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ON A REDUCTION OF NONLINEAR FIRST-ORDER DIFFERENTIAL EQUATION WITH OSCILLATING COEFFICIENTS TO A SOME SPECIAL KIND

Щоголев С. А. Про зведення нелінійного диференціального рівняння першого порядку з коливними коефіцієнтами до одного спеціального вигляду. Для нелінійного диференціального рівняння першого порядку з коефіцієнтами коливного типу побудовано перетворення, яке зводить це рівняння до рівняння з повільно змінними коефіцієнтами.
Ключові слова: диференціальний, повільно змінний, ряди Фур'є.

Щёголев С. А. О приведении нелинейного дифференциального уравнения первого порядка с осциллирующими коэффициентами к одному специальному виду. Для нелинейной колебательной системы второго порядка построено преобразование, приводящее эту систему к близкой системе с медленно меняющимися коэффициентами.
Ключевые слова: дифференциальный, медленно меняющийся, ряды Фурье.

Shchogolev S. A. On a reduction of nonlinear first-order differential equation with oscillating coefficients to a some special kind. For nonlinear oscillating second-order differential system construct the transformation which reducing this system close to a system with slowly varying coefficients.
Key words: differential, slowly varying, Fourier series.

Introduction. In the theory of nonlinear oscillations is an important problem of reducing a system defined on a s-dimensional torus, to so-called pure rotation, allowing you to explore the behavior of the system trajectories on this torus. In the case $s > 1$ we obtain multi-frequency system. Theory of quasi-periodic solutions of such systems is the subject of numerous studies $[1-4]$. In the case $s = 1$ torus degenerates into a circle, the system is a single frequency, and becomes an one first-order equation, which greatly simplifies the study. At the same time, if this equation is nonautonomous, in general, it is not integrated in quadratures, and then the task of bringing this equation to a simpler form is relevant. In this paper we consider the first-order differential equation, right part of which are represented by an absolutely and uniformly convergent Fourier series with slowly varying coefficients. The purpose of this paper is to obtain conditions for the existence of a similar structure transformation, this equation leads to an equation with a slowly varying right-hand side.

Auxiliary arguments. Let $G = \{t, \varepsilon : \ t \in \mathbf{R}, \ \varepsilon \in [0, \varepsilon_0], \ \varepsilon_0 \in \mathbf{R}^+\}$.

Definition 1. *We say, that a function $f(t, \varepsilon)$, in general a complex-valued, belongs to the class $S_m(\varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$, if $t, \varepsilon \in G$ and*

1) $f(t,\varepsilon)\in C^m(G)$ *with respect* t*,*
2) $d^kf(t,\varepsilon)/dt^k=\varepsilon^k f_k^*(t,\varepsilon)$ $(0\leqslant k\leqslant m)$*,*

$$\|f\|_m\overset{def}{=}\sum_{k=0}^{m}\sup_G|f_k^*(t,\varepsilon)|.$$

Under the slowly varying function we mean a function of class $S_m(\varepsilon_0)$.

Definition 2. *We say, that a function* $f(t,\varepsilon,\theta)$ *belongs to the class* $F^\theta_{m,l}(\varepsilon_0)$ $(m,l\in\mathbf{N}\cup\{0\})$*, if this function can be represented as:*

$$f(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)\exp{(in\theta)},$$

and:
1) $f_n(t,\varepsilon)\in S_m(\varepsilon_0)$*,* $\theta\in\mathbf{R}$*;*
2)

$$\|f\|_{m,l}\overset{def}{=}\|f_0\|_m+\sum_{n=-\infty}^{\infty}|n|^l\|f_n\|_m<+\infty,$$

particular

$$\|f\|_{m,0}=\sum_{n=-\infty}^{\infty}\|f_n\|_m;$$

If the function $f(t,\varepsilon,\theta)$ are real, then $f_{-n}(t,\varepsilon)\equiv\overline{f_n(t,\varepsilon)}$.
Obviously, the functions of class $F^\theta_{m,l}(\varepsilon_0)$ are 2π-periodic with respect θ.
If

$$u=\sum_{n=-\infty}^{\infty}u_n\mathrm{e}^{in\theta},\ v=\sum_{n=-\infty}^{\infty}v_n\mathrm{e}^{in\theta}\in F^\theta_{m,l}(\varepsilon_0),$$

then $ku,u\pm v,uv\in F^\theta_{m,l}(\varepsilon_0)$, and
1) $\|ku\|_{m,l}=|k|\cdot\|u\|_{m,l}$;
2) $\|u\pm v\|_{m,l}\leqslant\|u\|_{m,l}+\|v\|_{m,l}$;
3) $\|uv\|_{m,l}\leqslant 2^m(2^l+1)\|u\|_{m,l}\cdot\|v\|_{m,l}$.

We prove the last property. From the definition of the norm $\|\cdot\|_{m,l}$ should be

$$\|u\|_{m,l}=\|u_0\|_m+\left\|\frac{\partial^l u}{\partial\theta^l}\right\|_{m,0}.$$

In [5] it was shown that $\forall p,q\in S_m(\varepsilon_0)$: $\|pq\|_m\leqslant 2^m\|p\|_m\|q\|_m$. Using Leibniz's formula, we can write:

$$\frac{\partial^l(uv)}{\partial\theta^l}=\sum_{\nu=0}^{l}C_l^\nu\frac{\partial^\nu u}{\partial\theta^\nu}\cdot\frac{\partial^{l-\nu}v}{\partial\theta^{l-\nu}}.$$

We denote: $(uv)_0=\sum\limits_{k=-\infty}^{\infty}u_kv_{-k}$. Hense $\|uv\|_{m,0}\leqslant 2^m\|u\|_{m,0}\cdot\|v\|_{m,0}$. Now we have:

$$\|uv\|_{m,l}=\|(uv)_0\|_m+\left\|\frac{\partial^l(uv)}{\partial\theta^l}\right\|_{m,0}\leqslant\|(uv)_0\|_m+\sum_{\nu=0}^{l}C_l^\nu 2^m\left\|\frac{\partial^\nu u}{\partial\theta^\nu}\right\|_{m,0}\cdot\left\|\frac{\partial^{l-\nu}v}{\partial\theta^{l-\nu}}\right\|_{m,0}\leqslant$$

$$\leqslant 2^m \|u\|_{m,l} \cdot \|v\|_{m,l} + 2^m \|u\|_{m,l} \cdot \|v\|_{m,l} \cdot 2^l = 2^m(2^l+1)\|u\|_{m,l} \cdot \|v\|_{m,l},$$

quod erat demonstrandum.

Main Results.

1. Statement of the Problem.

Consider the first-order differential equation:

$$\frac{d\theta}{dt} = \omega(t,\varepsilon) + \mu\Theta(t,\varepsilon,\theta) + \varepsilon b(t,\varepsilon,\theta), \tag{1}$$

where real functions $\omega(t,\varepsilon) \in S_m(\varepsilon_0)$, $\inf\limits_G \omega(t,\varepsilon) = \omega_0 > 0$, $\Theta \in F^{\theta}_{m,l}(\varepsilon_0)$, $b \in F^{\theta}_{m-1,l}(\varepsilon_0)$, $\mu \in (0,1)$.

We study the question of the existence of the transformation of kind

$$\theta = \Psi(t,\varepsilon,\varphi,\mu),$$

where $\Psi \in F^{\varphi}_{m_1,l_1}(\varepsilon_1)$ $(m_1 \leqslant m,\ l_1 \leqslant l,\ \varepsilon_1 \leqslant \varepsilon_0)$, which reducing the equation (1) to the form:

$$\frac{d\varphi}{dt} = \widetilde{\omega}(t,\varepsilon,\mu) + \varepsilon\beta(t,\varepsilon,\varphi,\mu),$$

where $\widetilde{\omega} \in S_{m_2}(\varepsilon_1)$ $(m_2 \leqslant m)$, $\beta \in F_{m_1,l_1}$.

The peculiarity of this problem is that there appear two small parameters – μ and ε, that perform different functions. Parameter μ characterizes the smallness of the nonlinearity $\Theta(t,\varepsilon,\theta)$ in right part of equation, and parameter ε characterizes the slow variability of function ω and coefficients of Fourier-series, which represents functions Ω and b. Therefore, restrictions on one of these parameters, in general, do not involve restrictions on another parameter. At the same time, most of the known results the smallness of the nonlinearity and the slow rate variability coefficients of the system are characterized by the same parameter.

Note that analogous problem has been considered by author in [6], but there equation (1) reduced to form:

$$\frac{d\varphi}{dt} = \omega^*(t,\varepsilon) + \mu^{r+1}\omega_r(t,\varepsilon,\varphi,\mu) + \varepsilon b_r(t,\varepsilon,\varphi,\mu),$$

where $r \in \mathbf{N}$, and thus oscillating terms, proportional to the small parameter μ in right part did not disappear completely, but only increases the order of their smallness relative μ. In this paper we prove the existence of a transformation that completely destroys these oscillating terms, and retains only oscillating terms proportional parameters ε.

2. Principal Results.

Theorem. *Let the function $\Theta(t,\varepsilon,\theta)$ in right part of equation (1) belongs to class $F^{\theta}_{m,l+2}(\varepsilon_0)$. Then exists $\mu_0 \in (0,1)$ such that for all $\mu \in (0,\mu_0)$ exists the transformation*

$$\theta = \varphi + v(t,\varepsilon,\varphi,\mu), \tag{2}$$

where $v(t,\varepsilon,\varphi,\mu) \in F^{\varphi}_{m,l}(\varepsilon_0)$, reducing the equation (1) to kind:

$$\frac{d\varphi}{dt} = \omega(t,\varepsilon) + \Phi(t,\varepsilon,\mu) + \varepsilon\beta(t,\varepsilon,\varphi,\mu), \tag{3}$$

where $\Phi(t,\varepsilon,\mu)\in S_m(\varepsilon_0)$, $\beta(t,\varepsilon,\varphi,\mu)\in F^{\varphi}_{m-1,l-1}(\varepsilon_0)$.

Proof. We define the function v from equation:

$$(\omega(t,\varepsilon)+\Phi(t,\varepsilon,\mu))\frac{\partial v}{\partial\varphi}=\mu\Theta(t,\varepsilon,\varphi+v)-\Phi(t,\varepsilon,\mu). \tag{4}$$

We introduce the operators:

$$\Gamma_n[\Theta(t,\varepsilon,\varphi)]=\frac{1}{2\pi}\int\limits_0^{2\pi}\Theta(t,\varepsilon,\varphi)\ e^{-in\varphi}d\varphi,\ n\in\mathbf{Z},$$

in particular

$$\Gamma_0[\Theta(t,\varepsilon,\varphi)]=\frac{1}{2\pi}\int\limits_0^{2\pi}\Theta(t,\varepsilon,\varphi)d\varphi;$$

$$I[\Theta(t,\varepsilon,\varphi)]=\sum_{\substack{n=-\infty\\(n\neq0)}}^{\infty}\frac{\Gamma_n[\Theta(t,\varepsilon,\varphi)]}{in}\ e^{in\varphi}.$$

Obviously $\Gamma_0[\Theta(t,\varepsilon,\varphi)]\in S_m(\varepsilon_0)$, $I[\Theta(t,\varepsilon,\varphi)]\in F^{\varphi}_{m,l+1}(\varepsilon_0)$, and

$$I\left[\frac{\partial\Theta(t,\varepsilon,\varphi)}{\partial\varphi}\right]=\Theta(t,\varepsilon,\varphi)-\Gamma_0[\Theta(t,\varepsilon,\varphi)]\in F^{\varphi}_{m,l}(\varepsilon_0).$$

If in particular $\Gamma_0[\Theta(t,\varepsilon,\varphi)]\equiv0$, then

$$I\left[\frac{\partial\Theta(t,\varepsilon,\varphi)}{\partial\varphi}\right]=\Theta(t,\varepsilon,\varphi).$$

The operators $\Gamma_0[\Theta(t,\varepsilon,\varphi)]$, $I[\Theta(t,\varepsilon,\varphi)]$, obviously, are linear.

Consider equation (4). We seek a solution $v\in F^{\varphi}_{m,l}(\varepsilon_0)$ of this equation and function $\Phi\in S_m(\varepsilon_0)$ by the method of successive approximations, defining the initial approximation v_0, Φ_0 from the equation:

$$\omega(t,\varepsilon)\frac{\partial v_0}{\partial\varphi}=\mu\Theta(t,\varepsilon,\varphi)-\Phi_0(t,\varepsilon,\mu), \tag{5}$$

and the subsequent approximations $v_k,\ \Phi_k\ (k=1,2,...)$ defining from the equations:

$$\omega(t,\varepsilon)\frac{\partial v_k}{\partial\varphi}=\mu\Theta(t,\varepsilon,\varphi+v_{k-1})-\Phi_{k-1}\frac{\partial v_{k-1}}{\partial\varphi}-\Phi_k(t,\varepsilon,\mu),\ k=1,2,\dots\ . \tag{6}$$

We denote:

$$\Phi_0(t,\varepsilon,\mu)=\mu\Gamma_0[\Theta(t,\varepsilon,\varphi)], \tag{7}$$

$$v_0(t,\varepsilon,\varphi,\mu)=\frac{\mu}{\omega(t,\varepsilon)}I[\Theta(t,\varepsilon,\varphi)], \tag{8}$$

$$\Phi_{k+1}(t,\varepsilon,\mu)=\mu\Gamma_0[\Theta(t,\varepsilon,\varphi+v_k)], \tag{9}$$

$$v_{k+1}(t,\varepsilon,\varphi,\mu)=\frac{\mu}{\omega(t,\varepsilon)}I[\Theta(t,\varepsilon,\varphi+v_k)]-\frac{\Phi_k(t,\varepsilon,\mu)}{\omega(t,\varepsilon)}\ v_k(t,\varepsilon,\varphi,\mu). \tag{10}$$

We show that all the approximations $\Phi_k(t,\varepsilon,\mu)$ $(k=0,1,2,...)$, defined by the formulas (7), (9), belongs to class $S_m(\varepsilon_0)$, and all the approximations $v_k(t,\varepsilon,\varphi,\mu)$ $(k=0,1,2,...)$ belongs to class $F^{\varphi}_{m,l}(\varepsilon_0)$.

Obviously $\Phi_0\in S_m(\varepsilon_0)$, $v_0\in F^{\varphi}_{m,l+1}(\varepsilon_0)\subset F^{\varphi}_{m,l}(\varepsilon_0)$, $v_0\in\mathbf{R}$ and $\Gamma_0[v_0]\equiv 0$. We show that function $\Theta_0(t,\varepsilon,\varphi,\mu)=\Theta(t,\varepsilon,\varphi+v_0(t,\varepsilon,\varphi,\mu))$ belongs to class $F^{\varphi}_{m,l}(\varepsilon_0)$. Since by hypothesis holds $\Theta(t,\varepsilon,\varphi)\in F^{\varphi}_{m,l+2}(\varepsilon_0)$, then

$$\sup_{t,\varepsilon\in G}\sup_{\varphi\in\mathbf{R}}\left|\frac{\partial^{s+r}\Theta(t,\varepsilon,\varphi)}{\partial t^s\partial\varphi^r}\right|<+\infty,\quad s=\overline{0,m},\ r=\overline{0,l+2}. \tag{11}$$

Converting expression

$$\Gamma_n[\Theta_0(t,\varepsilon,\varphi,\mu)]=\frac{1}{2\pi}\int\limits_0^{2\pi}\Theta_0(t,\varepsilon,\varphi)\ e^{-in\varphi}d\varphi,\ n\neq 0$$

by the formula $(l+2)$-fold integration by parts, and noting that

$$\frac{\partial v_0(t,\varepsilon,\varphi,\mu)}{\partial\varphi}=\mu(\Theta(t,\varepsilon,\varphi)-\Gamma_0[\Theta(t,\varepsilon,\varphi)]),$$

we obtain:

$$\Gamma_n[\Theta_0(t,\varepsilon,\varphi,\mu)]=\frac{1}{2\pi(in)^{l+2}}\int\limits_0^{2\pi}P(t,\varepsilon,\varphi,\mu)\ e^{-in\varphi}d\varphi,\ n\neq 0,$$

where $P(t,\varepsilon,\varphi,\mu)$ is polynom of degree $l+3$ with coefficients are belongs to class $S_m(\varepsilon_0)$ relatively derivatives $\frac{\partial^r\Theta(t,\varepsilon,\varphi)}{\partial\varphi^r}$ $(r=\overline{0,l+2})$, which are calculated by values of argument φ is equal φ or $\varphi+v_0$, where $v_0\in\mathbf{R}$. Given (11) we obtain, that $\Theta_0(t,\varepsilon,\varphi,\mu)\in F^{\varphi}_{m,l}(\varepsilon_0)$. Thus taking into account (9), (10), we obtain, that $\Phi_1\in S_m(\varepsilon_0)$, $v_1\in F^{\varphi}_{m,l}(\varepsilon_0)$.

Suppose by induction, that $\Phi_s\in S_m(\varepsilon_0)$, $v_s\in F_{m,l}(\varepsilon_0)$ $(s=\overline{2,k})$, and show, that then $\Phi_{k+1}\in S_m(\varepsilon_0)$, $v_{k+1}\in F_{m,l}(\varepsilon_0)$. For that we must to show, that function $\Theta_k(t,\varepsilon,\varphi,\mu)=\Theta(t,\varepsilon,\varphi+v_k(t,\varepsilon,\varphi,\mu))$ belong to class $F^{\varphi}_{m,l}(\varepsilon_0)$. Same as above, we transform the expression

$$\Gamma_n[\Theta_k(t,\varepsilon,\varphi,\mu)]=\frac{1}{2\pi}\int\limits_0^{2\pi}\Theta_k(t,\varepsilon,\varphi)\ e^{-in\varphi}d\varphi,\ n\neq 0$$

by the formula $(l+2)$-fold integration by parts, and using the equality (6), we obtain

$$\Gamma_n[\Theta_k(t,\varepsilon,\varphi,\mu)]=\frac{1}{2\pi(in)^{l+2}}\int\limits_0^{2\pi}Q(t,\varepsilon,\varphi,\mu)\ e^{-in\varphi}d\varphi,\ n\neq 0,$$

where $Q(t,\varepsilon,\varphi,\mu)$ is polynom of degree $l+3$ with coefficients are belongs to class $S_m(\varepsilon_0)$ relatively derivatives $\frac{\partial^r\Theta(t,\varepsilon,\varphi)}{\partial\varphi^r}$ $(r=\overline{0,l+2})$, which are calculated by values of argument φ is equal φ or $\varphi + v_s$, where $v_s \in \mathbf{R}$ $(s = \overline{1,k})$. Given (11) we obtain, that $\Theta_k(t,\varepsilon,\varphi,\mu) \in F^{\varphi}_{m,l}(\varepsilon_0)$. Thus taking into account (9), (10), we obtain, that $\Phi_{k+1} \in S_m(\varepsilon_0)$, $v_{k+1} \in F^{\varphi}_{m,l}(\varepsilon_0)$.

We introduce the sets:

$$\Omega_1 = \{\ \Phi \in S_m(\varepsilon_0):\ \|\Phi\|_m \leqslant d\ \},$$

$$\Omega_2 = \left\{\ v \in F^{\varphi}_{m,l}(\varepsilon_0):\ \|v\|_{m,l} \leqslant d\ \right\},\ d > 0.$$

We denote: $\sup\limits_{v\in\Omega_2} \|\Theta(t,\varepsilon,\varphi+v)\|_{m,l}$. We show that for sufficiently small values of parameter μ all the approximations Φ_k belongs to set Ω_1, and all the approximations v_k belongs to set Ω_2. On the basis of (7), (8) $\exists\ K \in (0,+\infty)$ such that $\|\Phi_0\|_m \leqslant \mu K M(d)$, $\|v_0\|_{m,l} \leqslant \mu K M(d)$. Suppose by induction, that

$$\|\Phi_k\|_m \leqslant \mu K M(d),\ \|v_k\|_{m,l} \leqslant m,l \leqslant \mu K M(d) \sum_{s=0}^{k} (2^m \mu K M(d))^s.$$

Then for sufficiently small μ: $\Phi_k \in \Omega_1$, $v_k \in \Omega_2$. Now:

$$\|\Phi_{k+1}\|_m \leqslant \mu K M(d),$$

$$\|v_{k+1}\|_{m,l} \leqslant \mu K M(d) + \mu K M(d) \sum_{s=0}^{k} (2^m \mu K M(d))^{s+1} = \mu K M(d) \sum_{s=0}^{k+1} (2^m \mu K M(d))^s.$$

We require that

$$\mu 2^m K M(d) < 1, \tag{12}$$

$$\frac{\mu K M(d)}{1 - \mu 2^m K M(d)} \leqslant d_0 < d. \tag{13}$$

Then all the he approximations Φ_k belongs to set Ω_1, and all the approximations v_k belongs to set Ω_2 $(k = 0,1,2,...)$.

Now we prove the convergence of the process (9), (10). We have:

$$\Phi_{k+1} - \Phi_k = \mu \Gamma_0 [\Theta(t,\varepsilon,\varphi+v_k) - \Theta(t,\varepsilon,\varphi+v_{k-1})], \tag{14}$$

$$v_{k+1} - v_k = \frac{\mu}{\omega(t,\varepsilon)}\ I[\Theta(t,\varepsilon,\varphi+v_k) - \Theta(t,\varepsilon,\varphi+v_{k-1})] -$$

$$-\frac{\Phi_k}{\omega(t,\varepsilon)}\ v_k + \frac{\Phi_{k-1}}{\omega(t,\varepsilon)}\ v_{k-1} = \frac{\mu}{\omega(t,\varepsilon)}\ I[\Theta(t,\varepsilon,\varphi+v_k) - \Theta(t,\varepsilon,\varphi+v_{k-1})] -$$

$$-\frac{\Phi_k}{\omega(t,\varepsilon)}\ (v_k - v_{k-1}) - \frac{1}{\omega(t,\varepsilon)}\ (\Phi_k - \Phi_{k-1}) v_{k-1}. \tag{15}$$

As performed $\Theta(t,\varepsilon,\varphi+v_k) \in F^{\varphi}_{m,l}(\varepsilon_0)$ $(k = 0,1,2,...)$, then

$$\Theta(t,\varepsilon,\varphi+v_k) - \Theta(t,\varepsilon,\varphi+v_{k-1}) = \frac{\partial \Theta(t,\varepsilon,\varphi+v_{k-1}+\nu(v_k - v_{k-1}))}{\partial \varphi}\ (v_k - v_{k-1}),$$

$(0 < \nu < 1)$, and $\partial\Theta(t,\varepsilon,\varphi+v_{k-1}+\nu(v_k-v_{k-1}))/\partial\varphi \in F^{\varphi}_{m,l-1}(\varepsilon_0)$. We denote:

$$L(d) = \sup_{v\in\Omega_2} \left\| \frac{\partial\Theta(t,\varepsilon,\varphi+v)}{\partial\varphi} \right\|_{m,l-1}.$$

Then from (14), (15) we obtain:

$$\|\Phi_{k+1}-\Phi_k\|_m \leqslant \mu K L(d)\|v_k - v_{k-1}\|_{m,l},$$

$$\|v_{k+1}-v_k\|_{m,l} \leqslant \mu K L(d)\|v_k - v_{k-1}\|_{m,l} + \frac{\mu K M(d)}{\omega_0}\, 2^m \|v_k - v_{k-1}\|_{m,l} +$$

$$+\frac{1}{\omega_0}\mu K L(d)\|v_k - v_{k-1}\|_{m,l}\cdot 2^m(2^l+1)d.$$

It follows that for the convergence of process (9), (10) is sufficient that the inequalities (12), (13) and also

$$\mu K L(d) + \frac{\mu K M(d)}{\omega_0}\, 2^m + \frac{\mu K L(d) 2^m (2^l+1)d}{\omega_0} < 1.$$

Thus equation (4) have a solution $v(t,\varepsilon,\varphi,\mu) \in F^{\varphi}_{m,l}(\varepsilon_0)$, and this solution belong to set Ω_2, therefore

$$\|v\|_{m,l} \leqslant \frac{\mu K M(d)}{1-\mu 2^m K M(d)},$$

As performed $v \in F^{\varphi}_{m,l}(\varepsilon_0)$, then $\frac{\partial v}{\partial t} = \varepsilon\widetilde{v}(t,\varepsilon,\varphi,\mu)$, where $\widetilde{v} \in F^{\varphi}_{m-1,l}(\varepsilon_0)$, and $\frac{\partial v}{\partial\varphi} \in F^{\varphi}_{m,l-1}(\varepsilon_0)$. Now we define the function $\beta(t,\varepsilon,\varphi,\mu)$ in (3) from the equation:

$$\left(1+\frac{\partial v}{\partial\varphi}\right)\, \beta(t,\varepsilon,\varphi,\mu) = b(t,\varepsilon,\varphi+v) - \widetilde{v}(t,\varepsilon,\varphi,\mu).$$

For the sufficiently small μ this equation has a unique solution $\beta \in F^{\varphi}_{m-1,l-1}(\varepsilon_0)$.

Theorem are proved.

Remark 1. *For the conditions of the above theorem is only necessary smallness of the parameter μ, but not parameter ε. Therefore the solution $v(t,\varepsilon,\varphi,\mu)$ of equation (4) and function $\Phi(t,\varepsilon,\mu)$ are defined in the same area G, that coefficients of this equation.*

Remark 2. *Using the chain of transformations analogous to the construction in [7], we can increase the order of smallness of the parameter ε of the oscillating term $\varepsilon\beta(t,\varepsilon,\varphi,\mu)$ in equation (3) and to transform this equation to the kind:*

$$\frac{d\psi}{dt} = \sigma(t,\varepsilon,\mu),$$

where $\sigma(t,\varepsilon,\mu) \in S_{m_1}(\varepsilon_1)$ $(m_1 < \min(m-1,l-1),\ \varepsilon_1 < \varepsilon_0)$.

Conclusion. Thus, for the equation (1) the sufficient conditions of the existence of the transformation, which reducing this equation close to a equation with slowly varying coefficients and the algorithm for constructing this transformation are obtained.

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