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ON A ONE CLASS OF THE SOLUTIONS OF THE NONLINEAR FIRST-ORDER DIFFERENTIAL EQUATION WITH OSCILLATING COEFFICIENTS

Щоголев С. А. Про один клас розв'язків нелінійного диференціального рівняння першого порядку з коливними коефіцієнтами. Для нелінійного диференціального рівняння першого порядку, коефіцієнти якого зображувані у вигляді абсолютно та рівномірно збіжних рядів Фур'є з повільно змінними коефіцієнтами та частотою, отримано умови існування частинного розв'язку аналогічної структури в резонансному випадку.
Ключові слова: диференціальний, повільно змінний, ряди Фур'є.

Щёголев С. А. Об одном классе решений нелинейного дифференциального уравнения первого порядка с осциллирующими коэффициентами. Для нелинейного дифференциального уравнения первого порядка, коэффициенты которого представимы в виде абсолютно и равномерно сходящихся рядов Фурье с медленно меняющимися коэффициентами и частотой, получены условия существования частного решения аналогичной структуры в резонапнсном случае.
Ключевые слова: дифференциальный, медленно меняющийся, ряды Фурье.

Shchogolev S. On a one class of the solutions of the nonlinear first-order differential equation with oscillating coefficients. For the nonlinear first-order differential equation, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency, the condidtions of existence of the particular solution of analogous structure are obtained at resonance case.
Key words: differential, slowly-varying, Fourier series.

Introduction. This paper is a continuation of research initiated in paper [1]. Here we using the definitions and designations from [1]. In this paper are considered the next system of the differential equations:

$$\frac{dx_j}{dt} = \sum_{k=1}^{2} a_{jk}(t,\varepsilon)x_k + f_j(t,\varepsilon,\theta(t,\varepsilon)) + \mu X_j(t,\varepsilon,\theta(t,\varepsilon),x_1,x_2),\ j=1,2, \quad (1)$$

where $t,\varepsilon \in G(\varepsilon_0) = \{t,\varepsilon:\ 0<\varepsilon<\varepsilon_0,\ -L\varepsilon^{-1} \le t \le L\varepsilon^{-1},\ 0<L<+\infty\}$, $\mathrm{colon}(x_1,x_2) \in D \subset \mathbf{R}^2$, $a_{jk} \in S(m,\varepsilon_0)$, $f_j \in F(m,l,\varepsilon_0,\theta)$, $X_1, X_2 \in F(m,l,\varepsilon_0,\theta)$ with respect t,ε,θ and analytic with respect $x_1,x_2 \in D$; $\mu \in (0,\mu_0) \subset \mathbf{R}^+$. Functions $a_{jk},\ f_j,\ X_j$ $(j,k=1,2)$ are real, and eigenvalues of matrix $(a_{jk}(t,\varepsilon))$ have a form $\pm i\omega(t,\varepsilon)$, where $\omega \in \mathbf{R}^+$.

In paper [1] the conditions of existence of the particular solutions belongs to class $F(m^*,l^*,\varepsilon^*,\theta)$ $(m^* \le m,,\ l^* \le l,\ \varepsilon^* \le \varepsilon_0)$ are obtained (the definitions of classes $S(m,\varepsilon_0)$, $F(m,l,\varepsilon_0,\theta)$ given in [1]). It was assumed that the conditions:

$$\inf_{G(\varepsilon_0)} |a_{12}(t,\varepsilon)| > 0,$$

$$\inf_{G(\varepsilon_0)} |k\omega(t,\varepsilon) - n\varphi(t,\varepsilon)| \geq \gamma > 0,\ k = 1, 2;\ n \in \mathbf{Z},$$

$\varphi(t,\varepsilon) = d\theta/dt$, means considered noresonance case. The purpose of this paper is to obtain analogous results in resonance case, means when eigenvalues of matrix $(a_{jk}(t,\varepsilon))$ have a form $\pm ir\varphi(t,\varepsilon)$, $r \in \mathbf{N}$. In order to simplify the presentation instead of system (1) we consider the first-order differential equation of special kind. The results for this equation can be easily extended to a system (1) and to the same systems of the more general kind [2].

Main Results

1. Statement of the Problem. We consider the next first-order differential equation:

$$\frac{dx}{dt} = f(t,\varepsilon,\theta(t,\varepsilon)) + \mu X(t,\varepsilon,\theta(t,\varepsilon),x), \tag{2}$$

where $t,\varepsilon \in G(\varepsilon_0)$, $|x| \leq d < +\infty$, $f \in F(m,l,\varepsilon_0,\theta)$, $X \in F(m,l,\varepsilon_0,\theta)$ with respect t,ε,θ and analytic with respect x, at $|x| \leq d$.

We study the problem of existence of the particular solutions of the classes $F(m^*,l^*,\varepsilon^*,\theta)$ $(m^* \leq m,, l^* \leq l,\ \varepsilon^* \leq \varepsilon_0)$ of the equation (2).

2. Auxiliary results.

Lemma 1. *Suppose we are given the following linear first-order differential equation*

$$\frac{dx}{dt} = \lambda(t,\varepsilon)x + u(t,\varepsilon,\theta(t,\varepsilon)), \tag{3}$$

where $\lambda \in S(m,\varepsilon_0)$, $u \in F(m,l,\varepsilon_0,\theta)$. Let condition:

$$\inf_{G(\varepsilon_0)} |\mathrm{Re}\lambda(t,\varepsilon)| = \gamma > 0. \tag{4}$$

Then the equation (3) has a particular solution $x(t,\varepsilon,\theta) \in F(m,l,\varepsilon_0,\theta)$ for any function $u \in F(m,l,\varepsilon_0,\theta)$, and exists $K_0 \in (0,+\infty)$ such that

$$\|x\|_{F(m,l,\varepsilon_0,\theta)} \leq \frac{K_0}{\gamma}\ \|u\|_{F(m,l,\varepsilon_0,\theta)}. \tag{5}$$

Proof. We represent the function u in the form of Fourier-series:

$$u(t,\varepsilon,\theta) = \sum_{n=-\infty}^{\infty} u_n(t,\varepsilon)\exp(in\theta).$$

The desired solution will be sought in the form of a Fourier series:

$$x(t,\varepsilon,\theta) = \sum_{n=-\infty}^{\infty} x_n(t,\varepsilon)\exp(in\theta). \tag{6}$$

Then for coefficients $x_n(t,\varepsilon)$ we obtain the following differential equations:

$$\frac{dx_n}{dt} = \sigma_n(t,\varepsilon)x_n + u_n(t,\varepsilon),\ n \in \mathbf{Z}, \tag{7}$$

where $\sigma_n(t,\varepsilon) = \lambda(t,\varepsilon) - in\varphi(t,\varepsilon)$.

We consider the following solution of equation (7):

$$x_n(t,\varepsilon) = \int\limits_{\pm\frac{L}{\varepsilon}}^{t} u_n(\tau,\varepsilon)\exp\left(\int\limits_{\tau}^{t}\sigma_n(s,\varepsilon)ds\right)d\tau, \tag{8}$$

where the sign in lower limit of integration coincides with the sign of $\mathrm{Re}\lambda(t,\varepsilon)$.

We consider the case $m = 0$ and $\mathrm{Re}\lambda(t,\varepsilon) \le -\gamma < 0$. We have:

$$x_n(t,\varepsilon) = \int\limits_{-\frac{L}{\varepsilon}}^{t} u_n(\tau,\varepsilon)\exp\left(\int\limits_{\tau}^{t}\sigma_n(s,\varepsilon)ds\right)d\tau,$$

$$\sup_{G(\varepsilon_0)}|x_n(t,\varepsilon)| \le \sup_{G(\varepsilon_0)}|u_n(t,\varepsilon)|\int\limits_{-\frac{L}{\varepsilon}}^{t}\exp\left(\int\limits_{\tau}^{t}\mathrm{Re}\lambda(s,\varepsilon)ds\right)d\tau \le$$

$$\le \sup_{G(\varepsilon_0)}|u_n(t,\varepsilon)|\int\limits_{-\frac{L}{\varepsilon}}^{t}\exp\left(-\gamma(t-\tau)\right)d\tau =$$

$$= \frac{1}{\gamma}\sup_{G(\varepsilon_0)}|u_n(t,\varepsilon)|\left(1-\exp\left(-\gamma\left(t+\frac{L}{\varepsilon}\right)\right)\right) < \frac{1}{\gamma}\sup_{G(\varepsilon_0)}|u_n(t,\varepsilon)|. \tag{9}$$

It is easy to show that a similar estimate holds in the case $\mathrm{Re}\lambda(t,\varepsilon) \ge \gamma > 0$. Thus in case $m = 0$ Lemma are proved. For the case $m \ge 1$ using arguments similar to those given in [3], and using estimation (9), we obtain the Lemma.

We suppose, that

$$\int\limits_{0}^{2\pi} f(t,\varepsilon,\theta)d\theta = 0 \;\forall\; (t,\varepsilon) \in G(\varepsilon_0). \tag{10}$$

We consider the function:

$$\xi_0(t,\varepsilon,\theta) = M_0(t,\varepsilon) + \widetilde{\xi}(t,\varepsilon,\theta),$$

where

$$\widetilde{\xi}(t,\varepsilon,\theta) = L[f(t,\varepsilon,\theta)] = \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty}\frac{\Gamma_n[f]}{in\varphi}\;\exp(in\theta),$$

and function $M_0(t,\varepsilon)$ are defined as the root of equation:

$$P(t,\varepsilon,M) = \int\limits_{0}^{2\pi} X(t,\varepsilon,\theta,M+\widetilde{\xi}(t,\varepsilon,\theta))d\theta = 0. \tag{11}$$

Lemma 2. *Let the equation (2) such that:*
1) *the function $f(t,\varepsilon,\theta)$ satisfy condition (10);*

2) *the equation (11) has a root* $M_0(t,\varepsilon)$ *such that*

$$\inf_{G(\varepsilon_0)}\left|\mathrm{Re}\frac{\partial P(t,\varepsilon,M_0)}{\partial M}\right| = \gamma_0 > 0. \tag{12}$$

Then exists $\mu_1 \in (0,\mu_0)$ *such that* $\forall\ \mu \in (0,\mu_1)$ *exists the non-degenerate transformation of kind*

$$x = \psi_1(t,\varepsilon,\theta,\mu) + \psi_2(t,\varepsilon,\theta,\mu)y,$$

where $\psi_1, \psi_2 \in F(m,l,\varepsilon_0,\theta)$, *reducing the equation (2) to kind:*

$$\frac{dy}{dt} = \mu\lambda_0(t,\varepsilon)y + \mu^2 r(t,\varepsilon,\theta,\mu)y + \mu\varepsilon v(t,\varepsilon,\theta,\mu)y +$$

$$+\varepsilon c(t,\varepsilon,\theta,\mu) + \mu^2 d(t,\varepsilon,\theta,\mu) + \mu Y(t,\varepsilon,\theta,y,\mu), \tag{13}$$

where $\lambda_0 \in S(m,\varepsilon_0)$, $r, d \in F(m,l,\varepsilon_0,\theta)$, $v \in F(m-1,l,\varepsilon_0,\theta)$, *function* Y *belong to class* $F(m,l,\varepsilon_0,\theta)$ *with respect* t,ε,θ *and contain terms not lower than second order with respect* y.

Proof. We make in the equation (2) the substitution:

$$x = \xi_0(t,\varepsilon,\theta) + z, \tag{14}$$

where z – the new unknown function, for which we obtain the equation:

$$\frac{dz}{dt} = \varepsilon g(t,\varepsilon,\theta) + \mu h(t,\varepsilon,\theta) + \mu p(t,\varepsilon,\theta)z + \mu Z(t,\varepsilon,\theta,z,\mu), \tag{15}$$

where

$$g = -\frac{1}{\varepsilon}\frac{\partial \xi_0}{\partial t} \in F(m-1,l,\varepsilon_0,\theta),\ h = X(t,\varepsilon,\theta,M_0+\widetilde{\xi}) \in F(m,l,\varepsilon_0,\theta),$$

$$p = \frac{\partial X(t,\varepsilon,\theta,M_0+\widetilde{\xi})}{\partial x} \in F(m,l,\varepsilon_0,\theta),\ Z = \frac{1}{2}\frac{\partial^2 X(t,\varepsilon,\theta,\xi_0+\nu z)}{\partial x^2}\ z^2\ (0<\nu<1).$$

By condition (11) we have:

$$\Gamma_0[h(t,\varepsilon,\theta)] \equiv 0.$$

We make in equation (15) the substitution:

$$z = \mu z_0(t,\varepsilon,\theta) + \widetilde{z}, \tag{16}$$

where $z_0 = L[h(t,\varepsilon,\theta)] \in F(m,l,\varepsilon_0,\theta)$, and $\widetilde{z}$ – new unknown function. We obtain:

$$\frac{d\widetilde{z}}{dt} = \varepsilon c_1(t,\varepsilon,\theta,\mu) + \mu^2 d_1(t,\varepsilon,\theta,\mu) + \mu p(t,\varepsilon,\theta)\widetilde{z} +$$

$$+\mu^2 q(t,\varepsilon,\theta,\mu)\widetilde{z} + \mu\widetilde{Z}(t,\varepsilon,\theta,\widetilde{z},\mu), \tag{17}$$

where

$$c_1 = -\frac{\mu}{\varepsilon}\frac{\partial z_0}{\partial t} + g \in F(m-1,l,\varepsilon_0,\theta),\ d_1 = pz_0 + \frac{1}{\mu}\ Z(t,\varepsilon,\theta,\mu z_0,\mu) \in F(m,l,\varepsilon,\theta),$$

$$q = \frac{1}{\mu} \frac{\partial Z(t, \varepsilon, \theta, \mu z_0, \mu)}{\partial z} \in F(m, l, \varepsilon_0, \theta),\ \widetilde{Z} = \frac{1}{2} \frac{\partial^2 Z(t, \varepsilon, \theta, \mu z_0 + \nu_1 \widetilde{z}, \mu)}{\partial z^2}\ \widetilde{z}^2\ (0 < \nu_1 < 1).$$

We make in equation (17) the transformation:

$$\widetilde{z} = (1 + \mu\widetilde{\psi}(t, \varepsilon, \theta))y, \tag{18}$$

where $\widetilde{\psi} = L[p(t, \varepsilon, \theta)]$. For sufficiently small μ this transformation is non-degenerate, and as result of its application we obtain the equation (13), in which:

$$\lambda_0(t, \varepsilon) = \Gamma_0[p(t, \varepsilon, \theta)], \tag{19}$$

$$c = (1 + \mu\widetilde{\psi})^{-1} c_1,\ d = (1 + \mu\widetilde{\psi})^{-1} d_1,\ r = (1 + \mu\widetilde{\psi})^{-1} \left(p\widetilde{\psi} - q(1 + \mu\widetilde{\psi}) - \widetilde{\psi}\lambda_0 \right),$$

$$v = -\frac{1}{\varepsilon}\ (1 + \mu\widetilde{\psi})^{-1} \frac{\partial\widetilde{\psi}}{\partial t},\ Y = (1 + \mu\widetilde{\psi})^{-1} \widetilde{Z}(t, \varepsilon, \theta, (1 + \widetilde{\psi})y, \mu).$$

Lemma 2 are proved.

3. Principal results.

Theorem. *Let the equation (2) satisfy condidtions of Lemma 2. Then exists* $\mu_2 \in (0, \mu_0)$, $\varepsilon_1(\mu) \in (0, \varepsilon_0)$ *such that* $\forall\ \mu \in (0, \mu_2)$, $\varepsilon \in (0, \varepsilon_1(\mu))$ *the equation (2) has a perticular solution* $x(t, \varepsilon, \theta, \mu) \in F(m - 1, l, \varepsilon_1(\mu), \theta)$.

Proof. Based on Lemma 2 for sufficienly small μ we reduce the equation (2) to the equation (13). In equation (13) we make the sustitution:

$$y = \frac{\varepsilon + \mu^2}{\mu}\ \widetilde{y}, \tag{20}$$

where $\widetilde{y}$ – new unknown function. Since the function Y contain the terms njt lower the second order with respect y, we obtain:

$$\frac{d\widetilde{y}}{dt} = \mu\lambda_0(t, \varepsilon)\widetilde{y} + \mu^2 r(t, \varepsilon, \theta, \mu)\widetilde{y} + \mu\varepsilon v(t, \varepsilon, \theta, \mu)\widetilde{y} +$$

$$+ \frac{\varepsilon\mu}{\varepsilon + \mu^2}\ c(t, \varepsilon, \theta, \mu) + \frac{\mu^3}{\varepsilon + \mu^2}\ d(t, \varepsilon, \theta, \mu) + \left(\varepsilon + \mu^2\right) \widetilde{Y}(t, \varepsilon, \theta, \widetilde{y}, \mu). \tag{21}$$

Consider corresponding to equation (21) the linear nonhomogeneous equation:

$$\frac{d\widetilde{y}_0}{dt} = \mu\lambda_0(t, \varepsilon)\widetilde{y}_0 + \frac{\varepsilon\mu}{\varepsilon + \mu^2}\ c(t, \varepsilon, \theta, \mu) + \frac{\mu^3}{\varepsilon + \mu^2}\ d(t, \varepsilon, \theta, \mu). \tag{22}$$

Based on (19) and condidtion (12) we have:

$$\inf_{G(\varepsilon_0)} |\mathrm{Re}\lambda_0(t, \varepsilon)| > 0.$$

Then based on Lemma 1 the equation (22) has a particular solution $\widetilde{y}_0(t, \varepsilon, \theta, \mu) \in F(m - 1, l, \varepsilon_0, \theta)$, and exists $K_1 \in (0, +\infty)$ such that:

$$\|\widetilde{y}_0\|_{F(m-1,l,\varepsilon_0,\theta)} \leq K_1 \left(\frac{\varepsilon}{\varepsilon + \mu^2}\ \|c\|_{F(m-1,l,\varepsilon_0,\theta)} + \frac{\mu^2}{\varepsilon + \mu^2}\ \|d\|_{F(m-1,l,\varepsilon_0,\theta)} \right).$$

We construct the process of successive approximations, defining as initial approximation $\widetilde{y}_0$, and subsequent approximations defining as solutions from class $F(m-1, l\varepsilon_0, \theta)$ of the equations:

$$\frac{d\widetilde{y}_{s+1}}{dt} = \mu\lambda_0(t,\varepsilon)\widetilde{y}_{s+1} + \mu^2 r(t,\varepsilon,\theta,\mu)\widetilde{y}_s + \mu\varepsilon v(t,\varepsilon,\theta,\mu)\widetilde{y}_s + + \frac{\varepsilon\mu}{\varepsilon+\mu^2}\, c(t,\varepsilon,\theta,\mu) +$$

$$+ \frac{\mu^3}{\varepsilon+\mu^2}\, d(t,\varepsilon,\theta,\mu) + \left(\varepsilon+\mu^2\right)\widetilde{Y}(t,\varepsilon,\theta,\widetilde{y}_s,\mu), \; s = 0,1,2,\ldots \; . \qquad (23)$$

Using techniques contraction mapping principle [4] it is easy to show that exists $\mu_2 \in (0,\mu_0)$ and $\varepsilon_2(\mu) = K_2\mu$, where K_2 – sufficiently small constant, such that $\forall\, \mu \in (0,\mu_2)$, $\forall\, \varepsilon \in (0,\varepsilon_2(\mu))$ the process (23) converges to the solution $\widetilde{y}(t,\varepsilon,\theta,\mu)$ of the equation (21), From its based on (21) and Lemma 2 we obtain the theorem.

Conclusion. Thus, for the equation (2) with the oscillating coefficients the sufficient conditions of the existence of the solution which represented by a Fourier-series with slowly varying coefficients and frequency are obtained in a one critical case.

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