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GENERALIZATIONS OF UNIVERSALITY FOR PERIODIC HURWITZ ZETA FUNCTIONS

It is well known that zeta-functions universal in the sense that their shifts uniformly on compact subsets of some region approximate any analytic functions form a rather wide class. In the paper, the universality for composite functions of the periodic Hurwitz zeta-functions is discussed.

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Introduction. In 1975, S. M. Voronin generalizing the Bohr-Courant denseness result [3] for the Riemann zeta-function $\zeta(s), s = \sigma + \mathrm{i}t$, obtained a remarkable universality theorem [17] on the approximation of analytic functions by shifts $\zeta(s + \mathrm{i}\tau), \tau \in \mathbb{R}$. We state a modern version of the Voronin theorem, for the proof, see, for example, [10]. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$.

Theorem 1. *Suppose that $K \subset D$ be a compact set with connected complement, and $f(s)$ is a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + \mathrm{i}\tau) - f(s)| < \varepsilon\} > 0$$

Voronin's theorem asserts that the set of shifts $\zeta(s + \mathrm{i}\tau)$ approximating a given analytic function is infinite and even has a positive lower density. On the other hand, the theorem is not effective in the sense that any $\tau \in \mathbb{R}$ with approximation property is not known. However, this effectivity problem does not hinder to apply the universality theorem which is useful for proof of functional independence, for study zero-distribution and moment problem of universal functions, is applied for estimation of complicated analytic functions. This is a motivation to extend the class of universal functions in the Voronin sense. After appearance of Voronin's paper, many authors obtained universality of various zeta and L-functions, and of some classes of Dirichlet series, for history and references, see [1, 5, 6, 9, 11, 13, 14, 16]. It turned out that some composite functions of universal functions are also universal. For example, $\log \zeta(s)$ defined by an usual manner [9], and $\zeta'(s)$ are universal functions. In [12], some classes of functions F such that $F(\zeta(s))$ preserve the universality property were introduced. The aim of this paper is the universality for composite functions of periodic Hurwitz zeta-functions.

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period k, and $\alpha, 0 < \alpha \leqslant 1$, be a fixed parameter. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by the series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}.$$

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If $a_m \equiv 1$, the function $\zeta(s, \alpha; \mathfrak{a})$ reduces to the classical Hurwitz zeta-function $\zeta(s, \alpha)$. The periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma > 1$, the equality

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{k^s} \sum_{l=0}^{k-1} a_l \zeta\left(s, \frac{l+\alpha}{k}\right),$$

which gives analytic continuation for $\zeta(s, \alpha; \mathfrak{a})$. If

$$a \stackrel{def}{=} \frac{1}{k} \sum_{l=0}^{k-1} a_l = 0,$$

then the periodic Hurwitz zeta-function is entire, while, in the case $a \neq 0$, $\zeta(s, \alpha; \mathfrak{a})$ is *a* meromorphic function, and the point $s = 1$ is unique simple pole with residue 1.

The universality of the function $\zeta(s, \alpha; \mathfrak{a})$ with transcendental parameter α has been began to study in [7], and proved unconditionally in [8]. Denote by $\mathcal{K}$ the set of compact subsets of the strip D with connected complement, and for $K \in \mathcal{K}$, denote by $H(K)$ the set of continuous functions on K which are analytic in the interior of K. Then the following universality theorem is true [8].

Theorem 2. *Suppose that α is transcendental, $K \in \mathcal{K}$ and $f \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon\} > 0.$$

Let $H(D)$ stand for the space of analytic functions on D equipped with the topology of uniform convergence on compacta. This paper is devoted to the universality of functions $F(\zeta(s, \alpha; \mathfrak{a}))$, where $F : H(D) \to H(D)$.

Main Results

1. Statement of results. In what follows, we suppose that the number α is transcendental.

Theorem 3. *Suppose that $F : H(D) \to H(D)$ is a continuous function such that, for every open set $G \subset H(D)$, the set $F^{-1}G$ is non-empty. Let $K \in \mathcal{K}$ and $f \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\{\tau \in [0, T] : \sup_{s \in K} |F(\zeta(s + i\tau, \alpha; \mathfrak{a})) - f(s)| < \varepsilon\} > 0.$$

The hypothesis of Theorem 3 that the set $F^{-1}G$ is non-empty is very general, however, it is difficult to check this hypothesis. In the next theorem, we replace the hypothesis of Theorem 3 by a stronger but simpler one.

Theorem 4. *Suppose that $F : H(D) \to H(D)$ is a continuous function such that, for each polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is non-empty. Let $K \in \mathcal{K}$ and $f \in H(K)$. Then the assertion of Theorem 3 is true.*

It is easily seen that, for every polynomial $p(s)$, there exits a polynomial $q(s)$ such that, for all $r \in \mathbb{N}$ and $c_1, ..., c_r \in \mathbb{C}$, $c_1 q'(s), ..., c_r q^{(r)}(s) = p(s)$. Therefore, by Theorem 4, the function $c_1\zeta'(s, \alpha, \mathfrak{a}), ..., c_r\zeta^{(r)}(s, \alpha; \mathfrak{a})$ is universal.

The continuity requirement for the function F in Theorem 4 can be replaced by an analogue of the Lipschitz condition in the space of analytic function. More precisely, we have the following theorem.

Theorem 5. *Suppose that $F : H(D) \to H(D)$ is a function such that, for each polynomial p=p(s), the set $F^{-1}\{p\}$ is non-empty, and for each $K \in \mathcal{K}$, there exist positive constants c and β, and $K_1 \in \mathcal{K}$ such that*

$$\sup_{s\in K} |F(g_1(s)) - F(g_2(s))| \leqslant c \sup_{s\in K_1} |g_1(s) - g_2(s)|^{\beta} \tag{1}$$

for all $g_1, g_2 \in H(D)$. Let $K \in \mathcal{K}$ and $f \in H(K)$. Then the assertion of Theorem 3 is true.

In view of the integral Cauchy formula, the function $F(g) = g^{(r)}$, $r \in \mathbb{N}$, satisfies hypotheses of Theorem 5 with $\beta = 1$.

Now we will restrict a class of approximated functions. For $a_1, ..., a_r$, denote

$$H_{a_1,...,a_r}(D) = \{g \in H(D) : (g(s) - a_j)^{-1} \in H(D),\ j = 1, ..., r\}.$$

Theorem 6. *Suppose that $F : H(D) \to H(D)$ is a continuous function such that $F(H(D)) \supset H_{a_1,...,a_r}(D)$. For $r = 1$, let $K \in \mathcal{K}$, $f \in H(K)$ and $f(s) \neq a_1$ on K. For $r \geqslant 2$, let $K \subset D$ be an arbitrary compact set, and $f \in H_{a_1,...,a_r}(D)$. Then the assertion of Theorem 3 is true.*

Solving the equation $\sin(g) = f$ in $g \in H(D)$, we easily find that if $f \in H_{-1,1}(D)$, then, by Theorem 6 with $r = 2$, $f(s)$ can be approximated by shifts $\sin(\zeta(s+i\tau, \alpha; \mathfrak{a}))$.

In general case, the following universality theorem is valid.

Theorem 7. *Suppose that $F : H(D) \to H(D)$ is a continuous function. Let $K \subset D$ be an arbitrary compact set, and $f \in F(H(D))$. Then the assertion of Theorem 3 is true.*

2. Proof of Theorem 5. Theorem 5 is a corollary of Theorem 2 and the Mergelyan theorem on the approximation of analytic functions by polynomials. We state the latter theorem in a convenient for us form as the next lemma.

Lemma 1. *Let $K \subset \mathbb{C}$ be a compact set with connected complement, and $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

Proof of the lemma is given in [15], see also [18].

Proof. [Proof of Theorem 5] By Lemma 1, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{2}$$

Let $\tau \in \mathbb{R}$ satisfy the inequality

$$\sup_{s\in K} |F(\zeta(s+\mathrm{i}\tau,\alpha;\mathfrak{a})) - g(s)| < c^{-\frac{1}{\beta}} \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}, \tag{3}$$

where $g \in F^{-1}\{p\}$, and $K_1 \in \mathcal{K}$ corresponds the set K in hypothesis of the theorem. Then, for the same τ, in view of (1),

$$\sup_{s\in K} |F(\zeta(s+\mathrm{i}\tau,\alpha;\mathfrak{a})) - p(s)| \leqslant c \sup_{s\in K_1} |F(\zeta(s+\mathrm{i}\tau,\alpha;\mathfrak{a})) - g(s)|^{\beta} < \frac{\varepsilon}{2}. \tag{4}$$

By Theorem 2, the set of τ satisfying (3) has a positive lower density. This and (4) show that

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |F(\zeta(s+\mathrm{i}\tau,\alpha;\mathfrak{a})) - p(s)| < \frac{\varepsilon}{2} \right\} > 0.$$

Combining this with (2) proves the theorem.

3. Proof of other Theorems. The proofs of Theorems 4–7 is based on limit theorems in the sense of weak convergence of probability measures in the space $H(D)$. Denote by $\mathcal{B}(S)$ the class of Borel set of the space S.

Define $\Omega = \prod_{m=0}^{\infty} \gamma_m$, where $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, the infinite-dimensional torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined, and this leads to the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathbb{N}_0$, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s,\alpha,\omega;\mathfrak{a})$ by the formula

$$\zeta(s,\alpha,\omega;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^s}.$$

Note that the latter series, for almost all ω, converges uniformly on compact subsets of D. Let P_ζ be the distribution of the random element $\zeta(s,\alpha,\omega;\mathfrak{a})$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s,\alpha,\omega;\mathfrak{a}) \in A), \quad A \in \mathcal{B}(H(D)).$$

Proof. [Proof of Theorem 3] By a theorem of [7], the probability measure

$$P_T(A) \stackrel{def}{=} \frac{1}{T} \mathrm{meas}\{\tau \in [0,T] : \zeta(s+\mathrm{i}\tau,\alpha;\mathfrak{a}) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_ζ as $T \to \infty$. Define

$$P_{T,F}(A) \stackrel{def}{=} \frac{1}{T} \mathrm{meas}\{\tau \in [0,T] : F(\zeta(s+\mathrm{i}\tau,\alpha;\mathfrak{a})) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Then, clearly, we have that $P_{T,F} = P_T F^{-1}$, where $P_T F^{-1}$ is defined, for $A \in \mathcal{B}(H(D))$, by $P_T F^{-1}(A) = P_T(F^{-1}A)$. Therefore , the continuity of F, weak convergence of P_T, and Theorem 5.1 of [2] show that $P_{T,F}$ converges weakly to $P_\zeta F^{-1}$ as $T \to \infty$.

The space $H(D)$ is separable. Therefore, the support of a probability measure P on $(H(D), \mathcal{B}(H(D)))$ is a minimal closed set $S_P \subset H(D)$ such that $P(S_P) = 1$. The set S_P consists of all elements x such that every open neighbourhood G of x has a positive P-measure.

In [8], it is proved that the support of the measure P_ζ is the whole of $H(D)$. We will prove that this is also true for the measure $P_\zeta F^{-1}$. Really, let g be an arbitrary element of $H(D)$, and G be an open neighbourhood of g. By the hypothesis of the theorem, the set $F^{-1}G$ is non-empty, and, because of the continuity of F, it is open, too. Thus, $F^{-1}G$ is an open neighbourhood of some element $g_1 \in H(D)$. Since the support of P_ζ is the whole of $H(D)$, we obtain that $P_\zeta(F^{-1}G) > 0$. Hence, $P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0$. Since g and G are arbitrary, this shows that the support of $P_\zeta F^{-1}$ is the whole of $H(D)$.

By Lemma 1, there exists a polynomial $p = p(s)$ such that (2) holds. Define

$$\mathcal{G} = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

The set $\mathcal{G}$ is an open neighbourhood of p which, in view of the above remark, is an element of the support of the measure $P_\zeta F^{-1}$. Therefore, $P_\zeta F^{-1}(\mathcal{G}) > 0$. Using the weak convergence of $P_{T,F}$ to $P_\zeta F^{-1}$ as $T \to \infty$, and applying an equivalent of weak convergence of probability measures in terms of open sets, see Theorem 3 of [2], we obtain that

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |F(\zeta(s + \mathrm{i}\tau, \alpha; \mathfrak{a})) - p(s)| < \frac{\varepsilon}{2} \right\}$$

$$\geqslant P_\zeta F^{-1}(\mathcal{G}) > 0.$$

This together with (2) proves the theorem.

Proof. [Proof of Theorem 4] The space $H(D)$ is metrisable. It is well known, see, for example, [4], that there exists a sequence $\{K_l : l \in \mathbb{N}\}$ of compact subsets of the strip D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then

$$\varrho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D),$$

is a metric in $H(D)$ which induces the topology of uniform convergence on compacta. In the case of $H(D)$, obviously, we may choose the sets K_l to be with connected complements. It is easily seen that $\varrho(g_1, g_2)$ is small if $\sup_{s \in K_l} |g_1(s) - g_2(s)|$ is small enough for sufficiently large l. Thus, approximation in the space $H(D)$ reduces to that on compact subsets with connected complements.

We will prove that, for every open set $G \subset H(D)$, the set $F^{-1}G$ is non-empty. Let $\emptyset \neq G \subset H(D)$ be arbitrary open set, and $g \in G$. Suppose that $K \in \mathcal{K}$. Then, by Lemma 1, for every $\varepsilon > 0$, these exists a polynomial $p = p(s)$ such that

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon.$$

Therefore, if ε is small enough, we may assume that $p \in G$, too. Thus, by hypothesis of the theorem, the set $F^{-1}G$ is non-empty. Therefore, the theorem follows from Theorem 3.

Proof. [Proof of Theorem 6] First we observe that the support of the measure $P_\zeta F^{-1}$ is the closure of the set $F(H(D))$. Really, let g be an arbitrary element of $F(H(D))$, and G be any open neighbourhood of g. Then there exists $g_1 \in H(D)$ such that $F(g_1) = g$. Therefore, the set $F^{-1}G$ is an open neighbourhood of g_1 by continuity of F. Since the support of the measure P_ζ is the whole of $H(D)$, this shows that $P_\zeta(F^{-1}G) > 0$. Thus,

$$P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0. \tag{5}$$

Moreover, $P_\zeta F^{-1}(F(H(D))) = P_\zeta(H(D)) = 1$. Since the support is a closed set, this together with (5) proves that the support of $P_\zeta F^{-1}$ is the closure of $F(H(D))$.

The case $r = 1$. By Lemma 1, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{6}$$

Since $f(s) \neq a_1$ on K, we have that $p(s) \neq a_1$ on K as well provided ε is small enough. Therefore, we can define on K a continuous branch of $\log(p(s) - a_1)$ which will be analytic in the interior of K. Applying Lemma 1 once more, we find a polynomial $p_1(s)$ such that

$$\sup_{s\in K} |p(s) - a_1 - e^{p_1(s)}| < \frac{\varepsilon}{4}. \tag{7}$$

Obviously, $g_1(s) \stackrel{def}{=} e^{p_1(s)} + a_1 \in H(D)$, and $g_1(s) \neq a_1$. Thus, $g_1 \in H_{a_1}(D)$. Since $H_1(D) \subset F(H(D))$, by the above remark, g_1 is an element of the support of the measure $P_\zeta F^{-1}$. Define

$$G_1 = \{g \in H(D) : \sup_{s\in K} |f(s) - g_1(s)| < \frac{\varepsilon}{2}\}.$$

Then G_1 is an open neighbourhood of g_1, thus, we have that $P_\zeta F^{-1}(G_1) > 0$. Using the weak convergence of the measure $P_{T,F}$ to $P_\zeta F^{-1}$, hence we obtain that

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |F(\zeta(s + \mathrm{i}\tau, \alpha; \mathfrak{a})) - g_1(s)| < \frac{\varepsilon}{2}\right\} \geqslant P_\zeta F^{-1}(G_1) > 0. \tag{8}$$

Inequalities (6) and(7) imply

$$\sup_{s\in K} |f(s) - g_1(s)| < \frac{\varepsilon}{2}$$

which together with (8) proves the theorem in the case $r = 1$.

The case $r \geqslant 2$. Since $f \in H_{a_1,\dots,a_r}(D)$ and $H_{a_1,\dots,a_r}(D) \subset F(H(D))$, we have that f is an element of the support of the measure $P_\zeta F^{-1}$. Define

$$G_2 = \{g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon\}.$$

Then $P_\zeta F^{-1}(G_2) > 0$, and the weak convergence of the measure $P_{T,F}$ to $P_\zeta F^{-1}$ gives the inequality

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s \in K} |F(\zeta(s+\mathrm{i}\tau, \alpha; \mathfrak{a})) - f(s)| < \varepsilon \right\}$$

$$\geqslant P_\zeta F^{-1}(G_2) > 0.$$

The theorem is proved.

Proof. [Proof of Theorem 7] We may use the same arguments as in the proof of the case $r \geqslant 2$ of Theorem 6, since, by the observation in the beginning of the proof of Theorem 6, $f(s)$ is an element of the support of the measure $P_\zeta F^{-1}$.

Conclusion. It is well known that zeta-functions universal in the sense that their shifts uniformly on compact subsets of some region approximate any analytic functions form a rather wide class. In the paper, the universality for composite functions of the periodic Hurwitz zeta-functions was discussed.

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Лаурінчікас А., Мочов Д.
Узагальнення універсальності зета–функцій Гурвіца

Резюме

Добре відомо, що зета-функції універсальні в тому сенсі, що їхні здвижки апроксимують аналітичні функції з доволі широкого класу рівномірно на компактних підмножинах деякої області. В статті обговорюється універсальність композиції періодичних зета-функцій Гурвіца.
Ключові слова: теорема про ліміт, періодична зета–функція Гурвіца, універсальність.

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Обобщения универсальности периодических зета-функций Гурвица

Резюме

Хорошо известно, что зета-функции универсальны в том смысле, что их сдвиги аппроксимируют аналитические функции из довольно широкого класса равномерным образом на компактных подмножествах некоторой области. В статье обсуждается универсальность композиций периодических зета–функций Гурвица.
Ключевые слова: предельная теорема, периодическая зета–функция Гурвица, универсальность.