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SMARANDACHE CEIL FUNCTION OVER $\mathbb{Z}[i]$

Сергєєв С. С. Функція Смарандаче над $\mathbb{Z}[i]$. Вивчені арифметичні власти-вости функції Смарандаче $S_k(\omega)$ та двоїстої функции Смарандаче $\bar{S}_k(\omega)$ над цілими гаусовими числами. Отримані асимптотичні оцінки суматорної функції для функції Смарандаче у секторі та двоїстої функції Смарандаче.
Ключові слова: функція Смарандаче, двоїста функція Смарандаче, асимптотичні оцінки.

Сергеев С. С. Функция Смарандаче над $\mathbb{Z}[i]$. Изучены арифметические свойства функции Смарандаче $S_k(\omega)$ и двойственной функции Смарандаче $\bar{S}_k(\omega)$ над целыми гауссовыми числами. Получены асимптотические оценки сумматорной функции для функции Смарандаче в секторе, а также для двойственной функции Смарандаче.
Ключевые слова: функция Смарандаче, двойственная функция Смарандаче, асимптотические оценки.

Sergeev S. S. Smarandache ceil function over $\mathbb{Z}[i]$. We use analytic method to study the arithmetic properties of the Smarandache ceil function $S_k(\omega)$ and its dual $\bar{S}_k(\omega)$ over the ring of Gaussian integers $\mathbb{Z}[i]$. Constructed asymptotic formula summatory functions for Smarandache ceil function $S_k(\omega)$ in sector and for its dual $\bar{S}_k(\omega)$.
Key words: Smarandache ceil function, dual function to Smarandache ceil function, asymptotic estimates.

Introduction. For any fixed positive integer $k \geq 2$, the Smarandache ceil function of order k were introduced by F. Smarandache [9] and has the following definition.

$$S_k(n) = min\left\{m \in \mathbb{N} : m|m^k\right\}, \forall n \in \mathbb{N}$$

The dual function of $S_k(n)$ is defined as

$$\bar{S}_k(n) = max\left\{m \in \mathbb{N} : m^k|n\right\}, \forall n \in \mathbb{N}$$

There are many papers on the Smarandache ceil function and its dual. Ding Liping [4] studied the mean value properties of the Smarandache ceil function $S_k(n)$, and obtained a sharp asymptotic formula for it. Xiaoyan Li [11] adn P. Varbanets and S. Kirabt [10] estimated an error term in asymptotic formulae for the mean value of the Smarandache dual function $\bar{S}_k(n)$.

In this paper, we use the analytic method to study the arithmetic properties of the Smarandache ceil function and its dual over the ring of Gaussian integers $\mathbb{Z}[i]$.

Notation. For $\alpha \in \mathbb{Z}[i]$ we define

$$S_k(\alpha) := min\left\{N(\omega), \omega \in \mathbb{Z}[i], 0 \leq \arg\omega \leq \frac{\pi}{2} : \alpha|\omega^k\right\},$$

$$\bar{S}_k(\alpha) := max\left\{N(\omega), \omega \in \mathbb{Z}[i], 0 \le \arg\omega \le \frac{\pi}{2} : \omega^k|\alpha\right\}.$$

Its easy to check that functions are multiplicative. Also we use some common notations sush as $\mathbb{Z}[i]$ for a ring of Gaussian integers, $\alpha \in \mathbb{Z}[i], \alpha = a+bi, N(\alpha) = a^2+b^2$, $\xi(s)$ — Hecke's zetta-function, $L(s,\chi_4)$ — Dirichlet L-function with non-principal character χ_4, $\Gamma(z)$ — totient gamma function, O, $<<$ — Vinogradov's symbol, $\sum^*, (\prod^*)$ — sum (or product) over unassociated Gaussian integers.

Auxiliary arguments. Let $m \in \mathbb{Z}, s \in \mathbb{C}$. Consider Hecke's Z-function defined in half–plane $Res > 1$ by absolutely convergent series

$$Z_m(s) = \sum_{\omega}{}^* e^{4mi\arg\omega} N(\omega)^{-s},$$

symbol * means summarizing over not-associated Gaussian integers $\omega \neq 0$.

Lemma 1. *Hecke's Z-function allow analytic extension over all complex s-plane and is integer function, if $m \neq 0$, and on $m = 0$ $Z_0(s)\zeta(s)L(s,\chi_4)$, where $\zeta(s)$ is Riemann zeta-function, $L(s,\chi_4)$ — Dirichlet L-function with non-principal character χ_4 modulo 4. Moreover we have the following functional equation*

$$\pi^{-s}\Gamma\left(2|m|+s\right)Z_m(s) = \pi^{-(1-s)}\Gamma\left(2|m|+1-s\right)Z_{-m}(1-s).$$

Lemma 2. *We have the following estimates*

(i) $Z_m(s) << \left(m^2+t^2\right)^{\frac{1}{2}-\sigma}\log^4(m^2+t^2+3), s = \sigma+it,$

(ii) $Z_m(\frac{1}{2}+it) << (m^2+t^2+3)^{\frac{1}{6}}\log^4(m^2+t^2+3),$

(iii) $\int\limits_{-T}^{T}\left|Z_m\left(\frac{1}{2}+\delta+it\right)\right|^2 dt << (T+|m|)\left(\log\left(T+|m|\right)\right)^a, a > 0 - const$

if δ is real and $|\delta| < (\log(|t|+|m|+3)^{-1}$.

Proof. For the proof of (i) we can use functional equestion for $Z_m(s)$, apply Stirling formulae for $\Gamma(z)$ and Phragmen-Lindeloef principle (using trivial estimation for $Z_m\left(1+\frac{1}{\log(T+|m|)}+it\right) << \log\left(T+|m|\right)$ and $Z\left(-\frac{1}{\log(T+|m|)}+it\right) << (m^2+t^2+3)^{\frac{1}{2}}\log^4(m^2+t^2+3)$.

Equation (ii) proved by P. Kaufman [7], and estimation (iii) was obtained in the work on M. D. Coleman [3].

Lemma 3. *([2]). There are absolute constants $c_1 > 0$ and $c_0, 0 < c_0 < 1$ so in area*

$$Res > 1 - c(\log(t^2+m^2+3))^{-c_0}$$

$Z_m(s) \neq 0$. *Moreover in that area we have the estimate*

$$\left(Z_m(s)\right)^{-1} << \log^2\left(m^2+t^2+3\right).$$

Lemma 4. *Let $f(\alpha)$ be an arbitrary function on $(Z)[i]$, and let $\mathcal{F} \subset \mathbb{Z}[i]$ be any set. Then every $\bar{M} \geq 1$ and $0 \leq \varphi_1 < \varphi_2 < \frac{\pi}{2}$ we have*

$$\sum_{\substack{\alpha \in \mathcal{F} \\ \varphi_1 \leq \arg\alpha < \varphi_2}} f(\alpha) = (\varphi_2 - \varphi_1) \sum_{\alpha\in\mathcal{F}} f(\alpha) + O\left(\frac{1}{M} \sum_{\alpha\in\mathcal{F}} |f(\alpha)|\right) + +O\left((\varphi_2-\varphi_1) \sum_{0\leq|m|\leq M} \left|\sum_{\alpha\in\mathcal{F}} f(\alpha) e^{4mi\arg\alpha}\right|\right).$$

Proof. This statement is analogue on Vinogradov's lemma. See [1].

Main Results. In view of the obvious inequalities

$$S_k(p^m) \leq N(p^m), \bar{S}_k(p^m) \leq N(p^m)^{\frac{1}{k}},$$

where p - Gaussian integer, $m \in \mathbb{N}$, and because of $S_k(\alpha)$ and $\bar{S}_k(\alpha)$ are multiplicative, we easily get in half-plane $Res > 2$

$$\begin{aligned} \sum_{\omega}{}^* \frac{S_k(\omega)e^{4mi\arg\omega}}{N(\omega)^s} &:= \\ := \prod_p{}^* &\left(1 + \frac{e^{4mi\arg p}}{N(p)^{s-1}} + \frac{e^{4mi\arg p^2}}{N(p)^{2s-1}} + \dots + \frac{e^{4mi\arg p^k}}{N(p)^{ks-1}} + \frac{e^{4mi\arg p^{k+1}}}{N(p)^{(k+1)s-1}} + \dots\right) = \\ = \prod_p{}^* &\left(1 + \left(1 + \frac{e^{4mi\arg p}}{N(p)^s} + \frac{e^{4mi\arg p^2}}{N(p)^2 s} + \dots + \frac{e^{4mi\arg p^{k-1}}}{N(p)^{(k-1)s}}\right) \times \right. \\ &\left.\left(\frac{e^{4mi\arg p}}{N(p)^{s-1}} + \frac{e^{4mi\arg p^{k+1}}}{N(p)^{(k+1)s-2}} + \dots\right)\right) = \\ = &\frac{Z_m(s-1)Z_m(ks-1)}{Z_m(2s-2)} G_m^{(k)}(s) = \frac{Z_m(s)Z_m(ks-1)}{Z_m(ks)} H_m^{(k)}(s) \end{aligned} \tag{1}$$

where $G_m^{(k)}$ and $H_m^{(k)}$ are functions defined by Dirichlet series and absolutely convergent in half-plane $Res \geq \frac{5}{4}$.

$$\sum_{\omega}{}^* \frac{S_k(\omega)e^{4mi\arg\omega}}{N(\omega)^s} = \frac{Z_m(s)Z_m(ks-1)}{Z_m(ks)}. \tag{2}$$

Relation (1) allows to prove next theorem.

Theorem 1. *Let $0 \leq \varphi_1 < \varphi_2 < \frac{\pi}{2}$. Then for all $k = 2, 3, \dots$ we have asymptotic formulae*

$$\sum_{\substack{\omega \in \mathbb{Z}[i] \\ \varphi_1 \leq \arg\omega \leq \varphi_2 \\ N(\omega) \leq x}} S_k(\omega) = c_k(\varphi_2-\varphi_1)x^2 + O\left(x^{\frac{3}{2}} (\log x)^4\right) + O\left((\varphi_2-\varphi_1)x^2(\log x)^{-a_1}\right), \tag{3}$$

where $a_1 > 0, c_k = \frac{\pi}{4} \cdot \frac{Z_0(2k-1)}{Z_0(2)} \prod_p{}^* \left(1 + \frac{1 - N(p)^{-2(k-1)}}{(N(p)+1)(N(p^2)-1)}\right).$

This asymptotic formula is non-trivial if

$$\varphi_2 - \varphi_1 >> x^{-\frac{1}{2}} (\log x)^b, \quad b > 5.$$

Proof. For $m=0$ using (1) and Peron's formula we obtain for $c>2, T>1$

$$\sum_{N(\omega)\le x}{}^{*}S_k(\omega)=\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\frac{Z_0(s-1)}{Z_0(2(s-1))}H_0^{(k)}(s)\frac{x^s}{s}ds+O\left(\frac{x^c}{T(c-2)}\right). \qquad (4)$$

We replace segment $\{Res=c, |Ims|\le T\}$ with polygon consisting of 2 horizontal parts

$$\mathcal{I}_1=\left\{\frac{3}{2}-\frac{c_1}{\log(T^2+m^2)^{c_0}}\le Res\le c, Ims=T\right\},$$

$$\mathcal{I}_2=\left\{\frac{3}{2}-\frac{c_1}{\log(T^2+m^2)^{c_0}}\le Res\le c, Ims=-T\right\},$$

and one vertical

$$\mathcal{I}_0=\left\{Res=\frac{3}{2}-\frac{c_1}{\log(T^2+m^2)^{c_0}}, T\le Ims\le T\right\},$$

where c_0, c_1 are constants from lemma 5.

Now using residue theorem

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\tfrac{Z_0(s-1)}{Z_0(2(s-1))}H_0^{(k)}(s)\tfrac{x^s}{s}ds&=\frac{1}{2\pi i}\left(\int\limits_{\mathcal{I}_1}-\int\limits_{\mathcal{I}_2}+\int\limits_{\mathcal{I}_0}\right)\tfrac{Z_0(s-1)}{Z_0(2(s-1))}H_0^{(k)}(s)ds+\\ +\operatorname*{res}_{s=2}\left(\tfrac{Z_0(s-1)}{Z_0(2(s-1))}H_0^{(k)}\tfrac{x^s}{s}\right)&=I_1-I_2+I_0+\operatorname*{res}_{s=2}\left(\tfrac{Z_0(s-1)}{Z_0(2(s-1))}H_0^{(k)}(s)\tfrac{x^s}{s}\right).\end{aligned} \qquad (5)$$

So from (4)-(5) and using lemmas 2 and 3, we found

$$\sum_{N(\omega)\le x}{}^{*}S_k(\omega)=c_kx^2+O\left(\frac{x^c}{T(c-2)}\right)+O\left(x^{\frac{3}{2}-\delta}(\log T)^{a+1}\right). \qquad (6)$$

Setting $c=2+\frac{1}{\log x}, T=x^{\frac{1}{2}}, \delta=\frac{c_1}{2\log T}$ we immediately have

$$\sum_{N(\omega)\le x}{}^{*}=c_kx^2+O\left(x^{\frac{3}{2}}(\log x)^{a_1}\right).$$

For $m\neq 0$ using similar considerations we have that

$$\sum_{N(\omega)\le x}{}^{*}S_k(\omega)e^{4mi\arg\omega}=O\left(\frac{x^c}{T(c-2)}\right)+O\left(x^{\frac{3}{2}-\delta}\left(\log(T+|m|)\right)^{a+1}\right) \qquad (7)$$

with $\delta=\frac{c_1}{2\log(T+|m|)}$

Using lemma 4 and (6) and (7) we finish proof of theorem 1 considering $M=T(\log T)^{-Q_1}, T=x^{\frac{1}{2}}$. ■

Now we are ready to explore function $\bar{S}_k(\omega)$

Theorem 2. *Let $x\to\infty$. Then for $k=2,3,\dots$ we have*

$$\bar{S}_k(x):=\sum_{N(\omega)\le x}{}^{*}\bar{S}_k(\omega)=A_k(x)+\Delta_k(x) \qquad (8)$$

where

$$A_k(x) = \begin{cases} \frac{3}{46}\left(L(2,\chi_4)\right)^{-1} x\log x + \frac{6}{\pi^2 L(2,\chi_4)}\left(1 + \frac{z_0'(2)}{z(2)} + \frac{\pi^2\cdot\gamma}{6}\right), & k=2 \\ \frac{\pi Z_0(2)}{Z_0(s)}x + \frac{\pi Z_0\left(\frac{2}{3}\right)}{12 Z_0(2)}x^{\frac{2}{3}}, & k=3 \\ \frac{\pi Z_0(3)}{Z_0(4)}x + \frac{\pi Z_0\left(\frac{1}{2}\right)}{16 Z_0(2)}x^{\frac{1}{2}}, & k=4 \\ \frac{\pi Z_0(k-1)}{Z_0(2k)}x, & k=5 \end{cases}$$

$$\Delta_k(x) << \begin{cases} x^{\frac{3}{4}}(\log x)^3, & k=2 \\ x^{\frac{1}{2}}(\log x)^4, & k=3 \\ x^{\frac{1}{2}}e^{-c_2(\log x)^{c_3}}, & k\geq 4 \end{cases}$$

$c_2 > 0, 0 < c_3 < 1$ — *absolute constants.*

Proof Lets use equality (2) with $m=0$. For $Res > 1$ we have

$$\sum_{\omega}{}^{*}\frac{\bar{S}_k(\omega)}{N(\omega)}^{s} = \frac{Z_0(s)Z_0(ks-1)}{Z_0(ks)} := F_k(s).$$

Function $F_2(s)$ has a pole in $s=1$ and in zeros of function $Z_0(2s)$. For $k>2$ singular points will be $s=1, s=\frac{2}{k}$ and zeros of $Z_0(ks)$. Therefore further we will distinguish 4 cases $k=2, k=3, k=4$ and $k\geq 5$.

For the $k=2$ case point $s=1$ is double pole and therefore using Peron's formulae ([8], application, theorem 3.1) we get

$$\sum_{\substack{\omega \\ N(\omega)\leq x}}{}^{*}\bar{S}_k(\omega) = \operatorname*{res}_{s=1}\left(\frac{Z_0(s)Z_0(2s-1)}{Z_0(2s)}\cdot\frac{x^s}{s}\right) + \frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\frac{Z_0(s)Z_0(2s-1)}{Z_0(2s)}\cdot\frac{x^s}{s}ds + O\left(\frac{x^c}{T\left(c-\frac{3}{2}\right)^2}\right), \tag{9}$$

$c > \frac{3}{2}, T > 1.$

Lets move path of integration on line $Res = \frac{3}{4}$ and take into account that on segment $|t|\leq T$ of this line

$$Z_0^{-1}(2s) = Z_0^{-1}\left(\frac{3}{2}+2it\right) << 1.$$

Moreover using Cauchy–Schwarz inequality

$$\left|\int\limits_{-T}^{T}\frac{Z_0(s)Z_0(2s-1)}{Z_0(2s)}\cdot\frac{x^s}{s}ds\right| << x^{\frac{3}{4}}\left(\int\limits_1^T\left|Z_0\left(\frac{3}{4}+it\right)\right|^2\frac{dt}{t}\cdot\int\limits_1^T\left|Z_0\left(\frac{1}{2}+2it\right)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}$$

$$<< x^{\frac{3}{4}}\left(\int\limits_1^T\left|\xi\left(\frac{3}{4}+it\right)\right|^4\frac{dt}{t}\right)^{\frac{1}{4}}\cdot\left(\int\limits_1^T\left|L\left(\frac{3}{4}+it,\chi_4\right)\right|^4\frac{dt}{t}\right)^{\frac{1}{4}}\cdot\left(\int\limits_1^T\left|\xi\left(\frac{1}{2}+2it\right)\right|^4\frac{dt}{t}\right)^{\frac{1}{4}}\cdot$$

$$\cdot\left(\int\limits_1^T\left|L\left(\frac{1}{2}+it,\chi_4\right)\right|^4\frac{dt}{t}\right)^{\frac{1}{4}} << x^{\frac{3}{4}}\log^3 T. \tag{10}$$

Further, given that $Z_0(s) = \xi(s)L(s,\chi_4)$ we easily discover

$$\operatorname*{res}_{s=1}\left(\frac{Z_0(s)Z_0(2s-1)}{Z_0(2s)}\cdot\frac{x^s}{s}\right) = \frac{3}{46}\left(L(2,\chi_4)\right)^{-1} x\log x + \\ + \frac{6}{\pi^2 L(2,\chi_4)}\left(1+\frac{Z_0'(2)}{Z(2)}+\frac{\pi^2\cdot\gamma}{6}\right), \tag{11}$$

where γ — Euler constant.

From (9)–(11) we get theorem for $k = 2$.

In case $k = 3$ we move path of integration on the line $Res = \frac{1}{2}$ wherein we move through 2 poles in points $s = 1$ and $s = \frac{2}{3}$.

Reasoning similar to the above gives

$$\begin{aligned} \bar{S}_3(x) &= \operatorname*{res}_{s=\frac{2}{3}}\left(\frac{Z_0(s)Z_0(3s-1)}{Z_0(3s)}\cdot\frac{x^s}{s}\right) + \operatorname*{res}_{s=1}\left(\frac{Z_0(s)Z_0(3s-1)}{Z_0(3s)}\cdot\frac{x^s}{s}\right) + \\ &+ O\left(\int_1^T \left|\frac{Z_0\left(\frac{1}{2}+it\right)Z_0\left(\frac{3}{2}+3it\right)}{Z_0\left(\frac{3}{2}+3it\right)}\right| \frac{x^{\frac{1}{2}}}{2}dt\right) + \\ &+ O\left(\int_{\frac{1}{2}}^{c} \left|\frac{Z_0(\sigma+iT)Z_0(3\sigma+iT)}{Z_0\left(\frac{3}{2}\pm 3iT\right)}\right| \cdot \frac{x^\sigma}{T}d\sigma\right) + O\left(\frac{x^c}{T\left(c-\frac{3}{2}\right)}\right) \end{aligned} . \tag{12}$$

And so using estimates of four moments of the $\xi(s)$ and $L(s,\chi_4)$ on the half-plane (see [Montgomeri]) leads to the asymptotic formulae

$$\bar{S}_3(x) = \frac{\pi Z_0(2)}{Z_0(3)}x + \frac{\pi Z_0\left(\frac{2}{3}\right)}{12 Z_0(2)}x^{\frac{4}{3}} + O\left(x^{\frac{1}{2}}(\log x)^4\right). \tag{13}$$

If $k = 4$ then integrand has 2 simple poles in points $s = 1$ and $s = \frac{1}{2}$ and pole in zeros of $Z_0(2s)$ places to the left of $s = \frac{1}{2}$.

We move path of integration in a region free of zeros $Z_0(2s)$ namely on the line

$$Res = \frac{1}{2} - \frac{1}{2}c\left(\log(T^2+1)\right)^{c_0}.$$

Then we obtain

$$\bar{S}_4(x) = \frac{\pi Z_0(3)}{Z_0(4)}x + \frac{\pi Z_0\left(\frac{1}{2}\right)}{16 Z_0(2)}x^{\frac{1}{2}} + O\left(x^{\frac{1}{2}}e^{-c_2(\log x)^{c_3}}\right). \tag{14}$$

Finally for $k \geq 5$ lets take same path as in case $k = 4$. This leads to asymptotic formulae

$$\bar{S}_k(x) = \frac{\pi Z_0(k-1)}{Z_0(2k)}x + O\left(x^{\frac{1}{2}}e^{-c_2(\log x)^{c_3}}\right). \tag{15}$$

Relations (13)-(15) proves our theorem. ■

Proof of the Theorem 1 shows that using Theorem 2 we can obtain asymptotic formula for the distribution of values of the function $\bar{S}_k(\omega)$ in narrow sectors $N(\omega) \leq x, \varphi_1 \leq \arg\omega < \varphi_2, \varphi_2 - \varphi_1 >> x^{-\alpha_k}(\log x)^4$, where $\alpha_2 = \frac{1}{4}, \alpha_k = \frac{1}{2}, k = 3, 4, \ldots$

Conclusion. In our work we use analytic method to study the arithmetic properties of the Smarandache ceil function $S_k(\omega)$ and its dual $\bar{S}_k(\omega)$ over the ring of Gaussian integers $\mathbb{Z}[i]$. An asymptotic formula for summatory functions for Smarandache ceil function $S_k(\omega)$ in sector and for its dual $\bar{S}_k(\omega)$ is obtained.

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