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APPROXIMATION OF SOLUTIONS TO THE OPTIMAL CONTROL PROBLEM FOR THE IMPULSIVE SYSTEM WITH MAXIMUM

This paper presents the averaging method for two problems: impulsive system with maximum and for the optimal control problem of this kind of system. For the first problem the Krylov–Bogolyubov's theorem is generalized. For the second one we are interested not only in approximation of the solution for optimal control problems with impulsive perturbation and maximum but also in approximation of corresponding functionals. In this purpose the averaging method is obtained as well. In this case averaging scheme includes the algorithm of correspondence between control functions of original and averaged optimal control problems. A numerical-asymptotic algorithm for solving an optimal control problem with a small parameter of such kind of system is designed.
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Introduction. Initially, an averaging method were developed by [2]. It further generalization to the functional differential equation was obtained by [7], [6], [8], to the impulsive system by [1]. [3] developed the averaging method for neutral type of impulsive system with maximum in case averaged system is "frozen", by means it does not depend on maximum of unknown function. We propose the average scheme where averaged system also depends on maximum of x as original system.

Moreover, [10] offered to apply an averaging method to the optimal control problem which is based on the following steps to the impulsive optimal control problem with maximum:

1) average a controlled system;
2) establish the correspondence between controlled functions of both (averaged and original) systems;
3) estimate the quality of control function of averaged problem by the functional of the original problem.

We apply this approach to the impulsive optimal control problem with maximum. The impulsive optimal control problem without maximum of unknown function was considered by [5].

Auxiliary arguments. For a piecewise continuous function $x \in [0,\infty) \to \mathbb{R}^m$ and continuous functions $g, \gamma \in [0,\infty) \to \mathbb{R}$, such that $0 \leqslant g(t) \leqslant \gamma(t) \leqslant t$ for any

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$t \geqslant 0$ we denote the componentwise "maximal" value of x over the time interval $[g(t), \gamma(t)]$ by

$$\widetilde{x}_G(t) = \left\{ \sup_{s\in[g(t),\gamma(t)]} x_1(s), \cdots, \sup_{s\in[g(t),\gamma(t)]} x_m(s) \right\}. \quad (1)$$

For any function $f \in C([0,\infty);\mathbb{R}^n)$ and any matrix $A : [0,\infty) \to \mathbb{R}^{n\times n}$ we introduce

$$||f(t)|| = \max_{1\leqslant i\leqslant n} \sup_{t\geqslant 0} |f_i(t)|, \quad ||A(t)|| = \max_{1\leqslant i\leqslant n} \sum_{j=1}^{m} \sup_{t\geqslant 0} |a_{ij}(t)|.$$

Let X and Y be two non-empty subsets of $\mathbb{R}^n$. We define the Hausdorf distance between them by

$$d_H(X,Y) = \max\left\{ \sup_{x\in X} \inf_{y\in Y} ||x-y||, \sup_{y\in Y} \inf_{x\in X} ||x-y|| \right\}.$$

The following notion of average will be used in this paper.

Definition 1. *[4]. A continuous bounded function $f : [0,\infty) \times D \to \mathbb{R}^n$ is said to have an average $\overline{f}(x)$ if the limit*

$$\overline{f}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,x)dt, \quad (2)$$

exists and $\forall (t,x) \in [0,\infty) \times D' \times D'$

$$\left\| \frac{1}{T} \int_0^T f(t,x)dt - \overline{f}(x) \right\| \leqslant q\sigma(T),$$

for every compact set $D' \subset D$, where q is a positive constant (possibly dependent on D') and $\sigma : [0,\infty) \to [0,\infty)$ is a strictly decreasing, continuous, bounded function such that $\sigma(T) \to 0$ as $T \to \infty$. The function σ is called convergence function.

Main Results

1. Impulsive system with maximum and small parameter. Because of the presence of impulses maximum is not always attained we replace maximum with supremum. So let us consider the system in standard form with supremum and with fixed times of impulse actions

$$\begin{array}{ll} \dot{x}(t) = \varepsilon f(t,x,\widetilde{x}_G), & t \geqslant 0, t \neq \tau_k, \\ \Delta x|_{t=\tau_k} = \varepsilon\left(x(\tau_k+0) - x(\tau_k-0)\right) = \varepsilon I_k(x), & k = 0,1,2,\ldots \\ x(0) = x_0, & \end{array} \quad (3)$$

where $x \in \mathbb{R}^n$ is the phase vector, $f : [0, L\varepsilon^{-1}] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function; $L > 0$; ε is a small parameter; $g(t), \gamma(t)$: $[0, L\varepsilon^{-1}] \to \mathbb{R}$ are known, continuous functions and $0 \leqslant g(t) \leqslant \gamma(t) \leqslant t$;

$$\sup_{s\in[g(t),\gamma(t)]} x(s) = (\sup_{s\in[g(t),\gamma(t)]} x_1(s), \ldots, \sup_{s\in[g(t),\gamma(t)]} x_n(s))^T;$$

τ_k are fixed numbers such that $0=\tau_0<\tau_1<\dots<\tau_k<\dots$ and $\lim\limits_{k\to\infty}\tau_k=\infty$.

By a solution to problem (3) we mean a real valued function x defined on $[0,\infty)$ which is left continuous on $[0,\infty)$ and is differentiable on (τ_k,τ_{k+1}) $(k=0,1,2,\dots)$ satisfying

$$\dot{x}(t)=\varepsilon f(t,x,\widetilde{x}_G),\quad t\in\bigcup_{k=0}^{\infty}(\tau_k,\tau_{k+1}),$$
$$x(0)=x_0.$$

We associate the following averaged autonomous system with the original system (3):

$$\dot{y}=\varepsilon\left(\overline{f}(y,\widetilde{y}_G)+\overline{I}(y)\right),\quad t\geqslant 0y(0)=x_0, \tag{4}$$

where

$$\overline{I}(x)=\lim_{T\to\infty}\frac{1}{T}\sum_{0<\tau_k<T}I_k(x). \tag{5}$$

The following theorem establishes conditions for justification of averaging method

Theorem 1. *Let in domain $Q=[t\geqslant 0,\ x,\widetilde{x}_G\in D\subset\mathbb{R}^n]$ the following conditions hold:*

1) *functions $f(t,x,\widetilde{x}_G)$, $I_k(x),(k=0,1,2,\cdots)$ are continuous and there exist constants M, λ such that*

$$\|f(t,x,\widetilde{x}_G)\|\leqslant M,\quad \|I_k(x)\|\leqslant M;$$
$$\left\|f(t,x,\widetilde{x}_G)-f(t,x^1,\widetilde{x}^1_G)\right\|\leqslant\lambda\left[\left\|x-x^1\right\|+\left\|\widetilde{x}_G-\widetilde{x}^1_G\right\|\right];$$
$$\left\|I_k(x)-I_k(x^1)\right\|\leqslant\lambda\left\|x-x^1\right\|\ \ \forall x^1,\widetilde{x}^1_G\in D;$$

3) *functions $g(t)$ and $\gamma(t)$ are uniformly continuous;*

4) *there exist (2) and (5) uniformly with respect to $x,\widetilde{x}_G$;*

5) *there exists $\theta>0$ such that for $k=0,1,2,\cdots$, the following inequality holds $\theta\leqslant\tau_k-\tau_{k-1}$, where $\tau_0=0$.*

6) *there exists $\rho>0$ such that the solution $y=y(t)$ to the averaged system (4), where $y(0)=x(0)\in D'\subset D$ defined for any $t\geqslant 0$ and belongs together with ρ neighborhood to the domain D.*

Then for any $\eta>0$ and $L>0$ there exists $\varepsilon^*=\varepsilon^*(\eta,L)>0$ such that for any $\varepsilon\in(0,\varepsilon^*]$ and $t\in[0,L\varepsilon^{-1}]$ the following estimation holds:

$$\|x(t)-y(t)\|\leqslant\eta. \tag{6}$$

Proof. Let us notice that the function $\overline{f}(x,\widetilde{x}_G)$ is a bounded function and satisfies Lipschitz condition. Indeed, according to the assumption 4) and definition 1 one can indicate function σ such that the following estimation holds:

$$\left\|\overline{f}(x,\widetilde{x}_G)-\overline{f}(x^1,\widetilde{x}^1_G)\right\|\leqslant\left\|\overline{f}(x,\widetilde{x}_G)-\frac{1}{T}\int\limits_0^T f(t,x,\widetilde{x}_G)dt\right\|$$

$$+\left\| \frac{1}{T}\int\limits_0^T \left[f(t,x,\widetilde{x}_G)-f(t,x^1,\widetilde{x}_G^1)\right]dt \right\| + \left\| \frac{1}{T}\int\limits_0^T f(t,x^1,\widetilde{x}_G^1)dt - \overline{f}(x^1,\widetilde{x}_G^1)\right\|$$

$$\leqslant 2\sigma(T)+\frac{1}{T}\int\limits_0^T \left\|f(t,x,\widetilde{x}_G)-f(t,x^1,\widetilde{x}_G^1)\right\|dt$$

$$\leqslant 2\sigma(T)+\lambda\left(\left\|x-x^1\right\|+\left\|\widetilde{x}_G-\widetilde{x}_G^1\right\|\right) \quad \forall x,x^1,\widetilde{x}_G,\widetilde{x}_G^1\in D.$$

We have

$$\left\|\overline{f}(x,\widetilde{x}_G)-\overline{f}(x^1,\widetilde{x}_G^1)\right\|\leqslant\lambda\left(\left\|x-x^1\right\|+\left\|\widetilde{x}_G-\widetilde{x}_G^1\right\|\right)$$

when $\sigma(T)\to 0$ as $T\to\infty$.

From conditions 1) and 4) we obtain that for systems (3) and (4) there exist unique and extend solutions $x=x(t), y=y(t)$ for $t\geqslant 0$ while $x,y\in D$.

Let us use integrate form for equations (3) and (4)

$$x(t)=x_0+\varepsilon\left(\int\limits_0^t f(s,x(s),\widetilde{x}_G(s))\,ds+\sum_{0<\tau_k<t} I_k(x)\right),$$

$$y(t)=x_0+\varepsilon\left(\int\limits_0^t \left(\overline{f}(y(s),\widetilde{y}_G(s))+\overline{I_k}(y)\right)\,ds\right)$$

for $t\in[0,L\varepsilon^{-1}]$.

Let us estimate the difference

$$\|x(t)-y(t)\|\leqslant$$

$$\leqslant\varepsilon\left\|\int\limits_0^t\left[f(s,x(s),\widetilde{x}_G(s))-\overline{f}(y(s),\widetilde{x}_G(s))\right]ds\right\|+\varepsilon\left\|\sum_{0<\tau_k<t}I_k(x)-\int\limits_0^t\overline{I}(y)\,ds\right\|.$$

We divide the interval $[0,L\varepsilon^{-1}]$ in m equal parts $t_0=0, t_1=\frac{L}{\varepsilon m},\cdots,t_i=\frac{iL}{\varepsilon m},..,t_m=\frac{L}{\varepsilon}$. Then using estimation from [12] for the first term, estimation from [1] for the second term and according to the assumption 4) one can indicate monotonically decreasing function $\sigma(t)\to 0$ as $t\to\infty$ such that for all $x\in D$ we obtain

$$\|x(t)-y(t)\|\leqslant\frac{LM(\lambda L(5+\theta)+3\theta)}{\theta m}+\varepsilon t_i\sigma(t_i)+\varepsilon t\sigma(t).$$

Define $F(\varepsilon)=\sup\limits_{l\in[0,L]}\left[l\sigma(\frac{l}{\varepsilon})\right]$ and notice $\varepsilon t_i\sigma(t_i)\leqslant F(\varepsilon)$, $\varepsilon t\sigma(t)\leqslant F(\varepsilon)$. Then

$$\|x(t)-y(t)\|\leqslant C(m)+2F(\varepsilon),$$

where $C(m)=\frac{LM(\lambda L(5+\theta)+3\theta)}{\theta m}$. Observe, $F(\varepsilon)\to 0$, as $\varepsilon\to 0$. Hence, let us fix m and choose ε^* such that for all $\varepsilon\in(0,\varepsilon^*]$ the estimation (6) is true.

2. Optimal control problem for the impulsive system with maximum. Let us consider the following problem with supremum

$$\begin{array}{ll} \dot{x} = \varepsilon\left[f(t,x,\widetilde{x}_G) + A(x,\widetilde{x}_G)\zeta(t,u)\right], & t \geqslant 0, t \neq \tau_k, \\ \Delta x|_{t=\tau_k} = \varepsilon I_k(x), & k = 0,1,2,\ldots, \\ x(0) = x_0, & \end{array} \tag{7}$$

where $A \in \mathbb{R}^{n\times n}$, $a_{ij} \in \mathbb{R}$, $\zeta : [0, L\varepsilon^{-1}] \times U \to \mathbb{R}^r$ $\mathcal{U}$ is a set of all piecewise continuous functions u from $[0, L\varepsilon^{-1}]$ to $\mathbb{R}^r$, $u(t) \in U \subset comp(\mathbb{R}^r)$.
We are interested in a control function which provides the minimum of functional

$$J[u] = \Phi(x(L\varepsilon^{-1})). \tag{8}$$

Let us consider the corresponding averaged system

$$\begin{array}{l} \dot{y} = \varepsilon\left[\overline{f}(y,\widetilde{y}_G) + A(y,\widetilde{y}_G)v + \overline{I}(y)\right], \\ y(0) = x_0. \end{array} \tag{9}$$

with functional

$$\overline{J}[u] = \Phi(y(L\varepsilon^{-1})), \tag{10}$$

where $v \in V$ is a new control vector and set V is defined as

$$V = \lim_{T\to\infty} \frac{1}{T}\int_0^T \zeta(t,U)dt, \tag{11}$$

in (11) we understand integral of set-valued function as Aumann integral, convergence we understand the sense of Hausdorff metric.

2.1 The algorithm of correspondence of control functions. The control functions in original and averaged systems are different e.g. can belong to the space of different dimensions. That is why it is necessary to establish the correspondence between control functions of (7),(8) and (9),(10).

1. For admissible control $v \in \mathcal{V}$ find the correspondence admissible control $u \in \mathcal{U}$ in the following way:

   (a) calculate points $v_i = \frac{1}{T_0}\int\limits_{iT_0}^{(i+1)T_0} v(t)dt$, $i = 0,1,2,\ldots$, (T_0 is an arbitrary constant).

   (b) assign control $u(t) = \{u_i(t), iT_0 \leqslant t < (i+1)T_0, i = 0,1,2,..\}$, where $u_i = u_i(t)$ can be obtained from the conditions:

$$\min_{u(t)\in U}\left\|\frac{1}{T_0}\int_{iT_0}^{(i+1)T_0}\zeta(t,u(t))dt - v_i\right\| = \left\|\frac{1}{T_0}\int_{iT_0}^{(i+1)T_0}\zeta(t,u_i(t))dt - v_i\right\|. \tag{12}$$

The set-valued mapping $\zeta(t,U)$ is continuous and bounded then by the Lyapunov theorem (see [9]) the set

$$V^i_{T_0} = \left\{ \frac{1}{T_0} \int\limits_{iT_0}^{(i+1)T_0} \zeta(t,u_i(t))dt, \quad u_i(t) \in U \right\}$$

is convex and compact. According to (11) $\lim\limits_{T_0\to\infty} h(V^i_{T_0}, V) = 0$. Hence, there exist $\overline{v}_i \in V^i_{T_0}$ the nearest to the v_i, in other words there exist control function $u_i(t)$ in (12) such that

$$\frac{1}{T_0} \int\limits_{iT_0}^{(i+1)T_0} \zeta(t,u_i(t))dt = \overline{v}_i. \tag{13}$$

2. For an admissible control $u \in \mathcal{U}$ find the corresponding admissible control $v \in \mathcal{V}$ in the following way:

   (a) calculate $w_i(t) = \frac{1}{T_0} \int\limits_{iT_0}^{(i+1)T_0} \zeta(t,u_i(t))dt$, $i = 0,1,2,\ldots$, (T_0-is an arbitrary constant;

   (b) assign control $v(t) = \{v_i(t), iT_0 \leqslant t < (i+1)T_0, i = 0,1,2,..\}$, where v_i can be obtained from the condition:

   $$\operatorname*{argmin}_{v\in V} \|w_i - v\| = \|w_i - v_i\|.$$

   There exists v_i as a minimum of continuous function $\|w_i - v_i\|$ on a compact set V.

Remark 1. *Control functions $u = u(t)$ in 1(b) and $v = v(t)$ in 2(b) determined ambiguously.*

2.2 Justification of the averaging method. The following theorem provides justification of the averaging method for controlled system (7).

Theorem 2. *Suppose that in domain $Q = \{t \geqslant 0, x, \widetilde{x}_G \in D \subset \mathbb{R}^n, u(t) \in U \subset comp(\mathbb{R}^r)\}$ assumption 1),4)-6) of the theorem 1 are satisfied. Moreover,*

1) *matrix A is continuous and there exist M, λ such that the following inequalities hold*

$$\|A(x,\widetilde{x}_G)\| \leqslant M,$$

$$\left\|A(x,\widetilde{x}_G) - A(x^1,\widetilde{x}^1_G)\right\| \leqslant \lambda \left[\left\|x - x^1\right\| + \left\|\widetilde{x}_G - \widetilde{x}^1_G\right\|\right] \ \forall x^1, \widetilde{x}^1_G \in D;$$

2) *function $\zeta(t,u)$ is continuous with respect to t, u*

3) there exists $\rho > 0$ such that for any admissible control function $v \in \mathcal{V}$ the solution $y = y(t)$ to the averaged system (9), where $y(0) = x(0) = x_0 \in D' \subset D$ defined for any $t \geqslant 0$ and belongs together with ρ neighborhood to the domain D.

Then for any $\eta > 0$ and $L > 0$ there exists $\varepsilon^* = \varepsilon^*(\eta, L) > 0$ such that for any $\varepsilon \in (0, \varepsilon^*]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements hold

1) for any admissible control $u \in \mathcal{U}$ of system (7) there exists control function v of system (9), such that:

$$\|x(t) - y(t)\| \leqslant \eta, \tag{14}$$

2) for any admissible control $v \in \mathcal{V}$ of system (9) there exists control function u of system (7), such that (14) holds.

Remark 2. *By assumption 3) function ζ is a continuous so we denote $M := \max\limits_{t,u,\widetilde{u}_G} |\zeta(t, u, \widetilde{u}_G)|$.*

Proof. Let us proof the first statement of the theorem, the second part proved analogously. Using the integral equations for (7) and (9) we have:

$$\|x(t) - y(t)\| \leqslant$$

$$\leqslant \varepsilon \left\| \int_0^t [f(s, x(s), \widetilde{x}_G(s)) + A(x(s), \widetilde{x}_G(s))\zeta(s, u(s)))]\, ds + \sum_{0<\tau_k<t} I_k(x) - \right.$$

$$\left. - \int_0^t \left[\overline{f}(\widetilde{y}_G(s), \widetilde{y}_G(s)) + A(y(s), \widetilde{y}_G(s))v(s) + \overline{I}(y)\right] ds \right\| \leqslant$$

$$\leqslant \varepsilon \left\| \int_0^t \left[f(s, x(s), \widetilde{x}_G(s))) - \overline{f}(y(s), \widetilde{y}_G(s))\right] ds \right\| + \varepsilon \left\| \sum_{0<\tau_k<t} I_k(x) + \int_0^t \overline{I}(y) ds \right\| +$$

$$+ \varepsilon \left\| \int_0^t [A(x(s), \widetilde{x}_G(s))\zeta(s, u(s))) + A(y(s), \widetilde{y}_G(s))v(s)]\, ds \right\| =$$

$$= W_1 + W_2 + W_3.$$

According to [12] for any η_1 there exists $\varepsilon^*(\eta_1) > 0$ such that for any $\varepsilon \leqslant \varepsilon^*(\eta_1)$ the following holds:

$$W_1 \leqslant \frac{M}{m} \left[2\lambda(L + 2m\varepsilon) max\left\{\omega(g, \Delta), \omega(\gamma, \Delta)\right\} + L(\lambda L + 1)\right],$$

$$2m\eta_1 = \nu(m, \varepsilon),$$

where $\omega(g,\Delta) = \sup\limits_{|t''-t'|\leqslant\Delta} |g(t'') - g(t')|$, $\Delta = t_{i+1} - t_i = \frac{L}{\varepsilon m}$, $t'', t' \in [0,\infty)$ and analogously for the function γ.

For W_2 from [1] for sufficiently large $m \in \mathbb{N}$ we get

$$W_2 \leqslant \frac{LM(3\lambda L^2\theta + 3\theta + 1 + 2\lambda L)}{\theta m} + \frac{3\lambda L^2 M}{\theta m} = b(m,\theta).$$

For W_3 the following holds:

$$W_3 \leqslant \varepsilon \left\| \int\limits_0^t [A(x(s),\widetilde{x}_G(s)) - A(y(s),\widetilde{y}_G(s))]\,\zeta(s,u(s)))\,ds \right\| +$$
$$+ \left\| \varepsilon \int\limits_0^t A(y(s),\widetilde{y}_G(s))\,[\psi(s,u(s)) - v(s)]\;ds \right\| \leqslant$$
$$\leqslant \varepsilon\lambda M \int\limits_0^t [\|x(s) - y(s)\| + \|\widetilde{x}_G(s) - \widetilde{y}_G(s)\|]\;ds + W_4,$$

where

$$W_4 = \left\| \varepsilon \int\limits_0^t A(y(s),\widetilde{y}_G(s))\,[\psi(s,u(s)) - v(s)]\;ds \right\|.$$

For estimation W_4 we divide interval $[0, L\varepsilon^{-1}]$ into m equal parts by $t_i = \frac{iL}{\varepsilon m}$, $i = 1,..,m$. For any $t \in [t_p, t_{p+1})$ and for some $p \in \mathbb{N}$, $p < m$ we get

$$W_4 = \varepsilon \left\{ \sum_{i=0}^{p-1} \left\| \int\limits_{t_i}^{t_{i+1}} A(y(s),\widetilde{y}_G(s))\,[\zeta(s,u(s)) - v(s)]\;ds \right\| + \right.$$
$$\left. + \left\| \int\limits_{t_p}^{t} A(y(s),\widetilde{y}_G(s))\,[\zeta(s,u(s)) - v(s))]\,ds \right\| \right\} \leqslant$$

$$\leqslant \varepsilon \left\{ \sum_{i=0}^{p-1} \int\limits_{t_i}^{t_{i+1}} \|A(y(s),\widetilde{y}_G(s)) - A(y(t_i),\widetilde{y}_G(t_i))\,[\zeta(s,u(s)) - v(s))]\|\,ds + \right.$$

$$+ \sum_{i=0}^{p-1} \left\| \int\limits_{t_i}^{t_{i+1}} A(y(t_i),\widetilde{y}_G(s))\,[\zeta(s,u(s)) - v(s))]\;ds \right\| +$$
$$\left. + \int\limits_{t_p}^{t} \|A(y(s),\widetilde{y}_G(s))\,[\zeta(s,u(s)) - v(s))]\|\,ds \right\} \leqslant$$

$$\leqslant \varepsilon \left\{ \sum_{i=0}^{p-1} \int\limits_{t_i}^{t_{i+1}} \lambda \left(\|y(s) - y(t_i)\| + \|\widetilde{y}_G(s) - \widetilde{y}_G(t_i)\| \right) \|\zeta(s, u(s)) - v(s)\| \, ds + \right.$$

$$\left. + \int\limits_{t_p}^{t} \|A(y(s), \widetilde{y}_G(t_i)) \left(\zeta(s, u(s)) - v(s) \right)\| \, ds \right\} \leqslant 3M \frac{L}{m} \left(L\lambda + \max \{ \omega(g, \Delta), \omega(\gamma, \Delta) \} + 1 \right).$$

Then

$$W_3 \leqslant 2\varepsilon\lambda M \int\limits_0^t \delta(s) \, ds + 3M \frac{L}{m} \left(L\lambda + \max \{ \omega(g, \Delta), \omega(\gamma, \Delta) \} + 1 \right).$$

Here $\delta(t) = \max\limits_{s \in [0,t]} \|x(s) - y(s)\|$ is a uniform metrix. Let us collect estimations for W_1, W_2, W_3, W_4

$$\|x(t) - y(t)\| \leqslant$$

$$\leqslant \nu(m, \varepsilon) + b(m, \theta) + 3M \frac{L}{m} \left(L\lambda + \max \{ \omega(g, \Delta), \omega(\gamma, \Delta) \} + 1 \right) +$$

$$+ 2\varepsilon\lambda M \int_0^t \delta(s) \, ds = C(m) + 2\varepsilon\lambda M \int_0^t \delta(s) \, ds,$$

take maximum on $[0, t]$ from both sides and apply Gronwalla-Bellman inequality we obtain

$$\delta(t) \leqslant C(m) e^{2\lambda M L}.$$

Hence, for $\varepsilon \in (0, \varepsilon^*]$, $\quad t \in [0, L\varepsilon^{-1}]$, the trajectories belong to the domain D and by appropriate choise of sufficiently large m and sufficiently small ε we obtain the estimation (14). Thus the first part of theorem is proved.

Remark 3. *As an averaged system one can consider the following one*

$$\dot{y} = \varepsilon \left[\overline{f}(y, \widetilde{y}_G) + A(y, \widetilde{y}_G) \zeta(t, u) + \overline{I}(y) \right].$$

Owing to the average system depends on the same control that initial system we do not need the algorithm of correspondence of control functions.

2.3 Approximation of the functional of the optimal control problem by the impulsive system with maximum.

Theorem 3. *Suppose in domain* $Q = \{t \geqslant 0, x \in D \subset \mathbb{R}^n, u, \widetilde{u} \in U \subset comp(\mathbb{R}^m)\}$ *the assumptions of the theorem 2 hold. Moreover,*

1) there exits λ *such that*

$$\|\Phi(x) - \Phi(x')\| \leqslant \lambda \|x - x'\|;$$

2) there exist $u^*(t) \in U$ optimal control function of problem (7),(8), $x^*(t)$ — corresponding optimal trajectory and J^* — optimal value of functional.

Then for any $L > 0$ there exists $\eta_1 > 0$, and $\varepsilon^*(L) > 0$ such that for any $\varepsilon \in (0, \varepsilon^*]$ the following inequalities hold:

$$\left|\overline{J}[v^*] - J[u^*]\right| \leqslant \eta_1, \tag{15}$$

$$J[u_{v^*}] - J[u^*] \leqslant \eta_1, \tag{16}$$

where $\overline{J}[v^*]$ is the optimal value of functional of the problem (7),(8), u_{v^*} is a control function to the problem (9), (10) constructed by the algorithm and corresponding to the optimal control function v^* of problem (9), (10), v_{u^*} is the optimal control function of problem (9),(10) constructed by u^*.

Remark 4. *Note that Theorem 3 is valid if instead of the problem (7),(8) with unfixed right end we consider problem with flexed one, i.e. with restriction*

$$\psi_j(x(L\varepsilon^{-1})) \leqslant 0, \quad j = \overline{1, m}.$$

Hence, one can formulate numerical-asymptotic algorithm for solving optimal control problem for the impulsive system with supremum:

1) for known controlled problem with small parameter and supremum (7), (8) we define averaged problem (9), (10);

2) for known set of admissible control functions $\mathfrak{U}$ we construct the set of admissible control functions for averaged problem according to the algorithm of correspondence of control functions of original and averaged systems;

3) solve optimal control problem of averaged problem (9) with criterion (10) and find $v^*(t), y^*(t), \overline{J}^*$;

4) according to the algorithm by the found optimal control function $v^*(t)$ of averaged problem we find correspondence control function of original problem u^*_v which is asymptotically optimal for problem (7),(8);

5) for found control function u^*_v we create correspondence trajectory for the system (7) $x(t) = x(t, u^*_v)$;

6) calculate the value of functional (8) on the trajectory from the step (5).

Conclusion. In this paper the justification of the averaging method for the system of functional-differential equations with maximum and impulsive perturbation is presented. Also, the approximation of solutions for the optimal control problem for such kind of system is obtained.

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Кичмаренко О., Сапожникова Е., Дашковский С.
Апроксимація розв'язків задачі оптимального керування для імпульсних систем з максимумом

Резюме

Дана стаття представляє метод усереднення для двох типів задач: імпульсної задачі з максимумом та задачі оптимального керування такого типу систем. Для першої задачі отримане узагальнення теореми Крилова"– Боголюбова. Для другої ми зацікавлені не тільки в отриманні оцінки близкості розв'язків початкової та усередненої задач оптимального керування, а і у оцінці близкости відповідних функціоналів. З цією метою також отримано обгрунтування методу усередненя. В цьому випадку схема усередненя включає алгоритм відповідності функцій керування початкової та усередненої задач оптимального керування. Отримано алгоритм чисельно-асимптотичного розв'язку задачі оптимального керування системами такого типу з малим параметром.
Ключові слова: системи із запізненням, імпульсні системи, задача оптимального керування, метод усреднення.

Кичмаренко О., Сапожникова К., Дашковский С.
Аппроксимация решений задачи оптимального управления для импульсніх систем с максимумом

Резюме

Данная статья представляет метод усреднения для двух задач: импульсной задачи с максимумом и задачи оптимального управления такими типами систем. Для первой задачи получено обобщения теоремы Крылова—Боголюбова. Для второй мы заинтересованы не только в получинии оценки близости решений исходной и усредненой задач оптимального управления с максимумом и импульсным воздействием, а также в получении оценки для соответвующих функционалов. Для этой цели также получено обоснование метода усреднения. В этом случае схема усреднения включает алгоритм соответсвия функций управления исходной и усредненной задач оптимального управления. Получен алгоритм численно-асимптотического решения задачи оптимального управления системами такого типа с малым параметром.

Ключевые слова: системы с запаздыванием, импульсные системы, задача оптимального управления, метод усреднения.