10,14,2,15,13].

In [13] there are established the conditions of existence and uniqueness of positive solution for a Dirichlet-type problem of the nonlinear fractional differential equation

$$D_0^\alpha u(t) + f(t, u(t)) = 0, 0 < t < 1, 1 < \alpha < 2,$$
$$u(0) = u(1) = 0,$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and D_0^α is the fractional derivative of Riemann—Liouville.

In [14] it was proved the existence of positive solutions of the problem

$$\widetilde{D}_0^\alpha u(t) = f(t, u(t)), 0 < t < 1, 1 < \alpha < 2,$$
$$u(0) + u'(0) = 0, u(1) + u'(1) = 0,$$

where $\widetilde{D}_0^\alpha$ is the derivative of Caputo, and function f in $[0,1] \times [0,\infty)$ is nonnegative and continuous.

In [15] it was considered the boundary-value problem

$$\widetilde{D}_0^\alpha u(t) = f(t, u(t), \widetilde{D}_0^\beta u(t)), 1 < \alpha \le 2, 0 < \beta \le 1,$$
$$a_1 u(0) - a_2 u'(0) = A, b_1 u(1) + b_2 u'(1) = B,$$

where $a_i, b_i \geq 0, i = 1, 2, a_1 b_1 + a_1 b_2 + a_2 b_1 > 0, f : [0, 1] \times R \times R \to R$ is continuous function. The existence of solution was proved.

In this paper we consider the boundary-value problem

$$D_0^{1+\alpha} u(x) = F\left(x, u(x), D_0^{\alpha} u(x)\right), 0 < \alpha \leq 1, u(0) = u(a) = 0, \tag{1.1}$$

where function $F(x, y, z) : [0, a] \times R \times R \to R$ is measurable with respect to x for $(y, z) \in R \times R$ and continuous with respect to (y, z) for $x \in [0, a]$, and satisfies Lipschitz condition with respect to y and z as well. It was proved the existence and uniqueness of solution of this problem.

This paper is organized as follows. In Section 2 we introduce some preliminary results needed in the next sections. In Section 3 we present an existence and uniqueness result for the problem (1.1).

2. Preliminaries. In this section we introduce definitions and preliminary facts that will be used in this paper. Let $C(J), J = [0, a]$ be the Banach space of continuous functions $f : J \to R$ with the norm

$$\|f(x)\|_C = \{\max |f(x)| : 0 \leq x \leq a\}$$

and lets denote by $L(J)$ the Banach space of measurable functions $f(x)$ that are Lebesgue integrable with norm

$$\|f(x)\|_L = \int_0^a |f(x)| dx.$$

By $AC^n(J)$ we denote the set of continuously differentiable till the $(n-1)$ order in J functions , and $f^{(n-1)}(x) \in AC(J)$.

Let $\gamma \geq 0$ be a real number and $n = [\gamma] + 1$ where $[\gamma]$ is the integer part of γ. For a function $f : J \to R$ the expressions [1,2]

$$f_\gamma(x) = I_0^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x - t)^{(\gamma - 1)} f(t) dt, \tag{2.1}$$

$$D_0^\gamma f(x) = \frac{1}{\Gamma(n - \gamma)} \left(\frac{d}{dx}\right)^n \int_0^x (x - t)^{n - \gamma - 1} f(t) dt \tag{2.2}$$

are called, respectively, the Riemann—Liouville left-hand fractional integral and derivative of order γ.

Lemma 2.1. [3] *Assume that $(f_k(x))_{k=1}^{\infty}$ is a uniformly convergent to $f(x)$ sequence of continuous functions. Then $\lim_{k\to\infty} I_0^\gamma f_k(x) = I_0^\gamma f(x)$.*

Lemma 2.2. [12,3]. *Let $\gamma > 0, n = [\gamma] + 1$. Assume that $f(x)$ is such that $f_{n-\gamma}(x) \in AC^n(J)$. Then*

$$I_0^\gamma D_0^\gamma f(x) = f(x) - \sum_{k=0}^{n-1} \frac{x^{\gamma - k - 1}}{\Gamma(\gamma - k)} f_{n-\gamma}^{(n-k-1)}(0),$$

where $f_{n-\gamma}^{(n-k-1)}(0) = \lim_{x\to 0+} f_{n-\gamma}^{(n-k-1)}(x)$.

Lemma 2.3. [9] *Assume that $f : J \to R$ is measurable function and $|f(x)| \leq M$. Then $\mu(x) = I_0^\gamma f(x) \in C(J)$ and $\mu(0) = 0$.*

Lemma 2.4. [8] *Let σ_1 and σ_2 are any positive numbers and let $0 \le \mu \le 1$. Then $|\sigma_1^\mu - \sigma_2^\mu| \le |\sigma_1 - \sigma_2|^\mu$.*

It is considered boundary-value problem

$$D_0^{1+\alpha} y(x) = f(x), \tag{2.3}$$

$$y(0) = y(a) = 0, \tag{2.4}$$

where $0 < \alpha \le 1$, $f : J \to R$ is measurable function, and $|f(x)| \le M$.

Definition 2.1 *By solution of problem (2.3), (2.4) we name such function $y : J \to R$ that: (i) $y(x) \in C(J), y_{1-\alpha} \in AC^2[J]$; (ii) satisfies the boundary conditions (2.4); (iii) satisfies the differential equation (2.3) for a.a. $x \in J$.*

Lemma 2.5. *Let $f : J \to R$ is measurable function and $|f(x)| \le M$. Then the boundary-value problem (2.3), (2.4) has a unique solution*

$$y(x) = \int_0^a G(x,t) f(t) dt, \tag{2.5}$$

where

$$G(x,t) = \left[\begin{array}{c} -\frac{(x(a-t))^\alpha - (a(x-t))^\alpha}{a^\alpha \Gamma(1+\alpha)}, 0 \le t \le x \\ -\frac{(x(a-t))^\alpha}{a^\alpha \Gamma(1+\alpha)}, x \le t \le a. \end{array} \right. \tag{2.6}$$

Here $G(x,t)$ is the Green's function of boundary-value problem (2.3), (2.4).

Proof. Suppose that the solution of problem (2.3), (2.4) exists. Then corresponding to (2.2)

$$D_0^{1+\alpha} y(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dx}\right)^2 \int_0^x (x-t)^{-\alpha} y(t) dt = y''_{1-\alpha}(x) \in L(J).$$

Consequently

$$I_0^{1+\alpha} D_0^{1+\alpha} y(x) = I_0^{1+\alpha} f(x).$$

As a consequence of Lemma 2.2

$$I_0^{1+\alpha} D_0^{1+\alpha} y(x) = y(x) - \frac{x^\alpha}{\Gamma(1+\alpha)} y'_{1-\alpha}(0) - \frac{x^{\alpha-1}}{\Gamma(\alpha)} y_{1-\alpha}(0),$$

at that in accord with lemma 2.3 $y_{1-\alpha}(0) = 0$. Consequently

$$y(x) - \frac{x^\alpha}{\Gamma(1+\alpha)} y'_{1-\alpha}(0) = I_0^{1+\alpha} f(x). \tag{2.7}$$

As $y(a) = 0$, then from (2.7) at $x = a$ it follows that

$$y'_{1-\alpha}(0) = -\frac{1}{a^\alpha} \int_0^a (a-t)^\alpha f(t) dt.$$

Then

$$y(x) = -\frac{x^\alpha}{a^\alpha \Gamma(1+\alpha)} \int_0^a (a-t)^\alpha f(t) dt + \frac{1}{\Gamma(1+\alpha)} \int_0^x (x-t)^\alpha f(t) dt. \tag{2.8}$$

Lets represent (2.8) as following:

$$y(x) = -\frac{x^\alpha}{a^\alpha\Gamma(1+\alpha)}\left(\int_0^x (a-t)^\alpha f(t)dt + \int_x^a (a-t)^\alpha f(t)dt\right) +$$

$$+\frac{1}{\Gamma(1+\alpha)}\int_0^x (x-t)^\alpha f(t)dt = \int_0^x \left[-\frac{(x(a-t))^\alpha - (a(x-t))^\alpha}{a^\alpha\Gamma(1+\alpha)}\right] f(t)dt +$$

$$+\int_x^a \left(-\frac{(x(a-t))^\alpha}{a^\alpha\Gamma(1+\alpha)}\right) f(t)dt = \int_0^a G(x,t)f(t)dt.$$

Main Results. Lets consider the differential equation

$$D_0^{1+\alpha}y(x) = F[y(x)] \equiv F(x, y(x), D_0^\alpha y(x)), 0 < \alpha \le 1, \tag{3.1}$$

which solutions satisfy boundary conditions (2.4). Let $F(x,y,z) : J \times R \times R \to R$ satisfies conditions: (a) continuous with respect to $(y,z) \in R \times R$ for fixed $x \in J$ and measurable with respect to $x \in J$ for fixed $(y,z) \in R \times R$; (b) $|F(x,y,z)| \le M$ for $(x,y,z) \in J \times R \times R$.

Definition 3.1 *As the solution of boundary-value problem (3.1), (2.4) we name function $y : J \to R$, which satisfies conditions (i), (ii) of definition 2.1 and differential equation (3.1) for a.a. $x \in J$.*

Theorem 3.1 *Let function $F(x,y,z) : J \times R \times R$ satisfies conditions (a), (b). A function $y(x) \in C(J)$ will be the solution of boundary-value problem (3.1), (2.4) if and only if it is a solution of the integral equation*

$$y(x) = \int_0^a G(x,t)F\left(t, y(t), D_0^\alpha y(t)\right)dt. \tag{3.2}$$

Proof. Let $y(x) \in C(J)$ is a solution of boundary-value problem (3.1), (2.4). Then function $F(x, y(x), D_0^\alpha y(x)) : J \to R$ is measurable and $|F(x, y(x), D_0^\alpha y(x))| \le M$. By lemma 2.5 $y(x)$ is the solution of integral equation (3.2). Now let $y(x) \in C(J)$ be a solution of integral equation (3.2) and lets prove that $y(x)$ is the solution of boundary-value problem (3.1), (2.4). By (2.8) the solution of integral equation (3.2) is representable as following:

$$y(x) = -\frac{x^\alpha\delta}{a^\alpha\Gamma(1+\alpha)} + I_0^{1+\alpha}F[y(x)], \tag{3.3}$$

where $\delta = \int_0^a (a-t)^\alpha F[y(t)]dt$. Then

$$\begin{gathered} y_{1-\alpha}(x) = I_0^{1-\alpha}y(x) = \tfrac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}\left(\tfrac{-\delta t^\alpha}{a^\alpha\Gamma(1+\alpha)}\right)dt + \\ +I_0^{1-\alpha}I_0^{1+\alpha}F[y(x)] = -\tfrac{x\delta}{a^\alpha} + I_0^2 F[y(x)], \end{gathered} \tag{3.4}$$

where

$$I_0^2 F[y(x)] = \int_0^x (x-t)F[y(t)]dt.$$

Beside this,

$$D_0^\alpha y(x) = y'_{1-\alpha}(x) = -\frac{\delta}{a^\alpha} + \int_0^x F[y(t)]dt, x \in J, \tag{3.5}$$

$$D_0^{1+\alpha} y(x) = F[y(x)] = F(x, y(x), D_0^\alpha y(x)), \tag{3.6}$$

for a.a. $x \in J$.

From (3.4), (3.5) it follows that $y_{1-\alpha}(x) \in AC^2(J)$ and from (3.6) it follows that $y(x)$ satisfies the equation (3.1) for a.a. $x \in J$. From (3.3) follows that $y(0) = y(a) = 0$.

Theorem 3.2 *Let function $F(x, y, z) : J \times R \times R \to R$ satisfies the conditions (a), (b) and the condition of Lipschitz*

$$|F(x, y, z) - F(x, y_1, z_1)| \le L_1 |y - y_1| + L_2 |z - z_1|,$$

at that

$$\rho(a) = \frac{L_1 a^{1+\alpha}}{4^\alpha \Gamma(1+\alpha)} + L_2 a < \frac{\alpha + 1}{\alpha + 2}.$$

Then exists the unique solution of boundary-value problem (3.1), (2.4) at $x \in [0, a]$.

Proof. By $C_\alpha(J)$ we denote the set of functions $u : J \to R$ such that $u(x) \in \in C(J), D_0^\alpha u(x) \in C(J)$ with the norm

$$\|u(x)\|_\alpha = \max\left(\max_J |u(x)|, \frac{a^\alpha}{4^\alpha \Gamma(\alpha+1)} \max_J |D_0^\alpha u(x)| \right).$$

Lets prove that the space $C_\alpha(J)$ with the norm $\| \cdot \|_\alpha$ is full. Let $(u_k(x))_{k=1}^\infty$ is fundamental sequence in $C_\alpha(J)$. Then uniformly in J $u_k(x) \to u(x), D_0^\alpha u_k(x) \to \to v(x)$ at $k \to \infty$ and $u(x) \in C(J), v(x) \in C(J)$. By lemma 2.1 $\lim_{k\to\infty} u_{k,1-\alpha}(x) = = \lim_{k\to\infty} I_0^{1-\alpha} u_k(x) = I_0^{1-\alpha} u(x) = u_{1-\alpha}(x)$, at that by lemma 2.3 $u_{1-\alpha}(x) \in C(J)$, $u_{1-\alpha}(0) = 0$. As

$$u_{k,1-\alpha}(x) = \int_0^x D_0^\alpha u_k(t) dt,$$

at $k \to \infty$ will receive that

$$u_{1-\alpha}(x) = \int_0^x v(t) dt.$$

Consequently $v(x) = u'_{1-\alpha}(x) = D_0^\alpha u(x)$. So the sequence $(u_k(x))_{k=1}^\infty \subset C_\alpha(J)$ is convergent by norm $\| \cdot \|_\alpha$ to the function $u(x) \in C_\alpha(J)$.

For $u(x) \in C_\alpha(J)$ lets define the operator T, supposed that

$$Tu(x) = \int_0^a G(x, t) F(t, u(t), D_0^\alpha u(t)) dt. \tag{3.7}$$

Lets prove that $T : C_\alpha(J) \to C_\alpha(J)$. Let $w(x) = Tu(x)$ and $0 \le x_1 < x_2 \le a$. Then

$$|w(x_2) - w(x_1)| \le M\Big(\int_0^{x_1} |G(x_2, t) - G(x_1, t)| dt + \int_{x_1}^{x_2} |G(x_2, t) - G(x_1, t)| dt +$$

$$+ \int_{x_2}^a |G(x_2, t) - G(x_1, t)| dt\Big) = M(A_1 + A_2 + A_3).$$

Applying lemma 2.4 and (2.6) will receive that

$$A_1 = \frac{1}{a^\alpha \Gamma(\alpha+1)} \int_0^{x_1} | - (x_2(a-t))^\alpha + (a(x_2 - t))^\alpha + (x_1(a-t))^\alpha - (a(x_1 - t))^\alpha | dt \le$$

$$\leq \frac{1}{a^\alpha\Gamma(1+\alpha)}\int_0^{x_1}(|(x_2(a-t))^\alpha-(x_1(a-t))^\alpha|+$$

$$+|(a(x_2-t))^\alpha-(a(x_1-t))^\alpha|)dt\leq$$

$$\leq \frac{1}{a^\alpha\Gamma(1+\alpha)}\int_0^{x_1}\left((a-t)^\alpha(x_2^\alpha-x_1^\alpha)+a^\alpha|(x_2-t)^\alpha-(x_1-t)^\alpha|\right)dt\leq$$

$$\leq \frac{2a(x_2-x_1)^\alpha}{\Gamma(1+\alpha)}.$$

By analogy we prove that $A_k\leq\frac{2a(x_2-x_1)^\alpha}{\Gamma(1+\alpha)}, k=2,3$. Consequently

$$|w(x_2)-w(x_1)|\leq\frac{6Ma(x_2-x_1)^\alpha}{\Gamma(1+\alpha)}.$$

Therefore if $|x_2-x_1|\leq\delta_1,\delta_1=\left(\frac{\Gamma(1+\alpha)\varepsilon}{6Ma}\right)^{\frac{1}{\alpha}}$, then $|w(x_2)-w(x_1)|\leq\varepsilon$. So, $w(x)\in C(J)$.

In accord with (3.4), (3.5) will receive that

$$w_{1-\alpha}(x)=-\tfrac{x\lambda}{a^\alpha}+I_0^2F[u(x)], D_0^\alpha w(x)=w'_{1-\alpha}(x)= \\ =-\tfrac{\lambda}{a^\alpha}+\textstyle\int_0^x F[u(t)]dt, \tag{3.8}$$

where $\lambda=\int_0^a(a-t)^\alpha F[u(t)]dt$. From (3.8) follows that $D_0^\alpha w(x)\in C(J)$. Lets note that corresponding to (3.8) $w_{1-\alpha}(x)\in AC(J), D_0^\alpha w(x)\in AC(J)$. Therefore the fixed point of operator T will be the solution of boundary-value problem (3.1), (2.4). We need only to prove that the operator T is the contraction mapping in $C_\alpha(J)$. Suppose $u_k(x)\in C_\alpha(J), v_k(x)=D_0^\alpha u_k(x), w_k(x)=Tu_k(x), k=1,2$. Since $|G(x,t)|\leq a^\alpha/(4^\alpha\Gamma(1+\alpha))$, then

$$|w_1(x)-w_2(x)|\leq\int_0^a|G(x,t)||F[u_1(t)]-F[u_2(t)]|dt\leq$$

$$\leq\frac{a^\alpha}{4^\alpha\Gamma(1+\alpha)}\int_0^a\left(L_1|u_1(t)-u_2(t)|+L_2|v_1(t)-v_2(t)|\right)dt\leq$$

$$\leq\frac{a^{\alpha+1}L_1}{4^\alpha\Gamma(1+\alpha)}\max_J|u_1(x)-u_2(x)|+L_2a\left(\frac{a^\alpha}{4^\alpha\Gamma(1+\alpha)}\max_J|v_1(x)-v_2(x)|\right)\leq$$

$$\leq\rho(a)\|u_1(x)-u_2(x)\|_\alpha.$$

Lets write $w_k(x)$ in the form (3.3) $w_k(x)=-\frac{x^\alpha\lambda_k}{a^\alpha\Gamma(1+\alpha)}+I_0^{1+\alpha}F[u_k(x)]$, where $\lambda_k=\int_0^a(a-t)^\alpha F[u_k(t)]dt, k=1,2$. Then

$$D_0^\alpha w_k(x)=-\frac{\lambda_k}{a^\alpha}+\int_0^x F[u_k(t)]dt, k=1,2,$$

$$|D_0^\alpha w_1(x)-D_0^\alpha w_2(x)|\leq$$

$$\leq\frac{1}{a^\alpha}\int_0^a(a-t)^\alpha|F[u_1(t)]-F[u_2(t)]|dt+\int_0^x|F[u_1(t)]-F[u_2(t)]|dt\leq$$

$$\leq \frac{1}{a^\alpha}\left(L_1 \max_J |u_1(x) - u_2(x)| + L_2 \max_J |v_1(x) - v_2(x)|\right)\frac{a^{\alpha+1}}{\alpha+1}+$$

$$+a\left(L_1 \max_J |u_1(x) - u_2(x)| + L_2 \max_J |v_1(x) - v_2(x)|\right) \leq$$

$$\leq \frac{a(\alpha+2)}{\alpha+1}\left(L_1 \max_J |u_1(x) - u_2(x)| + L_2 \max_J |v_1(x) - v_2(x)|\right),$$

$$\frac{a^\alpha}{4^\alpha \cdot \Gamma(\alpha+1)}|D_0^\alpha w_1(x) - D_0^\alpha w_2(x)| \leq \frac{\alpha+2}{\alpha+1}\times$$

$$\times\left(\frac{a^{\alpha+1}L_1}{4^\alpha \cdot \Gamma(\alpha+1)}\max_J |u_1(x) - u_2(x)| + L_2 a\frac{a^\alpha}{4^\alpha \cdot \Gamma(\alpha+1)}\max_J |v_(x) - v_2(x)|\right) \leq$$

$$\leq \frac{\alpha+2}{\alpha+1}\rho(a)\|u_1(x) - u_2(x)\|_\alpha.$$

Consequently $\|w_1(x) - w_2(x)\|_\alpha = \|Tu_1(x) - Tu_2(x)\|_\alpha \leq \tau\|u_1(x) - u_2(x)\|_\alpha$, $\tau = \frac{\alpha+2}{\alpha+1}\rho(a)$. Since $\tau < 1$, then operator T is a contracting mapping in $C_\alpha(J)$. Then by Banach contraction fixed point theorem, the boundary-value problem (3.1), (2.4) has a unique solution.

Remark 1. *Let boundary conditions (2.4) look like $y(0) = 0, y(a) = B$. Lets find the solution of boundary-value problem $D_0^{1+\alpha}z(x) = 0, z(0) = 0, z(a) = B$. Applying lemma 2.2, we receive*

$$z(x) - \frac{x^\alpha}{\Gamma(1+\alpha)}z'_{1-\alpha}(0) = 0. \tag{3.9}$$

From (3.9) at $x = a$ follows that $z'_{1-\alpha}(0) = (B \cdot \Gamma(1-\alpha))/a^\alpha$. Consequently

$$z(x) = \frac{x^\alpha B}{a^\alpha}, z_{1-\alpha}(x) = \frac{x \cdot B \cdot \Gamma(1+\alpha)}{a^\alpha}, D_0^\alpha z(x) = \frac{B \cdot \Gamma(1+\alpha)}{a^\alpha}.$$

The change of variable $y(x) = u(x) + z(x)$ leads to boundary-value problem

$$D_0^{1+\alpha}u(x) = g(x, u(x), D_0^\alpha u(x)), u(0) = u(a) = 0,$$

where $g(x, u(x), D_0^\alpha u(x)) = f\left(x, u(x) + \frac{x^\alpha B}{a^\alpha}, D_0^\alpha u(x) + \frac{B\cdot\Gamma(1+\alpha)}{a^\alpha}\right)$.

Conclusion. Sufficient conditions for the existence and uniqueness of solution of boundary-value problem for fractional differential equation with Riemann—Liouville derivative were establised in this paper.

1. **Aleroev T. S.** The Sturm–Liouville problem for a second order ordinary differential equation with fractional derivatives in the lower terms (Russian) / T. S. Aleroev // Differentsial'nye Uravneniya. – 1982. – V. 18, N 2. – P. 341–342.
2. **Bai Z. B.** Positive solution for boundary value problem of nonlinear fractional differential equation / Z. B. Bai, H. S. Lü // J. Math. Anal. Appl. – 2005. – V. 311, N 2. – P. 495–505.

3. **Diethelm Kai.** The analysis of fractional difeerential equation / Kai Diethelm. – Spring-Verlag Heidelberg, 2010. – 247 p.
4. **Hilfer R.** Applications of Fractional Calculus in Physics / R. Hilfer // Word Scientific. – Singapore, 2000.
5. **Kilbas A. A.** Differential equations of fractional order: methods, results and problems / A. A. Kilbas, J. J. Trujillo // J. Appl. Anal. – 2001. – V. 78. – P. 153–192.
6. **Kilbas A. A.** Differential equations of fractional order: methods, results and problems / A. A. Kilbas, J. J. Trujillo // J. Appl. Anal. – 2002. – V. 81. – P. 435–493.
7. **Kilbas A. A.** Theory and Applications of Fractional Differential Equations / A. A. Kilbas, H. M. Srivastava, J. J. Trujillo // North-Holland Mathematics Studies, 204. – Elsevier Science B.V. – Amsterdam, 2006.
8. **Mushelishvili N. I.** Singular integral equations / N. I. Mushelishvili. – Sci., M., 1968. – 511 p.
9. **Mykhailenko A. V.** Semi-implicit differential equations of fractional order / A. V. Mykhailenko // J. Num. Appl. Math. – 2010. – N1 (100). – P. 1–6.
10. **Nakhushev A. M.** The Sturm—Liouville problem for a second order ordinary differential equation with fractional derivatives in the lower terms / A. M. Nakhushev // Dokl. Acad. Nauk SSSR. – 1977. – V. 234. – P. 308–311.
11. **Podlubny I.** Fractional Differential Equation / I. Podlubny. – Academic Press, San Diego, 1999. – 340 p.
12. **Samko S. G.** Fractional Integrals and Derivatives. Theory and Applications / S. G. Samko, A. A. Kilbas, O. I. Marichev. – Gorden and Breach, New York, 1993. – 687 p.
13. **Su X.** Solutions to boundary-value problems for nonlinear differential equations of fractional order / X. Su, S. Zhang // Electronic Journal of Differential equations. – 2009. – V. 26. – P. 1–15.
14. **Zhang S. Q.** Existence of solution for a boundary value problem of fractional order / S. Q. Zhang // Acta Mathematica Scietia. – 2006. – 26, B(2). – P. 220–228.
15. **Zhang S. Q.**Positive solution for boundary-value probles of nonlinear fractional differential equation / S. Q. Zhang // Electronic Journal of Differential Equations. – 2002. – V. 36. – P. 1–12.