UDC 511.2

J.-M. De Koninck, I. Kátai
Département de mathématiques et de statistique, Universit Laval, Canada
Computer Algebra Department, Eötvös Loránd University, Budapest, Hungary

PRIME FACTORIZATION AND NORMAL NUMBERS

Research of the first author was supported in part by a grant from NSERC.

Let $q \geqslant 2$ be a fixed integer. Given an integer $n \geqslant 2$ and writing its prime factorization as $n = p_1 p_2 \cdots p_r$, where $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_r$ stand for all the prime factors of n, we let $\ell(n) = \overline{p_1}\,\overline{p_2} \cdots \overline{p_r}$, that is the concatenation of the respective base q digits of each prime factor p_i, and set $\ell(1) = 1$. We prove that the real number $0.\ell(1)\ell(2)\ell(3)\ell(4)\ldots$ is a normal in base q. In fact, we show more, namely that the same conclusion holds if we replace each $\overline{p_i}$ by $\overline{S(p_i)}$, where $0S(x) \in \mathbb{Z}[x]$ is an arbitrary polynomial of positive degree such that $S(n) > 0$ for all integers $n \geqslant 1$. We prove analogous results and in particular that, given any fixed positive integer a, the real number $0.\ell(2 + a)\ell(3 + a)\ell(5 + a)\ldots\ell(p + a)\ldots$, where p runs through all primes, is a normal number in base q.
MSC: 11K16.
Key words: prime factorization, normal numbers .

Introduction. Given an integer $q \geqslant 2$, a q-normal number (or a *normal number*) is a real number whose q-ary expansion is such that any preassigned sequence of length $k \geqslant 1$, of base q digits from this expansion, occurs at the expected frequency, namely $1/q^k$.

The problem of determining if a given number is normal is unresolved. For instance, fundamental constants such as π, e, $\sqrt{2}$, $\log 2$ as well as the famous Apèry constant $\zeta(3)$, have not yet been proven to be normal numbers, although numerical evidence tends to indicate that they are. Interestingly, Borel [2] has shown that almost all real numbers are normal, that is that the set of those real numbers which are not normal has Lebesgue measure 0.

One of the first to come up with a normal number was Champernowne [3] who, in 1933, was able to prove that the number made up of the concatenation of the natural numbers, namely the number

$$0.1234567891011121314151617181920 21...,$$

is normal in base 10. In 1946, Copeland and Erdős [4] showed that the same is true if one replaces the sequence of natural numbers by the sequence of primes, namely for the number

$$0.23571113171923293137...$$

In the same paper, they conjectured that if $f(x)$ is any nonconstant polynomial whose values at $x = 1, 2, 3, \ldots$ are positive integers, then the decimal $0.f(1)f(2)f(3)\ldots$, where $f(n)$ is written in base 10, is a normal number. In 1952, Davenport and Erdős [5] proved this conjecture.

In 1997, Nakai and Shiokawa [15] showed that if $f(x)$ is any nonconstant polynomial taking only positive integral values for positive integral arguments, then the

Received 10.06.2015

number $0.f(2)f(3)f(5)f(7)...f(p)...$, where p runs through the prime numbers, is normal.

In a series of papers [8], [9], [11], [12], we created various families of normal numbers. In particular, we showed that the numbers

$$0.p(2)p(3)p(4)p(5)... \qquad \text{and} \qquad 0.P(2)P(3)P(4)P(5)...,$$

where $p(n)$ and $P(n)$ stand respectively for the smallest and largest prime factors of n, are normal numbers.

Also, in two papers [7], [10], we used the fact that the prime factorization of integers is locally chaotic but at the same time globally very regular in order to create very different families of normal numbers.

Here, we create a new family of normal numbers again using the factorization of integers but with a different approach. Write each integer $n \geqslant 2$ as $n = p_1 p_2 \cdots p_r$, where $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_r$ represent all the prime factors of n. Then, setting $\ell(1) = 1$ and, for each integer $n \geqslant 2$, letting $\ell(n)$ represent the concatenation of the primes $p_1, p_2, ..., p_r$, we show that by concatenating $\ell(1), \ell(2), \ell(3), ...$, we can create a normal number, that is that the real number $0.\ell(1)\ell(2)\ell(3)...$ is a normal number. Actually, we prove more general results.

Notation. The letters p and π with or without subscript will always denote prime numbers. We let $\wp$ stand for the set of all prime numbers, $\pi(x)$ for the number of prime numbers not exceeding x and $\pi(x; k, l)$ for the number of primes $p \leqslant x$ such that $p \equiv l \pmod{k}$. Moreover, we set $\mathrm{li}(x) := \int_2^x \frac{dt}{\log t}$. Also, we denote by ϕ the Euler totient function and by $\Omega(n)$ the number of prime factors of n counting their multiplicity. The letters c and C, with or without subscript, always denote a positive constant, but not necessarily the same at each occurrence. At times, we write x_1 for $\log x$, x_2 for $\log\log x$, and so on.

Let $q \geqslant 2$ be a fixed integer and let $A_q = 0, 1, 2, ..., q-1$. Given an integer $t \geqslant 1$, an expression of the form $i_1 i_2 ... i_t$, where each $i_j \in A_q$ is called a *word* of length t. Given a *word* α, we shall write $\lambda(\alpha) = t$ to indicate that α is a *word* of length t. We shall also use the symbol Λ to denote the *empty word*. For each $t \in \mathbb{N}$, we let A_q^t stand for the set of words of length t over A_q, while A_q^* will stand for the set of all words over A_q regardless of their length, including the empty word Λ. Observe that the concatenation of two words $\alpha, \beta \in A_q^*$, written $\alpha\beta$, also belongs to A_q^*. Finally, given a word α and a subword β of α, we will denote by $\nu_\beta(\alpha)$ the number of occurrences of β in α, that is, the number of pairs of words μ_1, μ_2 such that $\mu_1 \beta \mu_2 = \alpha$.

Given a positive integer n, we write its q-ary expansion as

$$n = \epsilon_0(n) + \epsilon_1(n) q + \cdots + \epsilon_t(n) q^t,$$

where $\epsilon_i(n) \in A_q$ for $0 \leqslant i \leqslant t$ and $\epsilon_t(n) \neq 0$. To this representation, we associate the word

$$\bar{n} = \epsilon_0(n)\epsilon_1(n)...\epsilon_t(n) \in A_q^{t+1}$$

For convenience, if $n \leqslant 0$, we let $\bar{n} = \Lambda$. Observe that the number of digits of such a number n will thus be $\lambda(\bar{n}) = \lfloor \frac{\log n}{\log q} \rfloor + 1$.

Finally, given a sequence of integers $a(1), a(2), a(3), \ldots$, we will say that the concatenation of their q-ary digit expansions $\overline{a(1)a(2)a(3)\ldots}$, denoted by $Concat(\overline{a(n)} : n \in \mathbb{N})$, is a q-normal sequence if the real number $0.Concat(a(n) : n \in \mathbb{N}) = 0.\overline{a(1)a(2)a(3)\ldots}$ is a q-normal number.

Main Results

1. Statement of the problem. Let $q \geqslant 2$ be a fixed integer. From here on, we let $S(x) \in \mathbb{Z}[x]$ be an arbitrary polynomial (of degree r_0) such that $S(n) > 0$ for all integers $n \geqslant 1$. Moreover, for each integer $n \geqslant 2$, we write its prime factorization as $n = p_1 p_2 \cdots pr$, where $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_r$ are all the prime factors of n and set

$$\ell(n) := \overline{S(p_1)S(p_2)}\ldots\overline{S(p_r)},$$

where each $S(p_i)$ is expressed in base q. For convenience, we set $\ell(1) = 1$.

Theorem 1. *The real number*

$$\xi := 0.\ell(1)\ell(2)\ell(3)\ell(4)\ldots$$

is a q-normal number.

Theorem 2. *Given an arbitrary positive integer a, the real number*

$$\eta := 0.\ell(2+a)\ell(3+a)\ell(5+a)\ldots\ell(p+a)\ldots,$$

where p runs through all primes, is a q-normal number

Let $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$ be the sequence of divisors of n and let $t(n) = \overline{S(d_1)S(d_2)}\ldots\overline{S(d_{\tau(n)})}$. Then, let

$$\begin{aligned} \theta &:= 0.Concat(t(n) : n \in \mathbb{N}), \\ \kappa &:= 0.Concat(t(p+a) : p \in \wp), \end{aligned}$$

where a is a fixed positive integer.

Theorem 3. *The above real numbers θ and κ are q-normal numbers*

Let $S(x)$ be as above and let $Q(x) \in \mathbb{Z}[x]$ be such that $Q(n) > 0$ for each integer $n \geqslant 1$. Then, consider the expression

$$Q(n) := \prod_{p^a || Q(n)} p^a = p_1 p_2 \cdots p_r,$$

where $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_r$ are all the prime factors of $Q(n)$, so that

$$\ell(Q(n)) = \overline{S(p_1)S(p_2)}\ldots\overline{S(p_r)}.$$

Then, let

$$\begin{aligned} \alpha &:= 0.Concat(\ell(Q(n)) : n \in \mathbb{N}), \\ \beta &:= 0.Concat(\ell(Q(p)) : p \in \wp), \end{aligned}$$

Theorem 4. *The above real numbers α and beta are both q-normal numbers.*

Let $Q(x)$ be as above. Then, let $1 = e_1 < e_2 < \cdots < e_{\delta(n)}$ be the sequence of all the divisors of $Q(n)$ which do not exceed n, consider the expression

$$h(Q(n)) := \overline{S(e_1)S(e_2)}...\overline{S(e_{\delta(n)})}$$

and set

$$\psi := 0.Concat(h(Q(n)) : n \in \mathbb{N})$$

Theorem 5. *The above real number ψ is a q-normal number.*

2. Preliminary lemmas.

Lemma 1. *Let $S \in \mathbb{Z}[x]$ be as above. Given a positive integer k, let β_1 and β_2 be any two distinct words belonging to A_q^k. Let $c_0 > 0$ be an arbitrary number and consider the intervals*

$$J_w := \left[w, w + \frac{w}{\log^{c_0} w}\right] \qquad (w > 1).$$

Further let $\pi(J_w)$ stand for the number of prime numbers belonging to the interval J_w. Then,

$$\frac{1}{\pi(J_w) \cdot \log w} \sum_{p \in J_w} \left|\nu_{\beta_1}(\overline{S(p)}) - \nu_{\beta_2}(\overline{S(p)})\right| \to 0 \qquad as\, w \to \infty$$

Proof. This result is a consequence of Theorem 1 in the paper of Bassily and Kátai [1].

Given an infinite sequence $\gamma = a_1a_2... \in A_q^{\mathbb{N}}$ and a positive integer T, we write γ^T for the word $a_1a_2...a_T$.

Lemma 2. *The infinite sequence γ is a q-normal sequence if for every positive integer k and arbitrary words $\beta_1, \beta_2 \in A_q^k$, there exists an infinite sequence of positive integers $T_1 < T_2 < \cdots$ such that*

$$\lim_{n\to\infty} \frac{\log T_{n+1}}{\log T_n} = 1, \tag{i}$$

$$\lim_{n\to\infty} \frac{1}{T_n} \left|\nu_{\beta_1}(\gamma^{T_n}) - \nu_{\beta_2}(\gamma^{T_n})\right| = 0 \tag{ii}$$

Proof. It follows from conditions (i) and (ii) that

$$\frac{1}{T} \left|\nu_{\beta_1}(\gamma^T) - \nu_{\beta_2}(\gamma^T)\right| \to 0 \text{ as } T \to \infty$$

and consequently that

$$\frac{1}{T} \left| q^k \nu_{\beta_1}(\gamma^T) - \sum_{\beta_2 \in A_q^k} \nu_{\beta_2}(\gamma^T) \right| \to 0 \text{ as } T \to \infty \tag{1}$$

But since

$$\sum_{\beta_2 \in A_q^k} \nu_{\beta_2}(\gamma^T) = T + O(1)$$

it follows from (1) that

$$\frac{1}{T}\left|\nu_{\beta_1}(\gamma^T) - \frac{T}{q^k}\right| \to 0 \text{ as } T \to \infty$$

thereby completing the proof of the lemma.

Lemma 3. *If $1 \leq k \leq x$ and $(k,l) = 1$,*

$$\pi(x; k, l) < \frac{3x}{\phi(k)\log(x/k)}.$$

Proof. This is Theorem 3.8 in the book of Halberstam and Richert [14].

Lemma 4 (Bombieri-Vinogradov Theorem)**.** *For every constant $A > 0$, there exists a constant $B = B(A)$ depending on A, such that for large values of x, the following estimate holds:*

$$\sum_{b<\sqrt{x}/(\log x)^B} \max_{\substack{1\leqslant a<b \\ (a,b)=1 \\ y\leqslant x}} \left|\pi(y;b,a) - \frac{\pi(y)}{\phi(b)}\right| < \frac{x}{\log^A x}$$

Proof. A proof of this result can be found in the book of Iwaniec and Kowalski [6].

3. Proof of Theorem 1. Let x be a large number and set

$$\xi(x) := \ell(1)\ell(2)\ell(3)...\ell(\lfloor x \rfloor).$$

Since $\log S(p) = (1 + o(1))r_0 \log p$ as $p \to \infty$, we find that

$$\begin{aligned}
\lambda(\xi^{(x)}) &= \sum_{n\leqslant x}\left(\left\lfloor \frac{\log \ell(n)}{\log q}\right\rfloor + 1\right) \\
&= \frac{1}{\log q}\sum_{n\leqslant x}\sum_{p^a || n} a \log S(p) + O(x) \\
&= \frac{1}{\log q}\sum_{\substack{p^a\leqslant x \\ a\geqslant 1}} a\log S(p)\left(\frac{x}{p^a} + O(1)\right) + O(x) \\
&= \frac{x}{\log q}\sum_{p\leqslant x}\frac{\log S(p)}{p} + O(x) \\
&= (1+o(1))r_0\frac{x\log x}{\log q} + O(x),
\end{aligned}$$

where we used the prime number theorem in the form $\sum_{p\leqslant x}\frac{\log p}{p} = \log p + O(1)$, thereby establishing that the number of digits of $\xi^{(x)}$ is of order $x \log x$, that is that

$$\lambda(\xi^{(x)}) \approx x \log x \tag{2}$$

Now, we easily obtain that

$$\nu_\beta(\xi^{(x)}) = \sum_{p^a\leqslant x} \nu_\beta(\overline{S(p)}) \left\lfloor \frac{x}{p^a} \right\rfloor + O(x) = x \sum_{p\leqslant x} \frac{\nu_\beta(\overline{S(p)})}{p} + O(x)$$

and therefore that, given any two distinct words $\beta_1, \beta_2 \in A_q^k$ and using (2), there exists a positive constant C such that, as $x \to \infty$,

$$\frac{1}{\lambda(\xi^{(x)})} \left| \nu_{\beta_1}(\xi^{(x)}) - \nu_{\beta_2}(\xi^{(x)}) \right| \leqslant \frac{C}{\log x} \sum_{p\leqslant x} \frac{\left| \nu_{\beta_1}(\overline{S(p)}) - \nu_{\beta_2}(\overline{S(p)}) \right|}{p} + o(1). \qquad (3)$$

On the other hand, it is clear from Lemma 1 that

$$\frac{1}{\pi([x,2x])\log x} \sum_{x\leqslant p<2x} \left| \nu_{\beta_1}(\overline{S(p)}) - \nu_{\beta_2}(\overline{S(p)}) \right| \to 0 \qquad (x \to \infty) \qquad (4)$$

Observe that, in light of (4), as $x \to \infty$,

$$\begin{aligned} \sum_{p\leqslant x} \frac{\left| \nu_{\beta_1}(\overline{S(p)}) - \nu_{\beta_2}(\overline{S(p)}) \right|}{p} &\leq \sum_{\substack{2^l\leqslant x \\ l\geqslant 1}} \frac{1}{2^l} \sum_{2^l\leqslant p<2^{l+1}} \left| \nu_{\beta_1}(\overline{S(p)}) - \nu_{\beta_2}(\overline{S(p)}) \right| \\ &= \sum_{\substack{2^l\leqslant x \\ l\geqslant 1}} \frac{1}{2^l}\, o\left(\frac{2^l \log 2^l}{l} \right) = o(\log x), \end{aligned}$$

which used in (3) along with (2) yields

$$\frac{1}{\lambda(\xi^{(x)})} \left| \nu_{\beta_1}(\xi^{(x)}) - \nu_{\beta_2}(\xi^{(x)}) \right| = o\left(\frac{1}{\log x} \log x \right) + o(1) = o(1),$$

thus completing the proof of Theorem 1.

4. Proof of Theorem 2. Let x be a large number and set

$$\eta(x) := Concat(\ell(p+a) : p \leqslant x).$$

First observe that the number of digits in the word $\eta^{(x)}$ is of order x, since

$$\lambda(\eta^{(x)}) \approx \pi(x) \log x \approx x. \qquad (5)$$

On the other hand, letting $\delta > 0$ be an arbitrary small number, it is known that there exists a positive constant $c > 0$ such that

$$\#\{\pi \leqslant x : P(\pi + a) > x^{1-\delta}\} \leqslant c\delta\pi(x) \qquad (6)$$

(see for instance the proof of Theorem 12.9 in the book of De Koninck and Luca [13]).

Arguing as in the proof of Theorem 1, we have that, given any two distinct words $\beta_1, \beta_2 \in A_q^k$, for some positive constant C_1,

$$\left|\nu_{\beta_1}(\eta^{(x)}) - \nu_{\beta_2}(\eta^{(x)})\right| \leqslant \sum_{p\leqslant x^{1-\delta}} \left|\nu_{\beta_1}(\overline{S(p)}) - \nu_{\beta_2}(\overline{S(p)})\right| \cdot \pi(x;p,-a)$$
$$+ C_1 \sum_{x^{1-\delta}<p\leqslant x} (\log p)\pi(x;p,-a) + O(\pi(x)\log\log x). \tag{7}$$

It follows from Lemma 3 that

$$\pi(x;p,-a) \ll \frac{x}{p\log(x/p)}, \tag{8}$$

which implies, in light of (6), that

$$\sum_{x^{1-\delta}<p\leqslant x} (\log p)\pi(x;p,-a) \ll \log x \cdot \delta\pi(x) \ll \delta x. \tag{9}$$

Using Lemma 1, it follows from (7), (8) and (9) that, for some positive constant C_2,

$$\lim_{x\to\infty} \frac{\left|\nu_{\beta_1}(\eta^{(x)}) - \nu_{\beta_2}(\eta^{(x)})\right|}{\lambda(\eta^{(x)})} \leqslant C_2\delta.$$

Since $\delta > 0$ was chosen to be arbitrarily small, it follows that the left hand side of last inequality must be 0. Combining this with observation (5), the result follows.

5. Proof of Theorem 3. The proof that θ is a normal number is somewhat similar to the proof that η is normal as shown in Theorem 2. Hence, we will focus our attention on the proof that κ is normal.

Let x be a large number and set $\kappa(x) := Concat(t(p+a) : p \leqslant x)$. First we observe that

$$\begin{aligned}
\lambda(\kappa^{(x)}) &= \sum_{d\leqslant x} \lambda(\overline{S(d)})\pi(x;d,-a) + O(\mathrm{li}(x)) \\
&= \sum_{d\leqslant x} \left(\left\lfloor \frac{\log S(d)}{\log q} \right\rfloor + 1 \right) \pi(x;d,-a) + O(\mathrm{li}(x)) \\
&= r_0 \sum_{d\leqslant x} \frac{\log d}{\log q}\pi(x;d,-a) + O\left(\sum_{p\leqslant x} \tau(p+a) \right) + O(\mathrm{li}(x)) \\
&= \frac{r_0}{\log q} \sum_{d\leqslant x} (\log d)\pi(x;d,-a) + O(x),
\end{aligned} \tag{10}$$

where we used the fact that $\sum_{p\leqslant x} \tau(p+a) = O(x)$.

Let $\delta > 0$ be an arbitrarily small number. On the one hand, for some positive constant C_1,

$$\begin{aligned}
\sum_{x^{1-\delta}<d\leqslant x} (\log d)\pi(x;d,-a) &\leqslant (\log x) \sum_{\substack{x^{1-\delta}<d\leqslant x \\ dv=p+a,\ p\leqslant x}} 1 \\
&\leqslant (\log x) \sum_{v\leqslant x^{\delta}} \pi(x;v,-a) \\
&\leqslant C_1(\log x) \sum_{v\leqslant x^{\delta}} \frac{x}{\phi(v)\log(x/v)} \leqslant \delta C_1 x\log x.
\end{aligned} \tag{11}$$

and, for some positive constant C_2

$$\sum_{d\leqslant x^{1-\delta}}(\log d)\pi(x;d,-a)\le(\log x)\sum_{d\leqslant x^{1-\delta}}\frac{C_2x}{\phi(d)\log(x/d)}\leqslant C_2x. \tag{12}$$

On the other hand, using Lemmas 3 and 4, for some positive constant C_3,

$$\begin{aligned}\sum_{d\leqslant x}(\log d)\pi(x;d,-a)&\geqslant\sum_{d\leqslant x^{1/3}}(\log d)\frac{\mathrm{li}(x)}{\phi(d)}-\sum_{d\leqslant x^{1/3}}(\log d)\left|\pi(x;d,-a)-\frac{\mathrm{li}(x)}{\phi(d)}\right|\\&=C_3(1+o(1))x\log x+O\left(\frac{x}{\log^A x}\right)\\&\gg x\log x.\end{aligned} \tag{13}$$

Hence combining relations (10), (11), (12) and (13), we find that

$$\lambda(\theta^{(x)})\approx x\log x. \tag{14}$$

Now, we easily obtain that, for any given distinct words $\beta_1,\beta_2\in A_q^k$,

$$\begin{aligned}\left|\nu_{\beta_1}(\theta^{(x)})-\nu_{\beta_2}(\theta^{(x)})\right|&\leqslant\sum_{d\leqslant x^{1-\delta}}\left|\nu_{\beta_1}(\overline{S(d)})-\nu_{\beta_2}(\overline{S(d)})\right|\pi(x;d,-a)+c\delta x\log x\\&\leqslant C_4\sum_{d\leqslant x^{1-\delta}}\frac{\left|\nu_{\beta_1}(\overline{S(d)})-\nu_{\beta_2}(\overline{S(d)})\right|}{\phi(d)\log(x/d)}+c\delta x\log x,\end{aligned} \tag{15}$$

where we used Lemma 3. Combining (15) with Lemma 1, we obtain that

$$\lim_{x\to\infty}\sup\left|\frac{\nu_{\beta_1}(\theta^{(x)})-\nu_{\beta_2}(\theta^{(x)})}{\lambda(\theta^{(x)})}\right|\leqslant\delta,$$

thereby implying, arguing as in the previous proofs and in light of (14), that

$$\lim_{x\to\infty}\sup\left|\frac{\nu_{\beta_1}(\theta^{(x)})-\nu_{\beta_2}(\theta^{(x)})}{\lambda(\theta^{(x)})}\right|=0,$$

thus completing the proof of Theorem 3.

6. Proof of Theorem 4. We will only consider the number β.

First, for each prime number π, we let $\rho(\pi)$ stand for the number of those residue classes n (with $(n,\pi)=1$) for which $Q(n)\equiv 0 \pmod{\pi}$, and we let

$$l_1^{(\pi)},l_2^{(\pi)},...,l_{\rho(\pi)}^{(\pi)}$$

be the list of these residue classes.

As before, setting $\beta^{(x)}:=Concat(\ell(Q(p)):p\leqslant x)$, we first observe that

$$\begin{aligned}\lambda(\beta^{(x)})&=\sum_{\pi\leqslant x}\lambda(\overline{S(\pi)})\sum_{\substack{p\leqslant x\\ Q(p)\equiv 0 \pmod{\pi}}}1+O\left(\sum_{p\leqslant x}\Omega(Q(p))\right)\\&=\sum_{\pi\leqslant x}\lambda(\overline{S(\pi)})\sum_{\nu=1}^{\rho(\pi)}\pi(x;\pi,l_\nu^{(\pi)})+O\left(x\frac{x_2}{x_1}\right).\end{aligned} \tag{16}$$

Since $\rho(\pi)$ is bounded, we obtain that

$$\lambda(\overline{S(\pi)}) = \left\lfloor \frac{\log S(\pi)}{\log q} \right\rfloor + O(1) = \frac{r_o \log \pi}{\log q} + O(1). \tag{17}$$

Hence, in light of (17), we have, given an arbitrarily small number $\delta > 0$,

$$\sum_{x^{1-\delta} < \pi \leqslant x} \lambda(\overline{S(\pi)}) \sum_{\nu=1}^{\rho(\pi)} \pi(x; \pi, l_\nu^{(\pi)}) \ll \delta x x_2 \tag{18}$$

and

$$\sum_{\pi \leqslant x^{1-\delta}} \lambda(\overline{S(\pi)}) \sum_{\nu=1}^{\rho(\pi)} \pi(x; \pi, l_\nu^{(\pi)}) \approx r_0 \sum_{\pi \leqslant x^{1-\delta}} \frac{\log \pi}{\log q} \sum_{\nu=1}^{\rho(\pi)} \pi(x; \pi, l_\nu^{(\pi)}) \approx r_0 \sum_{\pi \leqslant x^{1-\delta}} \frac{\rho(\pi)}{\pi} \approx x x_2. \tag{19}$$

Hence, combining (18) and (19) in (16), we get that

$$\lambda(\beta^{(x)}) \approx x x_2 \tag{20}$$

Then, using the same approach as in the proofs of the previous theorems, we find that

$$\left| \frac{\nu_{\beta_1}(\beta^{(x)}) - \nu_{\beta_2}(\beta^{(x)})}{x x_2} \right| \leq \delta + o(1) \qquad (x \to \infty)$$

and therefore that

$$\lim_{x \to \infty} \sup \left| \frac{\nu_{\beta_1}(\beta^{(x)}) - \nu_{\beta_2}(\beta^{(x)})}{x x_2} \right| = 0,$$

thus proving that β is a q-normal number.

7. Proof of Theorem 5. The proof is similar to that of Theorem 3 and we will therefore skip it.

8. Final remarks. Let $S, Q \in \mathbb{Z}[x]$ be as above and, given a prime number p, let $\rho(p)$ be the number of solutions n of $Q(n) \equiv 0 \pmod{p}$. Assume that $\rho(p) < p$ for all primes p.

Then, using the above techniques as well as those developed in our previous work [11], we can show that the real numbers

$$\theta_1 := 0.Concat(S(p(Q(n))) : n \in \mathbb{N})$$
$$\theta_2 := 0.Concat(S(p(Q(\pi))) : \pi \in \wp)$$

are q-normal numbers.

Conclusion. Using the concatenation of the prime factors of each positive integer, we created various new families of normal numbers.

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Де Конінк Ж.-М., Катаі І.

Факторізація на прості дільники та нормальні числа

Резюме

Нехай $q \geqslant 2$ — фіксоване ціле. Задавшись цілим числом $n \geqslant 2$ та записавши його розклад на прості дільники $n = p_1 p_2 \cdots p_r$, де $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_r$, ми вводимо позначення $\ell(n) = \overline{p_1}\,\overline{p_2} \cdots \overline{p_r}$ для конкатенації представлень простих дільників p_i за основою q, і за означенням кладемо $\ell(1) = 1$. Нами доведено, що дійсне число $0.\ell(1)\ell(2)\ell(3)\ell(4)\ldots$ є нормальним за основою q. Більш того, нами показано, що це твердження залишається справедливим навіть якщо кожний дільник $\overline{p_i}$ замінити на $\overline{S(p_i)}$, де $S(x) \in \mathbb{Z}[x]$ є таким

поліномом позитивного ступеню, що $S(n) > 0$ для всіх цілих $n \geqslant 1$. Також нами доведено, що для довільного фіксованого позитивного цілого числа a дійсне $0.\ell(2+a)\ell(3+a)\ell(5+a)\ldots\ell(p+a)\ldots$, де p пробігає всі прості числа, є нормальним за основою q.
Ключові слова: факторизація, прості числа, нормальні числа.

Де Конинк Ж.-М., Катаи И.
Факторизация на простые множители и нормальные числа

Резюме

Пусть $q \geqslant 2$ — фиксированное целое. Задавшись целим числом $n \geqslant 2$ и записав его разложение на простые делители $n = p_1 p_2 \cdots p_r$, где $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_r$, мы вводим обозначение $\ell(n) = \overline{p_1}\,\overline{p_2}\cdots\overline{p_r}$ для конкатенации представлений простых делителей p_i по основанию q, и по определению полагаем $\ell(1) = 1$. Нами доказано, что вещественное число $0.\ell(1)\ell(2)\ell(3)\ell(4)\ldots$ является нормальным по основанию q. Более того, нами показано, что это утверждение остается справедливым даже если каждый делитель $\overline{p_i}$ заменить на $\overline{S(p_i)}$, где $S(x) \in \mathbb{Z}[x]$ — такой многочлен положительной степени, что $S(n) > 0$ для всех целых $n \geqslant 1$. Также нами доказано, что для произвольного фиксированного положительного целого числа a вещественное число $0.\ell(2+a)\ell(3+a)\ell(5+a)\ldots\ell(p+a)\ldots$, где p пробегает все простые числа, является нормальным по основанию q.
Ключевые слова: быстро меняющиеся функции, нелинейные дифференциальные уравнения, асимптотика решений.

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