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AVERAGE NUMBER OF SQUARES DIVIDING mn

Лелеченко А. В. Середня кількість квадратів, що ділять mn. Вивчається асимптотична поведінка двовимірної суматорної функції $\sum_{m,n\leqslant x} \tau_{1,2}(mn)$, де $\tau_{1,2}(n) = \sum_{ab^2=n} 1$, з використанням недавнього результату Балазарда, Наими та Петермана й методу комплексного інтегрування. Отримано асимптотичну формулу з залишковим членом $O(x^{10/7})$, який за умови гіпотези Рімана покращується до $O(x^{7/5})$.
Ключові слова: асиметрична функція дільників, багатовимірна суматорна функція.

Лелеченко А. В. Среднее количество квадратов, делящих mn. Исследуется асимптотическое поведение двумерной сумматорной функции $\sum_{m,n\leqslant x} \tau_{1,2}(mn)$, где $\tau_{1,2}(n) = \sum_{ab^2=n} 1$, с использованием недавнего результата Балазарда, Наими, Петермана и метода комплексного интегрирования. Получена асимптотическая формула с остаточным членом $O(x^{10/7})$, который при условии гипотезы Римана улучшается до $O(x^{7/5})$.
Ключевые слова: асимметрическая функция делителей, многомерная сумматорная функция.

Lelechenko A. V. Average number of squares dividing mn. We study the asymptotic behaviour of the two-dimensional summatory function $\sum_{m,n\leqslant x} \tau_{1,2}(mn)$, where $\tau_{1,2}(n) = \sum_{ab^2=n} 1$, using recent result of Balazard, Naimi, Pétermann and the complex integration method. An asymptotic formula with an error term $O(x^{10/7})$ is obtained. Under the Riemann hypothesis the error term can be improved up to $O(x^{7/5})$.
Key words: asymmetric divisor function, multidimensional summatory function.

Introduction.

Let f be a multiplicative arithmetic function of one variable. The asymptotic behaviour of $\sum_{n\leqslant x} f(n)$ is a classic problem of analytic number theory, deeply studied for various specific functions and classes. Let us consider the problem of estimating of $\sum_{m,n\leqslant x} f(mn)$.

The divisor function τ is a simple, but non-trivial case. Applying Busche—Ramanujan identity

$$\tau(mn) = \sum_{d \mid \gcd(m,n)} \tau(m/d)\tau(n/d)\mu(d) \tag{1}$$

we split variables and obtain

$$\sum_{m,n\leqslant x} \tau(mn) = \sum_{\substack{j,k,l \\ j,k\leqslant x/l}} \tau(j)\tau(k)\mu(l) = \sum_{l\leqslant x} \mu(l) \biggl(\sum_{j\leqslant x/l} \tau(j) \biggr)^2 .$$

Using Huxley's estimate [4] $\sum_{j\leqslant y}\tau(j) = y\log y + (2\gamma-1)y + O(y^{\theta+\varepsilon})$, where $\theta = 131/416$, we regroup terms and get

$$\sum_{m,n\leqslant x}\tau(mn) = x^2\Biggl(\Biggl(\sum_{l=1}^{\infty}\frac{\mu(l)}{l^2}\Biggr)\Bigl(\log^2 x + 2(2\gamma-1)\log x + (2\gamma-1)^2\Bigr) - \\ - \Biggl(\sum_{l=1}^{\infty}\frac{\mu(l)\log l}{l^2}\Biggr)\Bigl(2\log x + 2(2\gamma-1)\Bigr) + \sum_{l=1}^{\infty}\frac{\mu(l)\log^2 l}{l^2}\Biggr) + O(x^{1+\theta+\varepsilon}). \quad (2)$$

It is natural to ask whether the main term can be derived analytically, by complex integration method. We will not go into details, but note that

$$\sum_{a,b=0}^{\infty}\tau(p^{a+b})x^a y^b = \sum_{a,b=0}^{\infty}(a+b+1)x^a y^b = \frac{1-xy}{(1-x)^2(1-y)^2}, \quad |x|,|y|<1.$$

The series $\sum\limits_{m,n=1}^{\infty}\tau(mn)m^{-z}n^{-w}$ converges absolutely for $\Re z, \Re w > 1$, so by multiplicativity in this region we have

$$\sum_{m,n=1}^{\infty}\frac{\tau(mn)}{m^z n^w} = \prod_p\sum_{a,b=0}^{\infty}\frac{\tau(p^{a+b})}{p^{az+bw}} = \prod_p\frac{1-p^{-z-w}}{(1-p^{-z})^2(1-p^{-w})^2} = \frac{\zeta^2(z)\zeta^2(w)}{\zeta(z+w)}. \quad (3)$$

Achieved representation allows to compute the coefficient of multiple Laurent series for $x^{z+w}z^{-1}w^{-1}\sum\limits_{m,n=1}^{\infty}\tau(mn)m^{-z}n^{-w}$ at $1/(z-1)(w-1)$, which appears coinciding with the main term of (2).

Notation. Letter p with or without indexes denotes a prime number. We write $f\star g$ for the Dirichlet convolution

$$(f\star g)(n) = \sum_{d|n}f(d)g(n/d).$$

In asymptotic relations we use $\sim$, $\asymp$, Landau symbols O and o, Vinogradov symbols $\ll$ and $\gg$ in their usual meanings. All asymptotic relations are given as an argument (usually x) tends to the infinity.

Letter γ denotes Euler—Mascheroni constant. Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same even in one equation).

As usual $\zeta(s)$ is the Riemann zeta-function. Real and imaginary components of the complex s are denoted as $\sigma := \Re s$ and $t := \Im s$, so $s = \sigma + it$.

For a fixed $\sigma \in [1/2, 1]$ define

$$\mu(\sigma) := \limsup_{t\to\infty}\frac{\log\bigl|\zeta(\sigma+it)\bigr|}{\log t}.$$

Auxiliary arguments. We say that a function is symmetric if any permutation of arguments does not change its value.

Let f be an arithmetic function of r variables. The associated Dirichlet series are defined as

$$F(s_1,\ldots,s_r)=\sum_{n_1,\ldots,n_r=1}^{\infty} f(n_1,\ldots,n_r)n_1^{-s_1}\cdots n_r^{-s_r}$$

and $(\sigma_1,\ldots,\sigma_r)$ is called abscissas of absolute convergence if $F(s_1,\ldots,s_r)$ converges absolutely in the region $\Re s_1>\sigma_1,\ldots,\Re s_r>\sigma_r$.

Lemma 1. *Let f be a symmetric arithmetic function of r variables and $(\sigma_a,\ldots,\sigma_a)$ are abscissas of absolute convergence of the associated Dirichlet series $F(s_1,\ldots,s_r)$. Define*

$$F_r^{\heartsuit}(\sigma,x,T):=\sum_{n_1,\ldots,n_r=1}^{\infty}\frac{|f(n_1,\ldots,n_r)|(n_1\cdots n_r)^{-\sigma}}{\min_{j=1,\ldots,r}(T|\log(x/n_j)|+1)}, \tag{4}$$

and let

$$\sideset{}{^*}\sum_{n_1,\ldots,n_r\leqslant x} f(n_1,\ldots,n_r):=\sum_{n_1,\ldots,n_r\leqslant x} f(n_1,\ldots,n_r)h(x/n_1)\cdots h(x/n_r), \tag{5}$$

where $h(y)=0$ for $0<y<1$, $h(1)=1/2$ and $h(y)=1$ otherwise.

For $x\geqslant 2$, $T\geqslant 2$, $\sigma\leqslant\sigma_a$, $\delta>0$, $\kappa=\sigma_a-\sigma+\delta/\log x$, $1=N_1\leqslant\cdots\leqslant N_r$, $1=M_1\leqslant\cdots\leqslant M_r$ and $N_0:=N_1+\cdots+N_r$ we have

$$\Biggl|\sideset{}{^*}\sum_{n_1,\ldots,n_r\leqslant x}\frac{f(n_1,\ldots,n_r)}{(n_1\cdots n_r)^s}- -\frac{1}{(2\pi i)^r}\int\limits_{N_1\kappa-iM_1T}^{N_1\kappa+iM_1T}\cdots\int\limits_{N_r\kappa-iM_rT}^{N_r\kappa+iM_rT}F(s+w_1,\ldots,s+w_r)x^{w_1+\cdots+w_r}\frac{dw_1\cdots dw_r}{w_1\cdots w_r}\Biggr|\ll \ll x^{N_0(\sigma_a-\sigma)}F_r^{\heartsuit}(\sigma_a+\delta/\log x,x,T). \tag{6}$$

Proof. This is a result of Balazard, Naimi and Pétermann [1, Prop. 6].

Lemma 2. *Let $f(t)\geqslant 0$. If*

$$\int_1^T f(t)\,dt\ll g(T),$$

where $g(T)=T^\alpha\log^\beta T$, $\alpha\geqslant 1$, then

$$I(T):=\int_1^T\frac{f(t)}{t}dt\ll\begin{cases}\log^{\beta+1}T & \text{if } \alpha=1,\\ T^{\alpha-1}\log^\beta T & \text{if } \alpha>1.\end{cases}$$

Proof. Let us divide the interval of integration into parts:

$$I(T)\leqslant\sum_{k=0}^{\lfloor\log_2 T\rfloor-1}\int_{T/2^{k+1}}^{T/2^k}\frac{f(t)}{t}dt+g(2)< <\sum_{k=0}^{\log_2 T}\frac{1}{T/2^{k+1}}\int_1^{T/2^k}f(t)\,dt+g(2)\ll\sum_{k=0}^{\lfloor\log_2 T\rfloor-1}\frac{g(T/2^k)}{T/2^{k+1}}.$$

Now the lemma's statement follows from elementary estimates.

Lemma 3. *Let $\eta > 0$ be arbitrarily small. Then for growing $|t| \geqslant 3$*

$$\zeta(s) \ll \begin{cases} |t|^{1/2-(1-2\mu(1/2))\sigma}, & \sigma \in [0, 1/2], \\ |t|^{2\mu(1/2)(1-\sigma)}, & \sigma \in [1/2, 1-\eta], \\ |t|^{2\mu(1/2)(1-\sigma)} \log^{2/3}|t|, & \sigma \in [1-\eta, 1], \\ \log^{2/3}|t|, & \sigma \in [1, 1+\eta], \\ 1, & \sigma \geqslant 1+\eta. \end{cases} \tag{7}$$

Proof. Estimates follow from Phragmén—Lindelöf principle and estimates of $\zeta(s)$ at $\sigma = 0, 1/2, 1$. See Titchmarsh [7, Ch. 5] or Ivić [5, Ch. 7.5] for details.

Lemma 4.

$$\int_1^T \left|\zeta(\sigma+it)\right|^2 dt \ll T, \qquad 1/2 < \sigma < 1.$$

Proof. See Ivić [5, (1.76)].

Main Results

Out paper is devoted to

$$\sum_{m,n \leqslant x} \tau_{1,2}(mn),$$

where $\tau_{1,2}(n) = \sum_{ab^2=n} 1$. This function is not as lucky as τ and does not posses representation like (1), so there is no easy way to split m and n.

The main result is

Theorem 1.

$$\sum_{m,n \leqslant x} \tau_{1,2}(mn) = C_1 x^2 + C_2 x^{3/2} + O(x^{10/7+\varepsilon}),$$

where $C_1 = 2.995\ldots$, $C_2 = -5.404\ldots$ are computable constants.

This theorem is analogous to the estimate by Graham and Kolesnik [2]

$$\sum_{n \leqslant x} \tau_{1,2}(n) = \zeta(2)x + \zeta(1/2)x^{1/2} + O(x^{\beta+\varepsilon}), \quad \beta = 1057/4785 \approx 0.2209.$$

1. Reduction to complex integration

Applying Lemma 1 with $r = 2$, $f(n_1, n_2) = \tau_{1,2}(n_1 n_2)$, $\sigma = s = 0$, $\sigma_a = 1$, $N_1 = N_2 = M_1 = M_2 = 1$, $\delta = 1$, $\log T \asymp \log x$ and writing (m, n, z, w, c) instead of $(n_1, n_2, w_1, w_2, \kappa)$ for convenience we deduce from (6) that

$$\sum_{m,n \leqslant x}{}^{*} \tau_{1,2}(mn) = \frac{1}{(2\pi i)^2} \iint\limits_{[c-iT, c+iT]^2} F(z,w) \frac{x^{z+w}}{zw} dz\, dw + O\left(x^2 F_2^{\heartsuit}(c, x, T)\right), \tag{8}$$

where $c = 1 + 1/\log x$ and

$$F(z,w) = \sum_{m,n=1}^{\infty} \frac{\tau_{1,2}(mn)}{m^z n^w}, \qquad \Re z, \Re w > 1. \tag{9}$$

By (4) for non-integer x

$$TF_2^{\heartsuit}(c,x,T) \ll \sum_{m,n} \frac{\tau_{1,2}(mn)}{(mn)^c \min\bigl(|\log\frac{x}{n}|, |\log\frac{x}{m}|\bigr)} \ll \sum_{\substack{|\log\frac{x}{n}|\geqslant 1\\ |\log\frac{x}{m}|\geqslant 1}} \frac{\tau_{1,2}(mn)}{(mn)^c} +$$
$$+ \sum_{\substack{|\log\frac{x}{n}|\leqslant 1\\ |\log\frac{x}{m}|\geqslant 1}} \frac{\tau_{1,2}(mn)}{(mn)^c |\log\frac{x}{n}|} + \sum_{\substack{|\log\frac{x}{n}|\leqslant 1\\ |\log\frac{x}{m}|\leqslant 1}} \frac{\tau_{1,2}(mn)}{(mn)^c \min\bigl(|\log\frac{x}{n}|, |\log\frac{x}{m}|\bigr)} :=$$
$$:= \Sigma_1 + \Sigma_2 + \Sigma_3. \quad (10)$$

We have $\Sigma_1 \ll \sum_{m,n=1}^{\infty} \tau_{1,2}(mn)/(mn)^c = F(c,c)$ and we will show below in (19) that

$$F(c,c) \ll \frac{1}{(c-1)^2} = \log^2 x. \quad (11)$$

Further, for x such that $|\log\frac{x}{n}| \leqslant 1$ we have $|\log\frac{x}{n}| \geqslant c|x-n|/x$ for $c = 1/(e-1)$. Then

$$\Sigma_2 \ll \sum_{x/e\leqslant n\leqslant xe} \sum_m \frac{\tau_{1,2}(mn)x}{(mn)^c|x-n|}.$$

Note that $\tau_{1,2}(mn) \leqslant \tau(mn) \leqslant \tau(m)\tau(n)$, because τ is completely submultiplicative. Thus

$$\Sigma_2 \ll x \sum_{x/e\leqslant n\leqslant xe} \frac{\tau(n)}{n^c|x-n|} \sum_m \frac{\tau(m)}{m^c}.$$

Here

$$\sum_{m=1}^{\infty} \tau(m)m^{-c} = \zeta^2(c) \ll (c-1)^{-2} = \log^2 x.$$

Let $M(y) = \max_{n\leqslant y} \tau(n)$. We have

$$\Sigma_2 \ll xM(xe)\log^2 x \sum_{x/e\leqslant n\leqslant xe} \frac{1}{n^c|x-n|},$$

where the last sum is $\ll x^{-c}\log x \ll x^{-1}\log x$, so finally

$$\Sigma_2 \ll M(xe)\log^3 x. \quad (12)$$

Now consider Σ_3. Defining $M_{1,2}(y) = \max_{n\leqslant y} \tau_{1,2}(n)$ we obtain

$$\Sigma_3 \ll \sum_{x/e\leqslant n\leqslant m\leqslant xe} \frac{\tau_{1,2}(mn)x}{(mn)^c \min\bigl(|x-n|, |x-m|\bigr)} \ll$$
$$\ll \frac{xM_{1,2}(x^2e^2)}{x^{2c}} \sum_{x/e\leqslant n\leqslant m\leqslant xe} \max\left(|x-n|^{-1}, |x-m|^{-1}\right) \ll M_{1,2}(x^2e^2)\log x. \quad (13)$$

Standard estimates [3, Th. 315] give $M_{1,2}(y) \leqslant M(y) \ll y^{\varepsilon}$, so substituting (11), (12) and (13) into (10) we obtain

$$F_2^{\heartsuit}(c,x,T) \ll T^{-1}\bigl(M(xe)\log^3 x + M_{1,2}(x^2e^2)\log x\bigr) \ll T^{-1}x^{\varepsilon}. \tag{14}$$

Note also that by definition (5)

$$\Biggl|\sum_{m,n\leqslant x}\tau_{1,2}(mn) - \sum_{m,n\leqslant x}{}^{*}\,\tau_{1,2}(mn)\Biggr| \ll \sum_{n\leqslant x}\tau_{1,2}(\lfloor x\rfloor n) \ll M(x^2)x. \tag{15}$$

Combining (8), (14) and (15) we get

$$\sum_{m,n\leqslant x}\tau_{1,2}(mn) = \frac{1}{(2\pi i)^2}\iint\limits_{[c-iT,c+iT]^2} F(z,w)\frac{x^{z+w}}{zw}dz\,dw + \\ + O\bigl(x^{1+\varepsilon} + T^{-1}x^{2+\varepsilon}\bigr). \tag{16}$$

2. Double Dirichlet series for $\tau_{1,2}$

Let us return to (9) and extract a product of zeta-functions from $F(z,w)$. Define

$$f(x,y) = \sum_{a,b=0}^{\infty}\tau_{1,2}(p^{a+b})x^a y^b, \qquad |x|,|y|<1. \tag{17}$$

Using identity

$$\tau_{1,2}(p^a) - \tau_{1,2}(p^{a-1}) - \tau_{1,2}(p^{a-2}) + \tau_{1,2}(p^{a-3}) = 0$$

multiply both sides of (17) by $(1-x)(1-x^2)$:

$$\begin{aligned}
&(1-x)(1-x^2)f(x,y) = \\
&\quad= \sum_{a=3}^{\infty}\sum_{b=0}^{\infty}\bigl(\tau_{1,2}(p^{a+b}) - \tau_{1,2}(p^{a+b-1}) - \tau_{1,2}(p^{a+b-2}) + \tau_{1,2}(p^{a+b-3})\bigr)x^a y^b + \\
&\qquad+ \sum_{b=0}^{\infty} y^b\left((1-x-x^2)\tau_{1,2}(p^b) + (1-x)\tau_{1,2}(p^{b+1})x + \tau_{1,2}(p^{b+2})x^2\right) = \\
&\qquad\quad= \sum_{b=0}^{\infty} y^b\left((1-x-x^2)\tau_{1,2}(p^b) + (x-x^2)\tau_{1,2}(p^{b+1})x + x^2\tau_{1,2}(p^{b+2})\right)
\end{aligned}$$

and further

$$\begin{aligned}
&(1-x)(1-x^2)(1-y)(1-y^2)f(x,y) = \\
&\qquad= (1-x-x^2)\bigl((1-y-y^2) + (1-y)y + 2y^2\bigr) + \\
&\qquad+ (x-x^2)\bigl((1-y-y^2) + 2(1-y)y + 2y^2\bigr) + \\
&\qquad+ x^2\bigl(2(1-y-y^2) + 2(1-y)y + 3y^2\bigr) = \\
&\qquad\qquad= 1 + xy - x^2y - xy^2,
\end{aligned}$$

which induces

$$f(x,y) = \frac{1+xy-x^2y-xy^2}{(1-x)(1-x^2)(1-y)(1-y^2)} = \\ = \frac{1-x^2y-xy^2-x^2y^2+x^3y^2+x^2y^3}{(1-x)(1-x^2)(1-y)(1-y^2)(1-xy)}. \quad (18)$$

Representation (18) immediately implies that

$$F(z,w) = \prod_p f(p^{-z}, p^{-w}) = \zeta(z)\zeta(2z)\zeta(w)\zeta(2w)\zeta(z+w)G(z,w) = \\ = \frac{\zeta(z)\zeta(2z)\zeta(w)\zeta(2w)\zeta(z+w)}{\zeta(2z+w)\zeta(2w+z)} H(z,w), \quad (19)$$

where series $H(z,w)$ converges absolutely in the region $\Re(2z+2w) > 1$. Definitely $G(z,w)$ converges absolutely for $(z,w) \in Q := \{\Re z \geqslant 1/3, \Re w \geqslant 1/3\}$.

Product of zeta-functions (19) shows that inside of the region Q function $F(z,w)$ has poles along lines $z=1$, $z=1/2$, $w=1$, $w=1/2$ and $z+w=1$. All of them are of the first order, except poles at $(1,1)$, $(1,1/2)$, $(1/2,1)$, which are of the second order, and a pole at $(1/2,1/2)$, which is of the third order.

Both (3) and (19) are partial cases of a general rule, which will be stated as a lemma.

Lemma 5. *Let* $\tau_{1,k}(n) = \sum_{ab^k=n} 1$. *Then for* $\Re z, \Re w > 1$ *we have*

$$\sum_{m,n=1}^{\infty} \frac{\tau_{1,k}(mn)}{m^z n^w} = \zeta(z)\zeta(w) \frac{\prod_{l=0}^{k} \zeta\bigl(lz+(k-l)w\bigr)}{\prod_{l=1}^{k} \zeta\bigl(lz+(k+1-l)w\bigr)} H_k(z,w), \quad (20)$$

where the series H_k *converges absolutely for* $\Re z, \Re w > 1/(k+2)$.

Proof. Cases $k=1$ and $k=2$ has been proven above, so we consider $k>2$ only. Let

$$f(x,y) = \sum_{a,b=0}^{\infty} \tau_{1,k}(p^{a+b}) x^a y^b, \qquad |x|,|y|<1.$$

For a monomial M let $[M]f(x,y)$ be a coefficient at M in the series f. Here

$$[x]f(x,y) = [y]f(x,y) = \tau_{1,k}(p) = 1,$$

so let us define

$$g(x,y) = (1-x)(1-y)f(x,y) = \\ = \sum_{a,b=1}^{\infty} \bigl(\tau_{1,k}(p^{a+b}) - 2\tau_{1,k}(p^{a+b-1}) + \tau_{1,k}(p^{a+b-2})\bigr) x^a y^b + \\ + \sum_{a=1}^{\infty} \bigl(\tau_{1,k}(p^a) - \tau_{1,k}(p^{a-1})\bigr)(x^a+y^a) + 1.$$

We have

$$\tau_{1,k}(p^a) = \begin{cases} 1, & a < k, \\ 2, & k \leqslant a < 2k, \end{cases}$$

so one can verify that

$$[x^a y^b]g(x,y) = \begin{cases} 0, & a+b < k, \\ 1, & a+b = k, \\ 0, & a+b = k+1,\ ab = 0 \\ -1, & a+b = k+1,\ ab > 0. \end{cases}$$

Thus

$$f(x,y) = \frac{1}{(1-x)(1-y)} \frac{\prod_{l=1}^{k}(1-x^l y^{k+1-l})}{\prod_{l=0}^{k}(1-x^l y^{k-l})} h(x,y),$$

where all monomials of the series $h(x,y)$ has degree at least $k+2$.

3. Path of integration and the main term

Our aim is to translate the domain of integration in (16) from $[c-iT, c+iT]^2$ till $[b-iT, b+iT]^2$, where $b = 1/3$. This is trickier than translating in the one-dimensional case, because a hyperrectangle R with opposite vertices $(b-iT, b-iT)$ and $(c+iT, c+iT)$ has 24 two-dimensional faces. Figure 1 contains a schematic plain projection of R with 16 vertices and 32 edges marked.

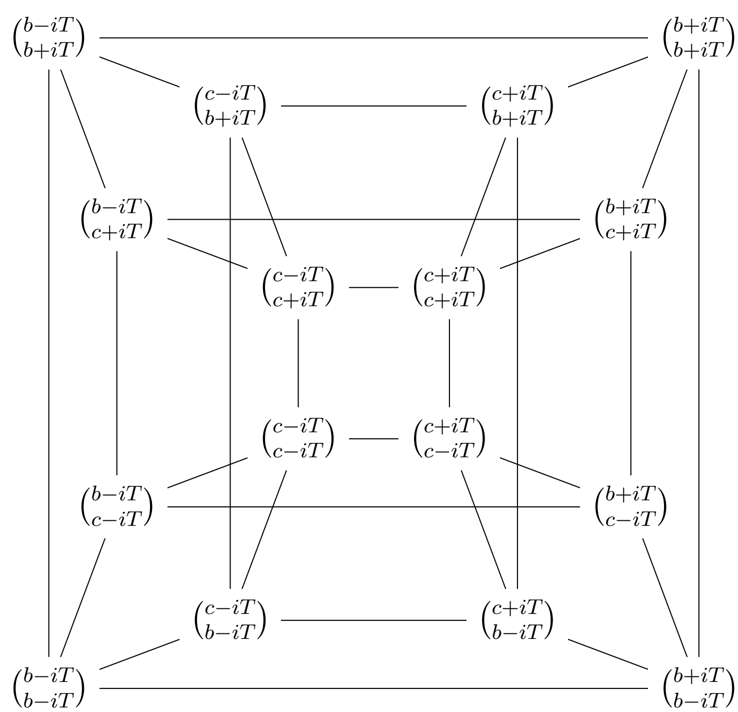

Fig 1. The hyperrectangle R with opposite vertices $(b-iT, b-iT)$ and $(c+iT, c+iT)$

Denote $L(z,w) = G(z,w)x^{z+w}z^{-1}w^{-1}$. This function has the same poles in R as $G(z,w)$ has. Note that (on contrary with integration by one-dimensional contour) poles of the first order do not induce divergence of integrals by plane domains: e. g., $\iint_{x^2+y^2\leqslant 1} \frac{dx\,dy}{\sqrt{x^2+y^2}} = 2\pi < \infty$, however $\int_{x^2\leqslant 1} \frac{dx}{x} = \infty$. Only poles of the second and higher orders are worth to pay attention.

Let $E(x)$ be the integral of $L(z,w)$ over all faces of R except $[c-iT, c+iT]^2$. By residue theorem [6]

$$\frac{1}{(2\pi i)^2} \iint\limits_{[c-iT,c+iT]^2} L(z,w)\,dz\,dw =$$
$$= \Big(\operatorname*{res}_{z=w=1} + \operatorname*{res}_{\substack{z=1\\ w=1/2}} + \operatorname*{res}_{\substack{z=1/2\\ w=1}} + \operatorname*{res}_{z=w=1/2} \Big) L(z,w) + O(E(x)). \tag{21}$$

Expanding $L(z,w)$ into Laurent series in two variables we get

$$\operatorname*{res}_{z=w=1} L(z,w) = \zeta^3(2)G(1,1)x^2, \tag{22}$$
$$\operatorname*{res}_{\substack{z=1\\ w=1/2}} L(z,w) = \operatorname*{res}_{\substack{z=1/2\\ w=1}} L(z,w) = \zeta(2)\zeta(\tfrac12)\zeta(\tfrac32)G(1,\tfrac12)x^{3/2}, \tag{23}$$
$$\operatorname*{res}_{z=w=1/2} L(z,w) \ll x\log x. \tag{24}$$

After substitution into (16) the residue at $(1/2, 1/2)$ will be absorbed by error term, so it is enough to have only upper bound. Inserting (22), (23) and (24) into (21) we get

$$\frac{1}{(2\pi i)^2} \iint\limits_{[c-iT,c+iT]^2} L(z,w)\,dz\,dw = C_1x^2 + C_2x^{3/2} + O(x\log x + E(x)), \tag{25}$$

where

$$C_1 = \frac{\pi^6}{216}G(1,1), \qquad C_2 = \frac{\pi^2}{3}\zeta(\tfrac12)\zeta(\tfrac32)G(1,\tfrac12).$$

Let us calculate numerical values of C_1 and C_2. Applying formal identity

$$\frac{F(z,w)}{\zeta(z)\zeta(w)} = \prod_p (1-p^{-z})(1-p^{-w}) \sum_{a,b=0}^{\infty} \frac{\tau_{1,2}(p^{a+b})}{p^{a+b}}$$

at $z=w=1$ we get

$$C_1 = \operatorname*{res}_{z=w=1} L(z,w) = \prod_p (1-p^{-1})^2 \sum_{a,b=0}^{\infty} \frac{\tau_{1,2}(p^{a+b})}{p^{a+b}} = 2.995\ldots$$

The product converges absolutely because

$$(1-p^{-1})^2 \sum_{a,b=0}^{\infty} \frac{\tau_{1,2}(p^{a+b})}{p^{a+b}} = \big(1-2p^{-1}+O(p^{-2})\big)\big(1+2p^{-1}+O(p^{-2})\big) =$$
$$= 1 + O(p^{-2}).$$

Similarly

$$\frac{F(z,w)}{\zeta(z)\zeta(w)\zeta(2z)} = \prod_p (1-p^{-z})(1-p^{-w})(1-p^{-2z}) \sum_{a,b=0}^{\infty} \frac{\tau_{1,2}(p^{a+b})}{p^{a+b}}$$

implies

$$C_2 = 2 \operatorname*{res}_{\substack{z=1\\ w=1/2}} L(z,w) =$$

$$= 2\zeta(1/2) \prod_p (1-p^{-1})^2(1-p^{-1/2}) \sum_{a,b=0}^{\infty} \frac{\tau_{1,2}(p^{a+b})}{p^{a+b/2}} = -5.404\ldots$$

4. The error term

Let us estimate $E(x)$. It was defined above to consist of integrals over 23 of 24 faces of the hyperectangle R, but due to the symmetry many of these integrals can be estimated in the same way.

In computations below we assume $x^{1/2} \ll T \ll x$, the exact value of T will be specified later in (28).

There are 2 faces of form $[b-iT, b+iT] \times [c-iT, c+iT]$. We have

$$I_1 := \int_{b-iT}^{b+iT} \int_{c-iT}^{c+iT} L(z,w)\, dz\, dw \ll \iint\limits_{[1,T]^2} \zeta(b+it_1)\zeta(2b+2it_1) \times$$

$$\times \zeta(c+it_2)\zeta(2c+2it_2)\zeta\big(b+c+i(t_1+t_2)\big) x^{b+c} t_1^{-1} t_2^{-1} dt_1 dt_2.$$

By (7) we can estimate

$$\zeta(c+it_2)\zeta(2c+2it_2)\zeta\big(b+c+i(t_1+t_2)\big) \ll \log^{2/3} T \cdot 1 \cdot 1.$$

As soon as $x^{1/\log x} \ll 1$ we have $x^{b+c} \ll x^{4/3}$. Also $\int_1^T t_2^{-1} dt_2 \ll \log T$. Thus I_1 can be estimated as

$$I_1 \ll x^{4/3} \log^{5/3} T \int_1^T \zeta(b+it)\zeta(2b+2it)t^{-1} dt.$$

By functional equation for ζ, Lemma 4 and Lemma 2

$$J := \int_1^T \zeta(b+it)\zeta(2b+2it)t^{-1}dt \ll \int_1^T t^{1/6}\zeta^2(2/3)t^{-1}dt \ll T^{1/6}\log T. \tag{26}$$

Then

$$I_1 \ll x^{4/3} T^{1/6} \log^{8/3} T. \tag{27}$$

We will show below in (40) that integrals over other faces (and so $E(x)$ as a whole) are less than either I_1 or $x^{2+\varepsilon}T^{-1}$, so T should be chosen to equalize this two magnitudes:

$$T = x^{4/7}. \tag{28}$$

Substitute it into (16) and (25) to obtain the final error term $x^{10/7+\varepsilon}$, which approves the statement of the Theorem 1.

From here and till the end of the section we will omit factors $\ll x^\varepsilon$ in asymptotic estimates for the brevity: they do not influence the resulting error term.

There are 4 faces of form $[b-iT, b+iT] \times [b \pm iT, c \pm iT]$. We have

$$\begin{aligned} I_2 := \int_{b-iT}^{b+iT} \int_{b+iT}^{c+iT} L(z,w)\, dz\, dw \ll \int_1^T \int_b^c \zeta(b+it)\zeta(2b+2it) \times \\ \times \zeta(\sigma+iT)\zeta(2\sigma+2iT)\zeta\big(b+\sigma+i(t+T)\big) x^{b+\sigma} t^{-1} T^{-1} d\sigma\, dt \ll \\ \ll x^{1/3} J T^{-1} \max_{\substack{\sigma\in[b,c] \\ t\in[1,T]}} \zeta(\sigma+iT)\zeta(2\sigma+2iT)\zeta\big(b+\sigma+i(t+T)\big) x^\sigma \ll \\ \ll x^{1/3} T^{-5/6} \max_{\sigma\in[b,c]} \zeta(\sigma+iT)\zeta(\sigma+1/3+iT)\zeta(2\sigma+iT) x^\sigma. \end{aligned}$$

Splitting $[b,c]$ into intervals $[1/3,1/2]$, $[1/2,2/3]$, $[2/3,c]$ and estimating $\zeta(\sigma+iT) \times \zeta(\sigma+1/3+iT)\zeta(2\sigma+iT)x^\sigma$ on each of them separately, we get

$$I_2 \ll x^{1/3} T^{-5/6} (T^{\mu(1/3)+2\mu(2/3)} x^{1/2} + T^{\mu(1/2)+\mu(5/6)} x^{2/3} + T^{\mu(2/3)} x).$$

Utilizing rough estimate $\mu(1/2) \leqslant 1/6$ from [7, Th. 5.5] we get by (7) that

$$\mu(\sigma) \leqslant \begin{cases} 1/2 - 2\sigma/3, & \sigma \in [0,1/2], \\ (1-\sigma)/3, & \sigma \in [1/2,1] \end{cases} \tag{29}$$

and

$$\mu(1/3) \leqslant 5/18, \quad \mu(2/3) \leqslant 1/9, \quad \mu(5/6) \leqslant 1/18, \tag{30}$$

so

$$I_2 \ll x^{1/3} T^{-5/6} (T^{1/2} x^{1/2} + T^{2/9} x^{2/3} + T^{1/9} x) \ll x^{4/3}. \tag{31}$$

There is 1 face of form $[b-iT, b+iT]^2$. Applying (30) we have

$$\begin{aligned} I_3 := \iint\limits_{[b-iT,b+iT]^2} L(z,w)\, dz\, dw \ll \iint\limits_{[1,T]^2} \zeta(b+it_1)\zeta(2b+2it_1) \times \\ \times \zeta(b+it_2)\zeta(2b+2it_2)\zeta\big(2b+i(t_1+t_2)\big) x^{2b} t_1^{-1} t_2^{-1} dt_1 dt_2 \ll \\ \ll x^{2/3} \iint\limits_{[1,T]^2} t_1^{5/18+1/9-1} t_2^{5/18+1/9-1} (t_1+t_2)^{1/9} dt_1 dt_2, \end{aligned}$$

which implies

$$I_3 \ll x^{2/3} T^{8/9}, \tag{32}$$

which is less than $x^{4/3}$ by our choice of T in (28).

There are 4 faces of form $[c-iT,c+iT]\times[b\pm iT,c\pm iT]$. We have

$$I_4 := \int_{c-iT}^{c+iT}\int_{b+iT}^{c+iT} L(z,w)\,dz\,dw \ll$$
$$\ll \int_1^T \int_b^c \zeta(c+it)\zeta(2c+2it)\zeta(\sigma+iT)\zeta(2\sigma+2iT)\zeta\big(c+\sigma+i(t+T)\big)\times$$
$$\times x^{c+\sigma}t^{-1}T^{-1}d\sigma\,dt \ll xT^{-1}\int_b^c \zeta(\sigma+iT)\zeta(2\sigma+2iT)x^\sigma d\sigma. \quad (33)$$

Here

$$\int_b^c \zeta(\sigma+iT)\zeta(2\sigma+2iT)x^\sigma d\sigma \ll \max_{\sigma\in[b,c]} \zeta(\sigma+iT)\zeta(2\sigma+iT)x^\sigma.$$

For $\sigma\in[b,1/2]$ we have

$$\zeta(\sigma+iT)\zeta(2\sigma+iT)x^\sigma \ll T^{\mu(1/3)+\mu(2/3)}x^{1/2} \ll Tx^{1/3}. \quad (34)$$

Taking into account (29) for $\sigma\in[1/2,1]$ we get

$$\zeta(\sigma+iT)\zeta(2\sigma+iT)x^\sigma \ll T^{\mu(\sigma)}x^\sigma \ll x^{\mu(\sigma)+\sigma} \ll x^{(1+2\sigma)/3} \ll x. \quad (35)$$

Returning to (33) we get

$$I_4 \ll x^2T^{-1}+x^{4/3}. \quad (36)$$

There are 4 faces of form $[b\pm iT,c\pm iT]^2$. We have

$$I_5 := \iint\limits_{[b+iT,c+iT]^2} L(z,w)\,dz\,dw \ll \max_{(z,w)\in[b+iT,c+iT]^2} L(z,w) \ll$$
$$\ll \max_{\sigma_1,\sigma_2\in[b,c]} \zeta(\sigma_1+iT)\zeta(2\sigma_1+2iT)\zeta(\sigma_2+iT)\zeta(2\sigma_2+iT)\zeta(\sigma_1+\sigma_2+2iT)\times$$
$$\times x^{\sigma_1+\sigma_2}T^{-2} \ll T^{2\mu(1/3)+3\mu(2/3)-2}x^2 \ll x^2T^{-1}. \quad (37)$$

Finally, there are 8 faces, which are parallel either to z- or w-plane, of form $[b-iT,c+iT]\times w$, where $w\in W:=\{b\pm iT,c\pm iT\}$. We have

$$I_6 := \iint\limits_{b-iT}^{c+iT} L(z,b+iT)\,dz \ll \int_1^T\int_b^c \zeta(\sigma+it)\zeta(2\sigma+2it)\zeta\big(\sigma+b+i(t+T)\big)\times$$
$$\times\zeta(b+iT)\zeta(2b+2iT)x^{\sigma+b}t^{-1}T^{-1}d\sigma\,dt \ll T^{\mu(1/3)+\mu(2/3)-1}x^{1/3}\times$$
$$\times\int_1^T\int_b^c \zeta(\sigma+it)\zeta(2\sigma+2it)\zeta(\sigma+1/3+iT)x^\sigma t^{-1}d\sigma\,dt.$$

Here

$$\zeta(\sigma+it)\zeta(2\sigma+2it)\zeta(\sigma+1/3+iT)x^\sigma t^{-1} \ll T^{\mu(1/3)+2\mu(2/3)-1}x,$$

so

$$I_6 \ll T^{\mu(1/3)+\mu(2/3)-1}x^{1/3}\int_1^T T^{\mu(1/3)+2\mu(2/3)-1}x\,dt \ll x^{4/3}. \quad (38)$$

Also

$$I_7 := \iint_{b-iT}^{c+iT} L(z, c+iT)\, dz \ll \int_1^T \int_b^c \zeta(\sigma+it)\zeta(2\sigma+2it) \times$$
$$\times \zeta\big(\sigma+c+i(t+T)\big)\zeta(c+iT)\zeta(2c+2iT)x^{\sigma+c}t^{-1}T^{-1} d\sigma\, dt \ll$$
$$\ll xT^{-1} \int_1^T \int_b^c \zeta(\sigma+it)\zeta(2\sigma+2it)x^{\sigma}t^{-1} d\sigma\, dt$$

We derive from (34) and (35) that

$$\int_b^c \zeta(\sigma+it)\zeta(2\sigma+2it)x^{\sigma} d\sigma \ll tx^{1/3} + x,$$

so

$$I_7 \ll xT^{-1} \int_1^T (x^{1/3} + xt^{-1}) dt \ll x^2 T^{-1} + x^{4/3}. \tag{39}$$

Now summing up (27), (31), (32), (36), (37), (38), (39) we get

$$E(x) \ll x^{4/3}T^{1/6} + x^{2+\varepsilon}T^{-1}. \tag{40}$$

Conclusion.

Our result can be slightly improved under the Riemann hypothesis. In such case we have $\zeta^{\pm 1}(s) \ll x^{\varepsilon}$ for $\sigma > 1/2$ and $\mu(1/2) = 0$ due to [7, (14.2.5)–(14.2.6)]. Then (19) immediately induces $F(z,w) \ll x^{\varepsilon}\zeta(z)\zeta(w)$ for $\Re z, \Re w > 1/4$ and all double integrals, incorporated in $E(x)$, can be split and estimated by a product of two one-dimensional integrals. For $b = 1/4 + 1/\log x$ we obtain

$$\int_{b-iT}^{b+iT} \zeta(z) \frac{x^z}{z} dz \ll x^{1/4+\varepsilon}T^{1/4},$$
$$\int_{c-iT}^{c+iT} \zeta(z) \frac{x^z}{z} dz \ll x^{1+\varepsilon},$$
$$\int_{b\pm iT}^{c\pm iT} \zeta(z) \frac{x^z}{z} dz \ll (x^{1/2+\varepsilon}T^{1/4} + x^{1+\varepsilon})/T.$$

Then $E(x) \ll x^{5/4+\varepsilon}T^{1/4}$ and choice $T = x^{3/5}$ provides us with $\alpha = 7/5 = 1.4$ in the statement of Theorem 1.

One should expect in the view of (20) that

$$\sum_{m,n \leqslant x} \tau_{1,k}(mn) = D_1 x^2 + D_2 x^{1+1/k} + O(x^{\alpha_k+\varepsilon}). \tag{41}$$

Translating the domain of integration till $[b-iT, b+iT]^2$, where $b = 1/(k+1)$, leads to the error term at least $x^{\frac{k+2}{k+1}+\varepsilon}T^{\frac12-\frac{1}{k+1}} + x^{2+\varepsilon}T^{-1}$, which corresponds to $\alpha_k = (4k+2)/(3k+1)$ for the best possible choice of T. Under the Riemann hypothesis for $b = 1/2k + 1/\log x$ we obtain $\alpha_k = (4k-1)/(3k-1)$. However, for $k > 2$ both of these estimates are bigger than $x^{4/3}$ and absorbs the term $D_2 x^{1+1/k}$ in (41). Such result can hardly be reckoned satisfactory.

One can consider the exponential divisor function $\tau^{(e)}$, which is multiplicative and defined by $\tau^{(e)}(p^a) = \tau(a)$. As far as $\tau^{(e)}(p^k) = \tau_{1,2}(p^k)$ for $k = 1, 2, 3, 4$, the Dirichlet series for $\tau^{(e)}$ also posesses the representation (19), so Theorem 1 remains valid for $\tau^{(e)}$ instead of $\tau_{1,2}$.

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