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AN APPROXIMATE NONLINEAR DYNAMIC PROBLEM SOLUTION OF FUNCTIONALLY GRADED MATERIAL SHALLOW SHELL STRUCTURE WITH IN TIME THICKNESS VARIATION

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This paper deals with an approximate analytical solution of nonlinear problem for dynamics of Functionally Graded Material (FGM) shallow shell on the basis of hybrid perturbation-WKB asymptotic approach. The motion of the structures is derived using classical shell theory. Special attention is paid to investigate the influence of material properties, which are graded in the thickness direction according to the power-law distribution in terms of volume fractures of the material, on dynamic behavior structures with given initial conditions. The non-linear strain-displacement relationships based upon the von Karman theory for moderately large normal deflections. By taking inertia forces into account and after some assumptions with respect to boundary conditions, discussed problem leads to a singular non-linear second order differential equation with variable in time coefficients. According to perturbation method with respect to parameter of nonlinearity, solution of initial differential equation is obtained in the form of two terms approximation. After the perturbation procedure the system of ordinary singular differential equations with variable in time coefficient is solved by two terms WKB – approximation as well. Final result is an approximate hybrid analytical solution of nonlinear problem on the basis of perturbation - WKB method. Solution of non-linear problem is presented in compact form, where first term corresponds to free vibration of system, second term – forced oscillation and third term – solution of nonlinear part of the problem. Analytical solution for some thickness functions in time of structure is compared with direct numerical integration of initial equation of the problem.

Key words: approximate analytical solution, nonlinear problem, dynamics, Functionally Graded Material, shallow shell, hybrid perturbation-WKB method.

ПРИБЛИЖЕННОЕ РЕШЕНИЕ ПРОБЛЕМЫ НЕЛИНЕЙНОЙ ДИНАМИКИ ПОЛОГОЙ ОБОЛОЧКИ ПЕРЕМЕННОЙ ВО ВРЕМЕНИ ТОЛЩИНЫ ИЗ ФУНКЦИОНАЛЬНО-ГРАДИЕНТНОГО МАТЕРИАЛА

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В статье рассматривается приближенное аналитическое решение нелинейной задачи динамики функционально-градиентных материалов (ФГМ) пологой оболочки на основе метода возмущений и ВКБ асимптотического подхода. Движение конструкций описывается с помощью классической теории оболочки. Особое внимание уделяется исследованию влияния свойств материалов, которые оцениваются по направлению толщины в соответствии с распределением по степенному закону с точки зрения объема материала, на динамические структуры поведения с заданными начальными условиями. Нелинейные зависимости деформации - смещения, основаны на теории Кармана для умеренно больших нормальных отклонений. Принимая силы инерции во внимание и некоторые граничные условия, обсуждаемая проблема приводит к особому нелинейному

дифференциальному уравнению второго порядка с переменными во времени коэффициентами. Проведено сравнение полученного аналитического решения для некоторых параметров конструкции с прямым численным интегрированием исходного уравнения задачи.
Ключевые слова: приближенное аналитическое решение, нелинейная задача, динамика, функционально-градиентные материалы, пологая оболочка, гибридный метод возмущений –ВКБ.

НАБЛИЖЕНИЙ РОЗВ'ЯЗОК ПРОБЛЕМИ НЕЛІНІЙНОЇ ДИНАМІКИ ПОЛОГОЇ ОБОЛОНКИ ЗМІННОЇ ЗА ЧАСОМ ТОВЩИНИ ІЗ ФУКЦІОНАЛЬНО-ГРАДІЄНТНОГО МАТЕРІАЛУ

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У статті розглядається наближений аналітичний розв'язок нелінійної задачі динаміки пологої оболонки із функціонально-градієнтного матеріалу (ФГМ) на основі методу збурень і ВКБ асимптотичного підходу. Рух конструкції описується на основі класичної теорії оболонок. Особлива увага приділяється дослідженню впливу властивостей матеріалу, який оцінюється за напрямом товщини відповідно до розподілу за степеневим законом з точки зору об'єму матеріалу, на динамічну поведінку при заданих початкових умовах. Нелінійні залежності деформації – переміщення базуються на теорії Кармана для досить невеликих нормальних відхилень. Беручи до уваги сили інерції і певні початкові умови, проблема зводиться до особливого нелінійного диференціального рівняння другого порядку зі змінними за часом коефіцієнтами. Проведено порівняння здобутого аналітичного розв'язку для деяких параметрів конструкції з прямим чисельним інтегруванням початкового рівняння задачі.
Ключові слова: наближений аналітичний розв'язок, нелінійна задача, динаміка, функціонально-градієнтні матеріали, полога оболонка, гібридний метод збурень – ВКБ.

1. INTRODUCTION

FGM thin-walled structures with metal inner surface and ceramic in outer surface widely used, for example, in modern air-space systems. In recent years important studies have been researched about vibration and stability FGM plates and shells with using numerical approaches. In [1] is given review of some recent studies in this sphere. Hear we refer just current papers which are deals with the subject of discussion. Nonlinear axisymmetric response of thin FGM shallow spherical shells with ceramic-metal-ceramic layers under uniform external pressure and temperature is discussed in [2]. Static and dynamic thermo mechanical buckling load of functionally graded plate (pulse of finite duration) is a subject of discussion in [3]. Nonlinear free vibration response of FGM cylindrical shell in thermal Environment on the basis of nonlinear finite element method is given in [4]. The present research is concerned with approximate analytical solution of dynamic problem of FGM shallow shells with time dependent thickness on the basis of hybrid asymptotic method, which was successfully applied to some mechanical problems [5-11].

2. GOVERNING EQUATION

Non-linear analytic analysis is given on the basic system of equations in terms of the stress and deflection following to the paper [1]. Suppose the FGM shallow shell is simply supported at its edges and subjected to a transverse load $q_0(t)$ and compressive edge loads $r_0(t)$, $p_0(t)$. We assume that modulus of elasticity and the mass density change in the thickness direction, while the Poisson ratio is assumed to be constant and thickness of shell is function of time.

Appling Bubnov–Galerkin procedure with assumption that initial imperfection of middle surface of shell has the form

$$w_0(x_1, x_2) = f_0 \sin\frac{m\pi x_1}{a}\sin\frac{m\pi x_2}{b}, \tag{1}$$

where f_0 is given amplitude, the non-linear second – order ordinary differential equation for function $f(t)$ with deflection function $w=(x_1, x_2, t)$

$$w(x_1, x_2, t) = f(t)\sin\frac{m\pi x_1}{a}\sin\frac{m\pi x_2}{b}, \tag{2}$$

that are correspond to simply support boundary conditions, is given in the form [1]:

$$\varepsilon^2\frac{d^2 f}{dt^2} + f\left(1 + 2f_0\bar{A}_2(t) - \bar{A}_3(t)f_0^2 - \bar{A}_1(t)\right) + f^2\left(-3\bar{A}_2(t)\right) + f^3\bar{A}_3(t) = \\ = Q_0 - \bar{A}_0(t) + f_0 - \bar{A}_2(t)f_0^2, \tag{3}$$

where

$$\varepsilon^2 = \frac{1}{\omega_{mn}^2}, \quad A_0(t) = \frac{16h(t)}{\pi^2 mn}(k_1 r_0 + k_2 p_0), \quad A_1(t) = \frac{\pi^2 h(t)}{a^2}\left(m^2 r_0 + n^2\lambda^2 p_0\right),$$

$$A_2(t) = \frac{16E_1(t)mn\lambda^2\left(k_1 n^2\lambda^2 + k_2 m^2\right)}{3a^2\left(m^2 + n^2\lambda^2\right)^2}, \quad A_3(t) = \frac{512E_1(t)m^2n^2\lambda^4}{9a^4\left(m^2 + n^2\lambda^2\right)^2}, \quad \bar{A}_i = \frac{A_i}{\omega_{mn}^2}, \tag{4}$$

$$\omega_{mn}^2 = \frac{1}{\rho_1(t)}\left[\frac{\left(E_1E_3 - E_2^2\right)}{E_1\left(1 - \nu^2\right)}\cdot\frac{\left(m^2 + n^2\lambda^2\right)\pi^2}{a^4} + \frac{E_1\left(k_1 n^2\lambda^2 + k_2 m^2\right)^2}{\left(m^2 + n^2\lambda^2\right)^2}\right],$$

$$E_1(t) = \left(E_m + \frac{E_c - E_m}{k+1}\right)h(t), \quad \rho_1 = \left(\rho_m + \frac{\rho_c - \rho_m}{k+1}\right)h(t),$$

k_1, k_2 – curvatures of middle surface shell in x_1 and x_2.

Initial differential equation (3) we rewrite in the form

$$\varepsilon^2\frac{d^2 f}{dt^2} + B_1(t)f + \mu\left(B_2(t)f^2 + B_3(t)f^3\right) = \bar{Q}_0, \tag{5}$$

where

$$B_1(t) = 1 + 2f_0\bar{A}_2(t) - \bar{A}_3(t)f_0^2 - \bar{A}_1(t), \quad B_2(t) = \frac{-3}{\mu}\bar{A}_2(t), \quad B_3(t) = \frac{1}{\mu}\bar{A}_3(t), \tag{6}$$

$$\bar{Q}_0 = Q_0 - \bar{A}_0(t) + f_0 - \bar{A}_2(t)f_0^2,$$

ε, μ are parameters.

According to perturbation method with respect to parameter of nonlinearity, solution of differential equation [5] we obtain in the form of two terms approximation [14]:

$$f(t) = \varphi_0(t) + \mu\varphi_1(t). \tag{7}$$

Substituting (7) into equation (5) and acquainted the terms with the same order of nonlinear parameter we obtain the system equations for unknown functions $\varphi_0(t)$ and $\varphi_1(t)$:

$$\mu^0: \ \varepsilon^2\varphi_0''(t) + B_1(t)\varphi_0 = \bar{Q}_0; \tag{8}$$

$$\mu^1: \ \varepsilon^2\varphi_1''(t) + B_1(t)\varphi_1 = -B_2(t)\varphi_0^2 - B_3(t)\varphi_0^3. \tag{9}$$

Ordinary singular differential equation with variable in time coefficient B_1 is solved by two terms WKB – approximation [5]:

$$\varphi_0^0(t) = 1/B_1(t)^{0,25}\left(c_1\sin K(t) + c_2\cos K(t)\right), \tag{10}$$

where

$$K(t) = \int \varepsilon^{-1} B_1^{\frac{1}{2}}(t)\,dt\,. \tag{11}$$

Particular solution of equation (8) can be represent in the form

$$\varphi_0^p(t) = \bar{c}_1(t)\sin K(t) + \bar{c}_2(t)\cos K(t), \tag{12}$$

where

$$\bar{c}_1(t) = \varepsilon \int \frac{\bar{Q}_0(t)\cos K(t)}{B_1^{\frac{1}{2}}(t)}dt, \quad \bar{c}_2(t) = -\varepsilon \int \frac{\bar{Q}_0(t)\sin K(t)}{B_1^{\frac{1}{2}}(t)}dt\,. \tag{13}$$

The solution of equation (8) in the first approximation is:

$$\varphi_0(t) = \varphi_0^0(t) + \varphi_0^p(t) = \sin K(t)\left(c_1 + \vec{c}_1(t)\right) + \cos K(t)\left(c_2 + \vec{c}_2(t)\right). \tag{14}$$

Second term in (9) for ordinary equation is obtained from

$$\varepsilon^2 \varphi_1''(t) + B_1(t)\varphi_1 = 0 \tag{15}$$

in the form:

$$\varphi_1^0(t) = d_1 \sin K(t) + d_2 \cos K(t)\,. \tag{16}$$

Corresponding particular solution of equation (9) is

$$\varphi_1^p(t) = \bar{d}_1(t)\sin K(t) + \bar{d}_2(t)\cos K(t), \tag{17}$$

where

$$\bar{d}_1(t) = \varepsilon \int \frac{\left(-B_2(t)\varphi_0^2 - B_3(t)\varphi_0^3\right)\bar{Q}_0(t)\cos K(t)}{B_1^{\frac{1}{2}}(t)}dt, \tag{18}$$

$$\bar{d}_2(t) = -\varepsilon \int \frac{\left(-B_2(t)\varphi_0^2 - B_3(t)\varphi_0^3\right)\bar{Q}_0(t)\sin K(t)}{B_1^{\frac{1}{2}}(t)}dt\,.$$

Finally we obtained the solution of nonlinear problem on the basis of perturbation – WKB method:

$$\begin{gathered} f = \varphi_0(t) + \mu\varphi_1(t) = \sin K(t)\left(c_1 + \vec{c}_1(t)\right) + \cos K(t)\left(c_2 + \vec{c}_2(t)\right) + \\ + \mu\Big(\sin K(t)\left(d_1 + \bar{d}_1(t)\right) + \cos K(t)\left(d_2 + \bar{d}_2(t)\right)\Big). \end{gathered} \tag{19}$$

Some numerical calculations for the shell with variable in time thickness supposed that thickness of shell without imperfection is, for example, linear function in time:

$$h(t) = h_0(1 - \eta t)\,. \tag{21}$$

Solution of non-linear problem (15) we can represent in form:

$$\begin{gathered} f(t) = \varphi_0(t) + \mu\varphi_1(t) = \sin K(t)\left[1/B_1(t)^{0,25} s_1 + \left(\bar{c}_1(t) + \mu\bar{d}_1(t)\right)\right] + \\ + \cos K(t)\left[1/B_1(t)^{0,25} s_2 + \left(\bar{c}_2(t) + \mu\bar{d}_2(t)\right)\right], \end{gathered} \tag{22}$$

where first term in square brackets corresponds to free vibration of system, seconds term – forced oscillation and third term – to solution of non-linear problems.

For this thickness function we obtain for coefficients of initial equation (5) of the following values:

$$\bar{a}_2 = 1, \quad \bar{a}_3 = 2, \quad B_1(t) = (1-0,1t)^3, \quad \bar{Q}_0 = -1 + (1-0,1t)^3,$$

$$h(t) = h_0(1-0,1t), \quad K(t) = -40(1-0,1t)^{\frac{5}{2}}. \tag{23}$$

An approximate analytical solution have obtained for the initial conditions

$$f(0) = 1, \tag{24}$$

$$f'(0) = 0.$$

Some results of numerical calculations are given in Fig. 1-4.

a) Two-terms analytical linear solution

$$f(t) = \sin K(t)\left[-0,745\Big/(1-0,1t)^{\frac{3}{4}}\right] + \cos K(t)\left[-0,667\Big/(1-0,1t)^{\frac{3}{4}}\right] \tag{25}$$

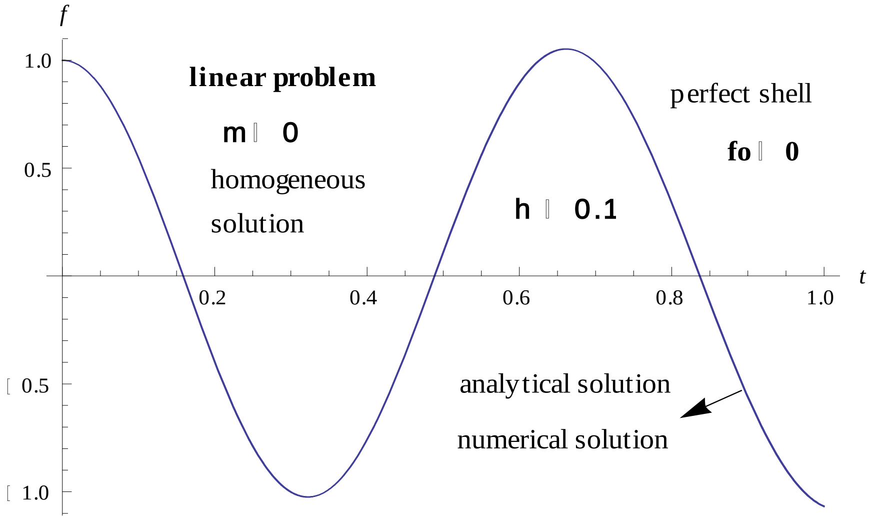

Fig. 1. Comparison of analytical and numerical linear solutions

b) Two-terms analytical nonlinear solution

$$f(t) = \sin K(t)\left[-0,265\Big/(1-0,1t)^{\frac{3}{4}} - 0,438\right] + \cos K(t)\left[-0,971\Big/(1-0,1t)^{\frac{3}{4}} + 0,256\right] \tag{26}$$

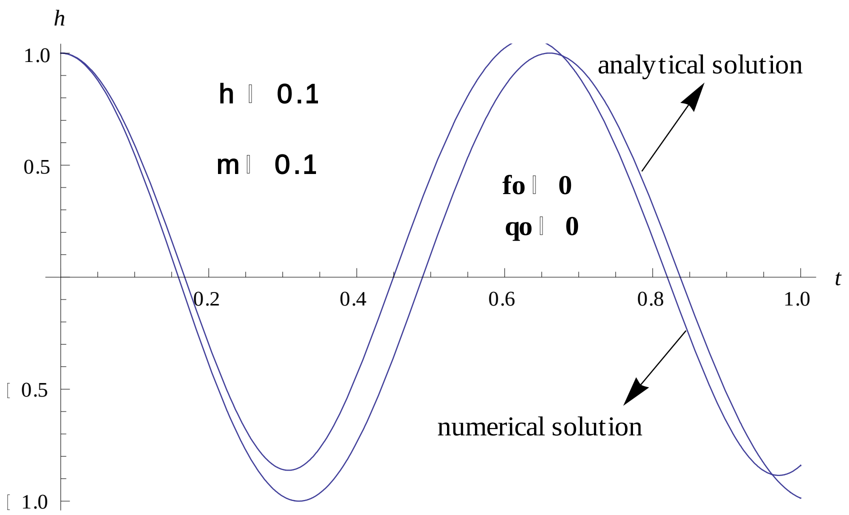

Fig. 2. Comparison of analytical and numerical nonlinear solutions

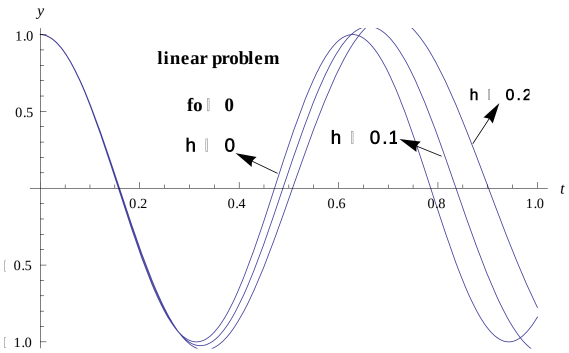

Fig. 3. Influence of thickness parameter values

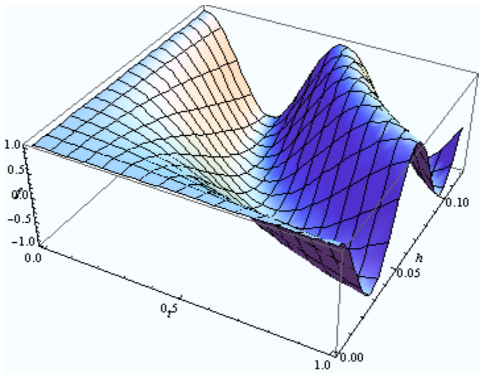
Fig. 4. Forced oscillation in time of perfect structure

CONCLUSION

An approximate analytical solution for forced oscillations of non-linear FGM shallow cylindrical shells with time dependent thickness on the basis of hybrid perturbation-two-terms WKB approximation method are obtained. For some particular parameters of structure analytical solutions are in good enough correlations with direct numerical solutions of initial singular nonlinear differential equations with variable coefficients. Father investigations will be devoted to influence of imperfections of middle surface of shells, different thickness function in time and functions of external forces as well on geometrically nonlinear structures behavior.

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