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# FINITE APPROXIMATION OF UNIT-HYPERCUBIC INFINITE NONCOOPERATIVE GAME VIA DIMENSION-DEPENDENT IRREGULAR SAMPLINGS AND RESHAPING THE PLAYER'S PAYOFFS INTO LINE ARRAY FOR SIMPLIFICATION AND SPEEDUP

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A method of converting the infinite noncooperative game on the unit hypercube into finite game is suggested. Purpose of the conversion is to obtain an approximate solution of the initial infinite game as the exact solution of the same type in the finite game. There are three steps to get the approximated solution. Firstly the players' payoff functions are sampled on the unit hypercube, what maps the initial infinite game into finite one. The samplings are dimension-dependent and irregular. Irregularity allows to sample quickly oscillating dimensions of payoff values tighter, whereas slowly oscillating dimensions are sampled sparser. The sampled players' payoff functions are multidimensional matrices, given at the second step of approximation. After having reshaped these matrices into matrices, whose number of dimensions is equal to the number of players, it is expected a favorable impact on simplification and speedup in finding the finite game exact solution. At the final third step, this solution is ascertained whether it is consistent, meaning its relative independence upon the sampling points over the unit hypercube. Consistency is formulated with its requirements, which rank the approximation accurateness. There are distinguished consistency and weak consistency. The consistency conception is generalized to cases, when the finite game solution is relatively independent upon the sampling points within the defined neighborhood of this solution. Thus, (weakly) $\lambda$-consistent solutions are defined for $\lambda \in \mathbb{N}$. For quickening consistency analysis, there is a theorem saying how weakly $\lambda$-consistent solution of finite noncooperative game appears $\lambda$-consistent one.

*Key words: infinite noncooperative game, unit hypercube, finite game approximation, finite noncooperative game, multidimensional matrix reshaping, finite approximation accurateness rank, approximate solution consistency.*

# КОНЕЧНАЯ АППРОКСИМАЦИЯ БЕСКОНЕЧНОЙ БЕСКОАЛИЦИОННОЙ ИГРЫ НА ЕДИНИЧНОМ ГИПЕРКУБЕ ПОСРЕДСТВОМ ЗАВИСИМЫХ ОТ ИЗМЕРЕНИЯ НЕРАВНОМЕРНЫХ ВЫБОРОК И ПРЕОБРАЗОВАНИЯ ВЫИГРЫШЕЙ ИГРОКА В ЛИНЕЙНЫЙ МАССИВ ДЛЯ УПРОЩЕНИЯ И УСКОРЕНИЯ

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Предлагается метод преобразования бесконечной бескоалиционной игры на единичном гиперкубе в конечную игру. Целью этого преобразования является получение некоего приближённого решения исходной бесконечной игры в качестве точного решения такого же типа в конечной игре. Аппроксимированное решение получается в три этапа. Сперва функции выигрыша игроков дискретизируются на единичном гиперкубе, что отображает исходную бесконечную игру в конечную. Выборки дискретизации являются зависимыми от измерения и неравномерны. Неравномерность позволяет дискретизировать быстро осциллирующие измерения значений выигрыша плотнее, тогда как медленно осциллирующие измерения дискретизируются более разрежёнными. Дискретизированные функции выигрыша игроков являются многомерными матрицами, получаемыми на втором этапе аппроксимации. После того, как эти матрицы преобразованы в матрицы, количество измерений которых равно числу игроков, ожидается упрощение и ускорение при нахождении точного решения конечной игры. На финальном, третьем, этапе устанавливается, является ли это решение согласованным, подразумевая его относительную независимость от точек выборок на единичном гиперкубе. Согласованность формулируется со своими требованиями, которые устанавливают ранг точности аппроксимации. Различают согласованность и слабую согласованность. Концепция согласованности обобщается на случаи, когда решение конечной игры является относительно независимым от точек выборок в определяемой окрестности этого решения. Таким образом для $\lambda \in \mathbb{N}$ определяются (слабые) $\lambda$-согласованные решения. Для ускорения анализа согласованности предлагается теорема о том, как из слабо $\lambda$-согласованного решения конечной бескоалиционной игры получается $\lambda$-согласованное.

*Ключевые слова: бесконечная бескоалиционная игра, единичный гиперкуб, конечная аппроксимация игры, конечная бескоалиционная игра, преобразование многомерной матрицы, ранг точности конечной аппроксимации, согласованность приближённого решения.*

# СКІНЧЕННА АПРОКСИМАЦІЯ НЕСКІНЧЕННОЇ БЕЗКОАЛІЦІЙНОЇ ГРИ НА ОДИНИЧНОМУ ГІПЕРКУБІ ЗА ДОПОМОГОЮ ЗАЛЕЖНИХ ВІД ВИМІРУ НЕРІВНОМІРНИХ ВИБІРОК І ПЕРЕТВОРЕННЯ ВИГРАШІВ ГРАВЦЯ У ЛІНІЙНИЙ МАСИВ ДЛЯ СПРОЩЕННЯ ТА ПРИСКОРЕННЯ

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Пропонується метод перетворення нескінченної безкоаліційної гри на одиничному гіперкубі в скінченну гру. Метою цього перетворення є отримання деякого наближеного розв’язку вихідної нескінченної гри у якості точного розв’язку такого ж типу в скінченній грі. Апроксимований розв’язок виходить у три етапи. Спершу функції виграшу гравців дискретизують на одиничному гіперкубі, що відображує вихідну нескінченну гру в скінченну. Вибірки дискретизації є залежними від виміру і нерівномірні. Нерівномірність дозволяє дискретизувати швидко осцилюючі виміри значень виграшу щільніше, тоді як повільно осцилюючі виміри дискретизують більш розрідженими. Дискретизовані функції виграшу гравців є багатовимірними матрицями, що отримуються на другому етапі апроксимації. Після того, як ці матриці перетворені в матриці, кількість вимірів яких дорівнює числу гравців, очікується спрощення і прискорення при знаходженні точного розв'язку скінченної гри. На фінальному, третьому, етапі встановлюється, чи є цей розв'язок узгодженим, маючи на увазі його відносну незалежність від точок вибірок на одиничному гіперкубі. Узгодженість формулюється зі своїми вимогами, які встановлюють ранг точності апроксимації. Розрізняють узгодженість і слабку узгодженість. Концепція узгодженості узагальнюється на випадки, коли розв'язок скінченної гри є відносно незалежним від точок вибірок у визначуваному околі цього розв'язку. Таким чином для $\lambda \in \mathbb{N}$ визначаються (слабкі) $\lambda$-узгоджені розв'язки. Для прискорення аналізу узгодженості пропонується теорема про те, як зі слабо $\lambda$-узгодженого розв'язку скінченної безкоаліційної гри виходить $\lambda$-узгоджений.

*Ключові слова: нескінченна безкоаліційна гра, одиничний гіперкуб, скінченна апроксимація гри, скінченна безкоаліційна гра, перетворення багатовимірної матриці, ранг точності скінченної апроксимації, узгодженість наближеного розв'язку.*

## NONCOOPERATIVE-GAME MODELING

In order to meet the growing demands and requirements against insufficient resources, noncooperative-game modeling is a means for allocating resources rationally and fairly. Insufficiency of resources is expressed as with inequalities between resources and pretensions to them, as well as with a group of persons or players participating in the allocation. Particularly, uncertainties or permanently altering circumstances provoke the insufficiency [1, 2]. Eventually, noncooperative-game modeling proposes equilibrium or Pareto-efficient strategies for participants. Using these strategies theoretically ensures rationality, fairness, equilibrium, and relative utility in removing discrepancies and uncertainties [1, 3, 4]. The question is only to find equilibrium or Pareto-efficient strategies as fast as possible, although it is exceptionally hard for infinite games [2, 5, 6].

## SOLVING INFINITE NONCOOPERATIVE GAMES

In particular, finding Nash equilibrium or Pareto-efficient solutions in even the finite noncooperative game (FNCG) is a computational difficulty [5, 7, 8]. Being more particular, solving dyadic games with three players takes some technique of visualization of the cube of situations in mixed strategies [5, 9], but dyadic games with more than three players are solved purely in analytics [5, 10]. Naturally that solving FNCG with more than two pure strategies at their players, whose number is three or greater, constitutes known computational problems [2, 5, 8, 11]. Nevertheless, the efficiency of solving infinite noncooperative games (INCG) is far behind the efficiency of solving FNCG. Solving INCG needs voluminous analytical substantiations, and it is never known certainly whether a solution exists or not [12, 13]. Even compact INCG, having solutions at least in mixed strategies for measurable payoff functions [1, 2, 5, 9], cannot be solved by a universal algorithmic approach. That is why INCG are substituted with their finite approximations [8, 14, 15]. In fact, finite approximation for INCG is either a mapping that converts INCG into FNCG [5, 9, 16] or an approach for finding an approximate solution of INCG whose strategies have finite supports [14, 17, 18]. However, finding an approximate finite support solution of INCG may need substantiations of sampling the players' action spaces. In a manner, this leads to the conversion of INCG into FNCG.

## THE ARTICLE GOAL AND ITS TASKS

The final goal of this article is to convert INCG into FNCG for obtaining an approximate solution of the initial INCG as the exact solution of the same type in FNCG. There is going to be considered unit-hypercubic INCG (UHINCG), which is isomorphic to compact INCG in Euclidean finite-dimensional subspaces of appropriate dimensions [5, 6, 12, 19, 20]. The conversion of UHINCG into FNCG must maintain the exact solution of FNCG being as close as needed to the exact solution of UHINCG. It should be mentioned that the exact solution of UHINCG does not always exist or just can be unknown [5, 9, 13], though. The closeness nonetheless is going to be expressed.

There are four tasks to meet the described goal:

1. To sample the players' payoff functions (PPF) on the unit hypercube, what maps the initial UHINCG into FNCG. The sampling mustn't be dependent upon all dimensions, and the sampling points are not necessarily to be equidistant along the dimension.

2. To reshape the sampled PPF as multidimensional matrices into matrices with minimally possible number of dimensions, letting get rid off dimensionalities and having the single dimension at each player. This will simplify FNCG and provide speedup in computing the exact solution of FNCG.

3. To state requirements to the exact solution of FNCG that every strategy from the solution wouldn't be too dependent upon the sampling points. This is believed to ensure the spoken closeness of UHINCG to its finite approximation in the form of FNCG, what provides also hypothetical similarity between the exact solution of FNCG and the same type solution of UHINCG.

4. To generalize the stated requirements to when the tolerable dependence of the solution upon the sampling points is possible to regard wider.

## DIMENSION-DEPENDENT IRREGULAR SAMPLINGS OF PPF

Let there be $N \in \mathbb{N} \setminus \{1\}$ players in INCG, in which the $n$-th player acts within the unit hypercube $U_n$ of $M_n$-dimensional pure strategies

$$\mathbf{X}_n = [x_{nm}]_{1 \times M_n} \in \mathop{\times}_{m=1}^{M_n} [0;\, 1] = U_n \subset \mathbb{R}^{M_n} \quad \text{by} \quad M_n \in \mathbb{N} \quad \forall\, n = \overline{1,\, N} \tag{1}$$

and in the situation

$$\mathbf{X} = \{\mathbf{X}_i\}_{i=1}^{N} \in \mathop{\times}_{i=1}^{N} U_i = \mathop{\times}_{d=1}^{\sum_{i=1}^{N} M_i} [0;\, 1] \subset \mathbb{R}^{\sum_{i=1}^{N} M_i} \tag{2}$$

gets the payoff $K_n(\mathbf{X})$. Therefore

$$\left\langle \{U_n\}_{n=1}^{N},\, \{K_n(\mathbf{X})\}_{n=1}^{N} \right\rangle \tag{3}$$

is UHINCG, where each of PPF $\{K_n(\mathbf{X})\}_{n=1}^{N}$ defined on the unit $\left(\sum_{i=1}^{N} M_i\right)$-dimensional hypercube

$$U = \mathop{\times}_{n=1}^{N} U_n = \mathop{\times}_{d=1}^{\sum_{n=1}^{N} M_n} [0;\, 1] \subset \mathbb{R}^{\sum_{n=1}^{N} M_n} \tag{4}$$

is measurable and all PPF are differentiable with respect to any of variables $\left\{\{x_{nm}\}_{m=1}^{M_n}\right\}_{n=1}^{N}$. Also there exist mixed derivatives of each of PPF $\{K_n(\mathbf{X})\}_{n=1}^{N}$ by any combination of variables $\left\{\{x_{nm}\}_{m=1}^{M_n}\right\}_{n=1}^{N}$ in any situation (2), where every variable is included no more than just once.

PPF $\{K_n(\mathbf{X})\}_{n=1}^{N}$ are sampled along each of dimensions of hypercube (4) with a specific sampling rule. For making the irregular sampling independent upon other dimensions, there are going to be applied dimension-dependent irregular samplings (DDIS). Thus let $S_m^{\langle n \rangle}$ be the number of intervals between the selected points in $m$-th dimension of hypercube $U_n$ in (1), where $S_m^{\langle n\rangle} \in \mathbb{N}$ for the utmost case of sampling. So, there is no fixed sampling step, although the endpoints of any unit segment are included into the sampling necessarily. Thus, in $m$-th dimension $n$-th player instead of the segment $[0; 1]$ of values of $m$-th component of its pure strategy (1) now possesses the set of points

$$D_m^{\langle n\rangle}\left(S_m^{\langle n\rangle}\right) = \left\{x_{nm}^{\langle s_m\rangle}\right\}_{s_m=1}^{S_m^{\langle n\rangle}+1} \text{ by } x_{nm}^{\langle 1\rangle} = 0, \; x_{nm}^{\left\langle S_m^{\langle n\rangle}+1\right\rangle} = 1,$$

$$x_{nm}^{\langle d_{nm}\rangle} < x_{nm}^{\langle d_{nm}+1\rangle} \;\; \forall\, d_{nm} = \overline{1, S_m^{\langle n\rangle}} \text{ for } m = \overline{1, M_n} \text{ and } n = \overline{1, N}. \tag{5}$$

Subsequently, the finite hypercubic irregular lattice (FHIL)

$$D^{\langle n\rangle}\left(\left\{S_m^{\langle n\rangle}\right\}_{m=1}^{M_n}\right) = \bigtimes_{m=1}^{M_n} D_m^{\langle n\rangle}\left(S_m^{\langle n\rangle}\right) = \bigtimes_{m=1}^{M_n} \left\{\left\{x_{nm}^{\langle s_m\rangle}\right\}_{s_m=1}^{S_m^{\langle n\rangle}+1}\right\} \tag{6}$$

substitutes the hypercube $U_n$, mapping the initial UHINCG (3) into FNCG

$$\left\langle \left\{D^{\langle n\rangle}\left(\left\{S_m^{\langle n\rangle}\right\}_{m=1}^{M_n}\right)\right\}_{n=1}^{N}, \left\{K_n(\mathbf{X})\right\}_{n=1}^{N} \right\rangle \text{ by } \mathbf{X}_i \in D^{\langle i\rangle}\left(\left\{S_m^{\langle i\rangle}\right\}_{m=1}^{M_i}\right) \tag{7}$$

on FHIL

$$D\left(\left\{\left\{S_m^{\langle n\rangle}\right\}_{m=1}^{M_n}\right\}_{n=1}^{N}\right) = \bigtimes_{n=1}^{N} D^{\langle n\rangle}\left(\left\{S_m^{\langle n\rangle}\right\}_{m=1}^{M_n}\right), \tag{8}$$

where hypersurface $K_n(\mathbf{X})$ is mapped into $\left(\sum_{i=1}^{N} M_i\right)$-dimensional array (matrix) with elements

$$K_n(\mathbf{X}) \;\; \forall\, \mathbf{X} \in D\left(\left\{\left\{S_m^{\langle i\rangle}\right\}_{m=1}^{M_i}\right\}_{i=1}^{N}\right) \text{ by } n = \overline{1, N}. \tag{9}$$

The sampling numbers

$$\left\{\left\{S_m^{\langle n\rangle}\right\}_{m=1}^{M_n}\right\}_{n=1}^{N} \tag{10}$$

shall clearly not be assigned arbitrarily, because DDIS mustn't remove information about local extremums and gradient over hypersurfaces $\left\{K_n(\mathbf{X})\right\}_{n=1}^{N}$. That is $\forall\, s_m = \overline{1, S_m^{\langle n\rangle}}$ there ought to be

$$\frac{\partial^{\sum_{i=1}^{N} M_i} K_n(\mathbf{X})}{\partial x_{11}\partial x_{12}\ldots\partial x_{1M_1}\partial x_{21}\partial x_{22}\ldots\partial x_{2M_2}\ldots\partial x_{N1}\partial x_{N2}\ldots\partial x_{NM_N}} \geqslant 0 \text{ or}$$

$$\frac{\partial^{\sum_{i=1}^{N} M_i} K_n(\mathbf{X})}{\partial x_{11}\partial x_{12}\ldots\partial x_{1M_1}\partial x_{21}\partial x_{22}\ldots\partial x_{2M_2}\ldots\partial x_{N1}\partial x_{N2}\ldots\partial x_{NM_N}} \leqslant 0$$

$$\forall\, x_{nm} \in \left[x_{nm}^{\langle s_m\rangle}; x_{nm}^{\langle s_m+1\rangle}\right], \;\; m = \overline{1, M_n}, \;\; n = \overline{1, N}, \tag{11}$$

and

$$\left| \frac{\partial^{\sum_{i=1}^{N} M_i} K_n(\mathbf{X})}{\partial x_{11}\partial x_{12}\dots\partial x_{1M_1}\partial x_{21}\partial x_{22}\dots\partial x_{2M_2}\dots\partial x_{N1}\partial x_{N2}\dots\partial x_{NM_N}} \right| \leqslant \alpha$$

$$\forall\ x_{nm} \in \left[ x_{nm}^{\langle s_m \rangle};\ x_{nm}^{\langle s_m+1 \rangle} \right],\ \ m = \overline{1, M_n}\,,\ \ n = \overline{1, N}\,, \tag{12}$$

for some $\alpha > 0$, implying tolerable oscillations of PPF $\left\{ K_n(\mathbf{X}) \right\}_{n=1}^{N}$ on every one of segments

$$\left\{ \left\{ \left\{ \left[ x_{nm}^{\langle s_m \rangle};\ x_{nm}^{\langle s_m+1 \rangle} \right] \right\}_{s_m=1}^{S_m^{\langle n \rangle}} \right\}_{m=1}^{M_n} \right\}_{n=1}^{N}.$$

In order to sample PPF under conditions (11) and (12) from the hypercube (4) down to FHIL (8), the sampling numbers (10) and points

$$\left\{ \left\{ \left\{ x_{nm}^{\langle s_m \rangle} \right\}_{s_m=2}^{S_m^{\langle n \rangle}} \right\}_{m=1}^{M_n} \right\}_{n=1}^{N} \tag{13}$$

can be chosen using the following assertion.

**Theorem 1.** If inequalities (12) hold $\forall\ s_m = \overline{1, S_m^{\langle n \rangle}}$ and if local extremums of PPF $\left\{ K_n(\mathbf{X}) \right\}_{n=1}^{N}$ are reached at points having only components (13), then PPF $\left\{ K_n(\mathbf{X}) \right\}_{n=1}^{N}$ are sampled through representing the unit hypercube $U_n$ as FHIL (6) with (5).

**Proof.** Since having local extremums only with components (13), none of PPF $\left\{ K_n(\mathbf{X}) \right\}_{n=1}^{N}$ has local extremums on every one of intervals

$$\left\{ \left\{ \left\{ \left( x_{nm}^{\langle s_m \rangle};\ x_{nm}^{\langle s_m+1 \rangle} \right) \right\}_{s_m=1}^{S_m^{\langle n \rangle}} \right\}_{m=1}^{M_n} \right\}_{n=1}^{N}.$$

Hence, for the differentiable PPF $\left\{ K_n(\mathbf{X}) \right\}_{n=1}^{N}$ conditions (11) hold as well. The theorem has been proved.

Based on practical reasonings,

$$\alpha \leqslant \beta \cdot \left( \max_{n \in \left\{ \overline{1, N} \right\}} \max_{\mathbf{X} \in U} K_n(\mathbf{X}) - \min_{n \in \left\{ \overline{1, N} \right\}} \min_{\mathbf{X} \in U} K_n(\mathbf{X}) \right)$$

by, say, $\beta = 0.05$, $\beta = 0.01$, $\beta = 0.005$ or $\beta = 0.001$ and other practically appropriate values for the scale factor $\beta$. That shall be sufficiently accurate for practice experience. However, some conditions below may cause need to resample PPF with the lowered parameter $\alpha$. These conditions, defining whether the approximate solution of the initial UHINCG (3) is consistent enough, are going to be applied to the exact solution of FNCG (7). For computing this solution faster, the sampled PPF $\left\{ K_n(\mathbf{X}) \right\}_{n=1}^{N}$ as $\left( \sum_{i=1}^{N} M_i \right)$-dimensional matrices must be reshaped into matrices with minimally possible number of dimensions, letting get rid off dimensionalities and having the single dimension at each player. Apparently, these reshaped matrices shall have $N$ dimensions.

## RESHAPING THE SAMPLED PPF INTO $N$ -DIMENSIONAL MATRICES

**Theorem 2.** Any sampled PPF in UHINCG (3) as $\left(\sum_{i=1}^{N} M_i\right)$-dimensional matrix can be reshaped into $N$-dimensional matrix so that each player's payoffs shall be assigned through the single dimension. The matrix mapping which realizes this reshaping is reversible.

**Proof.** Denote the sampled PPF $K_n(\mathbf{X})$ with elements (9) as $\left(\sum_{i=1}^{N} M_i\right)$-dimensional matrix $\mathbf{P}_n(0) = \left[ p_J^{\langle n \rangle} \right]_{\mathscr{F}}$ of the format

$$\mathscr{F} = \bigtimes_{i=1}^{N} \bigtimes_{m=1}^{M_i} \left( S_m^{\langle i \rangle} + 1 \right), \tag{14}$$

whose $\left(\sum_{i=1}^{N} M_i\right)$-position indices

$$J = \left\{ j_d \right\}_{d=1}^{\sum_{i=1}^{N} M_i} \text{ by } j_k \in \left\{ \overline{1, S_m^{\langle r \rangle} + 1} \right\} \tag{15}$$

at

$$k = m + \sum_{i=1}^{r-1} M_i \quad \forall\, m = \overline{1, M_r} \text{ and } \forall\, r = \overline{1, N} \tag{16}$$

determine the element

$$p_J^{\langle n \rangle} = K_n(\mathbf{X}) \text{ by } x_{rm} = x_{rm}^{\langle j_k \rangle}. \tag{17}$$

Indices numbered from $1 + \sum_{i=1}^{n-1} M_i$ to $\sum_{i=1}^{n} M_i$ in the set $J$ in (15) correspond to components of the pure strategy of the $n$-th player. These indices are convolved into value

$$u_n = \sum_{m=1}^{M_n} \left( \prod_{w=1}^{m-1} \left( S_{M_n - w + 1}^{\langle n \rangle} + 1 \right) \right) \cdot \left( j_{M_0 - m + 1} - \operatorname{sign}(m-1) \right) \text{ at } M_0 = \sum_{i=1}^{n} M_i \quad \forall\, n = \overline{1, N}. \tag{18}$$

Taking $j_k = \overline{1, S_m^{\langle r \rangle} + 1}$ at (16), there is

$$u_r = \overline{1, Q_r(0)} \text{ by } Q_r(0) = \prod_{m=1}^{M_r} \left( S_m^{\langle r \rangle} + 1 \right) \quad \forall\, r = \overline{1, N}. \tag{19}$$

Thus the set of indices (15) at (16) is mapped into the set of indices $I = \{u_r\}_{r=1}^{N}$ with the convolution (18). Therefore, matrix $\mathbf{P}_n(0) = \left[ p_J^{\langle n \rangle} \right]_{\mathscr{F}}$ of the format (14) by (15) — (17) is reshaped into $N$-dimensional matrix $\mathbf{G}_n(0) = \left[ g_I^{\langle n \rangle} \right]_{\mathscr{L}}$ of the format

$$\mathscr{L} = \bigtimes_{i=1}^{N} \prod_{m=1}^{M_i} \left( S_m^{\langle i \rangle} + 1 \right) \tag{20}$$

and its elements

$$g_I^{\langle n \rangle} = p_J^{\langle n \rangle} \quad \text{by} \quad n = \overline{1, N}\,.$$

For proving reversibility of this mapping, let the function $\rho(a, b)$ by $b \neq 0$ round the fraction $\frac{a}{b}$ to the nearest integer towards zero. And put another function

$$\eta(a, b) = a - b \cdot \rho(a, b)\,. \tag{21}$$

By the function (21), the last index in indicating the $n$-th player's aggregate index $u_n$ for matrix $\mathbf{P}_n(0) = \left[ p_J^{\langle n \rangle} \right]_{\mathscr{F}}$ is restored as

$$j_{M_0} = \eta\left(u_n, S_{M_n}^{\langle n \rangle} + 1\right) + \left(S_{M_n}^{\langle n \rangle} + 1\right)\left(1 - \operatorname{sign}\left[\eta\left(u_n, S_{M_n}^{\langle n \rangle} + 1\right)\right]\right) \text{ at } M_0 = \sum_{i=1}^{n} M_i\,. \tag{22}$$

The rest $M_n - 1$ indices are restored using the index (22):

$$j_{M_0 - m} = 1 + \eta\left( \frac{u_n - j_{M_0} - \sum_{w=1}^{m-1} \left( \prod_{w_1=1}^{w} \left( S_{M_n - w_1 + 1}^{\langle n \rangle} + 1 \right) \right) \cdot \left( j_{M_0 - w} - 1 \right)}{\prod_{w=1}^{m} \left( S_{M_n - w + 1}^{\langle n \rangle} + 1 \right)},\, S_{M_n - m}^{\langle n \rangle} + 1 \right) \quad \forall\, m = \overline{1, M_n - 1}. \tag{23}$$

Consequently, the matrix mapping $\mathbf{P}_n(0) \rightarrow \mathbf{G}_n(0)$ is accomplished along with the index mapping $J \rightarrow I$ via convolution (18), and the reverse index mapping $I \rightarrow J$ accomplished via expansion (22) and (23) gives the matrix mapping $\mathbf{G}_n(0) \rightarrow \mathbf{P}_n(0)$. The theorem has been proved.

The reshaping of the sampled PPF $\left\{\mathbf{P}_n(0)\right\}_{n=1}^{N}$ into $N$-dimensional matrices $\left\{\mathbf{G}_n(0)\right\}_{n=1}^{N}$ under Theorem 2 lets get rid off dimensionalities, and each player will act through its own single dimension. This simplifies ultimately formalism of FNCG (7) on FHIL (8), mapping FNCG (7)

$$\left\langle \left\{ D^{\langle n \rangle}\left( \left\{ S_m^{\langle n \rangle} \right\}_{m=1}^{M_n} \right) \right\}_{n=1}^{N},\, \left\{ \mathbf{P}_n(0) \right\}_{n=1}^{N} \right\rangle \tag{24}$$

into FNCG

$$\left\langle \left\{ \left\{ z_{u_n}^{\langle n \rangle}(0) \right\}_{u_n=1}^{Q_n(0)} \right\}_{n=1}^{N},\, \left\{ \mathbf{G}_n(0) \right\}_{n=1}^{N} \right\rangle \tag{25}$$

with the pure strategy $z_{u_n}^{\langle n \rangle}(0)$ of the $n$-th player corresponding to its strategy

$$\mathbf{X}_n = \left[ x_{nm}^{\langle j_k \rangle} \right]_{1\times M_n} \in D^{\langle n \rangle}\left( \left\{ S_m^{\langle n \rangle} \right\}_{m=1}^{M_n} \right) \subset U_n \tag{26}$$

at

$$k = m + \sum_{i=1}^{n-1} M_i \quad \text{for} \quad m = \overline{1,\, M_n} \tag{27}$$

in the initial UHINCG (3) after having sampled under numbers $\left\{ S_m^{\langle n \rangle} \right\}_{m=1}^{M_n}$. Besides, the mapping of FNCG (24) into FNCG (25) must presumably speed up computing the exact solution of FNCG (24), when $\exists\, n_1 \in \left\{ \overline{1,\, N} \right\}$ such that $M_{n_1} > 1$. Indeed, there is a known rule telling that manipulating with a many single-dimensional objects is more convenient than manipulating with single multidimensional object [21, 22]. Practically it is explained with that the greater supplementary dimensions of a matrix the longer computations might be. This computational retardation is easy exampled in Matlab environment. Suppose that a 12-dimensional $\bigtimes_{d=1}^{12} 4$-matrix represents the player's payoff values in 3-person game, where each player's pure strategy is of four dimensions with four points in every dimension. While operating on Intel Core i3-4150 CPU @ 3.50 GHz with 4 GB RAM within 64-bit Windows 7, 30000 Matlab operations of summing and extracting mean, finding minimal and maximal elements of three such matrices take about 10375 seconds, whereas the same takes about 8105 seconds over three six-dimensional $\bigtimes_{d=1}^{6} 16$-matrices, reshaped before. Moreover, reshaping these $\bigtimes_{d=1}^{12} 4$-matrices into three-dimensional $\bigtimes_{d=1}^{3} 256$-matrices under Theorem 2 reduces the operation time down to 7874 seconds. The computational time shortening gain is almost 1.32, what is pretty significant for the exampled reshaping (more than 2500 seconds).

## WEAK CONSISTENCY OF THE EXACT SOLUTION OF FNCG (25), APPROXIMATING A GENUINE SOLUTION IN UHINCG (3)

Denote by

$$\left\{ \left\{ p_*\left(u_n,\, 0\right) \right\}_{u_n=1}^{Q_n(0)} \right\}_{n=1}^{N} \tag{28}$$

a known Nash equilibrium or Pareto-efficient solution or other type equilibrium solution in FNCG (25), where $p_*\left(u_n,\, 0\right)$ is the optimal probability of applying the pure strategy $z_{u_n}^{\langle n \rangle}\left(0\right)$. Before calling (28) the approximate solution of UHINCG (3), every strategy from the solution (28) shouldn't be too dependent upon the sampling numbers (10) and the sampling points (13) or

$$\left\{ \left\{ \left\{ x_{nm}^{\langle s_m \rangle} \right\}_{s_m=1}^{S_m^{\langle n \rangle}+1} \right\}_{m=1}^{M_n} \right\}_{n=1}^{N}. \tag{29}$$

For controlling this dependence, along with FNCG (25) there is going to be considered FNCG

$$\left\langle \left\{ \left\{ z_{u_n}^{\langle n \rangle}(\delta) \right\}_{u_n=1}^{Q_n(\delta)} \right\}_{n=1}^{N}, \left\{ \mathbf{G}_n(\delta) \right\}_{n=1}^{N} \right\rangle \tag{30}$$

which is the result of mapping UHINCG (3) into FNCG under the sampling numbers

$$\left\{ \left\{ S_m^{\langle n \rangle} + \delta \right\}_{m=1}^{M_n} \right\}_{n=1}^{N} \tag{31}$$

by $\delta \in \mathbb{Z} \setminus \{0\}$, wherein

$$Q_n(\delta) = \prod_{m=1}^{M_n} \left( S_m^{\langle n \rangle} + 1 + \delta \right)$$

and with identifications

$$\left\{ S_m^{\langle n \rangle} \equiv S_m^{\langle n \rangle} + \delta \right\}_{m=1}^{M_n} \quad \forall\, n = \overline{1, N} \tag{32}$$

the re-settings (5) — (8) and re-mappings of the sampled under (31) PPF $\left\{ \mathbf{P}_n(\delta) \to \mathbf{G}_n(\delta) \right\}_{n=1}^{N}$ are fulfilled after re-formattings (14) and (20), whereupon the pure strategy of the $n$-th player $z_{u_n}^{\langle n \rangle}(\delta)$ corresponds to its strategy (26) at (27) in UHINCG (3) after having sampled under numbers $\left\{ S_m^{\langle n \rangle} + \delta \right\}_{m=1}^{M_n}$. Let there be a convention that the sampling numbers (31) and points

$$\left\{ \left\{ \left\{ x_{nm}^{\langle s_m \rangle}(\delta) \right\}_{s_m=1}^{S_m^{\langle n \rangle}+1} \right\}_{m=1}^{M_n} \right\}_{n=1}^{N} \tag{33}$$

chosen after them by $\delta \in \mathbb{N}$ are such against the sampling numbers (10) and points (29), that the inequalities

$$\max_{\overline{d_m = 1,\, S_m^{\langle n \rangle}}} \left( x_{nm}^{\langle d_m + 1 \rangle} - x_{nm}^{\langle d_m \rangle} \right) \geqslant \max_{\overline{d_m = 1,\, S_m^{\langle n \rangle} + \delta}} \left( x_{nm}^{\langle d_m + 1 \rangle}(\delta) - x_{nm}^{\langle d_m \rangle}(\delta) \right) \text{ at } m = \overline{1, M_n} \text{ by } n = \overline{1, N} \tag{34}$$

hold. This implies that density of the sampling points along each dimension by $\delta > 0$ doesn't decrease, and density of the sampling points along each dimension by $\delta < 0$ doesn't increase.

Denote the solution of FNCG (30) by

$$\left\{ \left\{ p_*(u_n, \delta) \right\}_{u_n=1}^{Q_n(\delta)} \right\}_{n=1}^{N} \tag{35}$$

similarly to denotation (28), in which $p_*(u_n, \delta)$ is the optimal probability of applying the pure strategy $z_{u_n}^{\langle n \rangle}(\delta)$. Thus by denoting the support

$$\operatorname{supp} \left\{ p_*(u_n, \delta) \right\}_{u_n=1}^{Q_n(\delta)} = \left\{ z_{u_n^*}^{\langle n \rangle}(\delta) \right\}_{u_n^* \in \mathfrak{U}_n(\delta) \subset \left\{ \overline{1, Q_n(\delta)} \right\}} \tag{36}$$

the $n$-th player gets payoff

$$v_n^*(\delta) = \sum_{\substack{I=\{u_l\}_{l=1}^N \\ u_l=\overline{1,\,Q_l(\delta)}}} \left( g_I^{\langle n \rangle} \cdot \prod_{i=1}^N p_*(u_i,\,\delta) \right) = \sum_{I^*=\left\{u_l^*:\,u_l^*\in\mathfrak{U}_l(\delta)\right\}_{l=1}^N} \left( g_{I^*}^{\langle n \rangle} \cdot \prod_{i=1}^N p_*(u_i^*,\,\delta) \right) \text{ for } n=\overline{1,\,N} \quad (37)$$

in situation (35). Henceforward, FNCG (25) can be compared to FNCG (30) by $\delta \in \mathbb{Z} \setminus \{0\}$ in two ways: through comparing payoffs $\left\{v_n^*(0)\right\}_{n=1}^N$ and $\left\{\left\{v_n^*(\delta)\right\}_{n=1}^N\right\}_{\delta\in\mathbb{Z}\setminus\{0\}}$, and through comparisons among supports (36) at each player. Comparisons among supports (36) at each player imply support cardinality comparisons, and support configuration comparisons regarding the hypercube of the player's pure strategies.

**Definition 1.** Situation (35) in FNCG (30) is called $\delta$-neighboring situation in relation to the situation (28) in FNCG (25).

**Definition 2.** Aggregate of situations (35) in FNCG (30) by $\delta = \overline{-\lambda,\,\lambda}$ and $\lambda \in \mathbb{N}$ is called $\lambda$-neighborhood of the situation (28) in FNCG (25).

When $N$ players take their payoffs $\left\{v_n^*(0)\right\}_{n=1}^N$ in the situation (28) and take payoffs $\left\{v_n^*(1)\right\}_{n=1}^N$ in 1-neighboring situation (for the minimally increased sampling numbers or the minimally increased number of the sampling points along each dimension), it is nonexceptional that there may be at least $\exists\, n_0 \in \left\{\overline{1,\,N}\right\}$ such that payoffs $v_{n_0}^*(0)$ and $v_{n_0}^*(1)$ will be significantly different. The same concerns payoffs $\left\{v_n^*(-1)\right\}_{n=1}^N$ taken in $(-1)$-neighboring situation (for the minimally decreased number of the sampling points along each dimension) in relation to $\left\{v_n^*(0)\right\}_{n=1}^N$. Meanwhile, it is unknown whether limits

$$\lim_{\delta\to\infty} v_n^*(\delta) \text{ by } n=\overline{1,\,N} \quad (38)$$

exist or not. Neither are known genuine payoffs $\left\{v_n^{**}\right\}_{n=1}^N$ in UHINCG (3), where $v_n^{**}$ is the $n$-th player's payoff in the situation, being approximated with (35) by $\delta \in \mathbb{Z}$. So it is apparent that the solution (28) of FNCG (25) can be said that it approximates a genuine solution in UHINCG (3) only if

$$\left|v_n^*(0) - v_n^*(1)\right| \leqslant \left|v_n^*(-1) - v_n^*(0)\right| \quad \forall\, n=\overline{1,\,N}. \quad (39)$$

Secondly, when the number of the sampling points along each dimension is minimally increased, any support cardinality shouldn't decrease:

$$\left|\mathfrak{U}_n(1)\right| \geqslant \left|\mathfrak{U}_n(0)\right| \quad \forall\, n=\overline{1,\,N}. \quad (40)$$

Inequalities (40) can be strengthened involving FNCG under the minimally decreased number of the sampling points along each dimension:

$$\left|\mathfrak{U}_n(1)\right| \geqslant \left|\mathfrak{U}_n(0)\right| \geqslant \left|\mathfrak{U}_n(-1)\right| \quad \forall\, n=\overline{1,\,N}. \quad (41)$$

Here also limits

$$\lim_{\delta\to\infty}\left|\mathfrak{U}_n(\delta)\right| \text{ by } n=\overline{1, N} \tag{42}$$

are non-proved for their existence, and it is unknown how they are close to the genuine solution in UHINCG (3), being approximated with (35) by $\delta\in\mathbb{Z}$.

And, thirdly, each player's strategies from solutions of the same type in FNCG must have nearly similar configurations. This feature isn't numerical, unlike (40) and (41). The study of the player's strategy configuration, if its support is not a singleton, needs consideration of the player's supports from different $\delta$-neighboring situations as hypersurfaces.

Let the $n$-th player's strategy support in the situation (28) be represented as piecewise linear hypersurface $h_n(u_n, 0)$. This hypersurface is a function of $u_n=\overline{1, Q_n(0)}$ and its nonzero vertices are those elements in the set (36) that are linearly linked to points on FHIL (6) not included into the support. The hypersurfaces in $\delta$-neighboring situation (35) are denoted $\left\{h_n(u_n, \delta)\right\}_{n=1}^{N}$. Vertices of hypersurface $h_n(u_n, 0)$ are in points

$$\left\{\left\{x_{nm}^{\langle j_k\rangle} : k=m+\sum_{i=1}^{n-1}M_i\right\}_{m=1}^{M_n},\ p_*(u_n, 0)\right\}_{u_n=1}^{Q_n(0)} \tag{43}$$

in the space $\mathbb{R}^{M_n+1}$. Hypersurfaces $\left\{h_n(u_n, \delta)\right\}_{n=1}^{N}$ are defined with identifications (32) analogously. Thus the solution (28) of FNCG (25) can be said that it approximates a genuine solution in UHINCG (3) only if

$$\max_{U_n}\left|h_n(u_n, 0)-h_n(u_n, 1)\right|\leqslant\max_{U_n}\left|h_n(u_n, -1)-h_n(u_n, 0)\right| \quad \forall\, n=\overline{1, N} \tag{44}$$

and

$$\left\|h_n(u_n, 0)-h_n(u_n, 1)\right\|\leqslant\left\|h_n(u_n, -1)-h_n(u_n, 0)\right\| \text{ in } \mathbb{L}_2(U_n) \quad \forall\, n=\overline{1, N} \tag{45}$$

similarly to inequalities (39). Deficiently, existence of limits

$$\lim_{\delta\to\infty}h_n(u_n, \delta) \text{ by } n=\overline{1, N} \tag{46}$$

and their convergence to the being approximated strategies in the solution of UHINCG (3) protrudes non-proved. Nonetheless hypersurfaces $\left\{h_n(u_n, \delta)\right\}_{n=1}^{N}$ feature strategies' configurations incompletely.

For completing the configuration feature, note that the $n$-th player matches the index $u_n^*\in\mathfrak{U}_n(0)$ to the point

$$\mathbf{X}_n^{\langle q\rangle}(0)=\left[x_{nm}^{\langle j_k(q, 0)\rangle}\right]_{1\times M_n}\in D^{\langle n\rangle}\left(\left\{S_m^{\langle n\rangle}\right\}_{m=1}^{M_n}\right)\subset U_n \text{ for } k=m+\sum_{i=1}^{n-1}M_i \text{ by } q=\overline{1, Q_n^*(0)} \tag{47}$$

at $Q_n^*(0)=\left|\mathfrak{U}_n(0)\right|$ through expanding the index $u_n^*\in\mathfrak{U}_n(0)$ via (22) and (23) into subset

$$\left\{ j_k(q, 0) : k = m + \sum_{i=1}^{n-1} M_i \right\}_{m=1}^{M_n} \subset J \quad (48)$$

by $n = \overline{1, N}$. Then let the set $\left\{ \mathbf{X}_n^{\langle q \rangle}(0) \right\}_{q=1}^{Q_n^*(0)}$ of the points (47) be sorted into the set

$$\left\{ \overline{\mathbf{X}}_n^{\langle q \rangle}(0) \right\}_{q=1}^{Q_n^*(0)} = \left\{ \left[ x_{nm}^{\langle \overline{j}_k(q, 0) \rangle} \right]_{1 \times M_n} \right\}_{q=1}^{Q_n^*(0)} \cap \left\{ \mathbf{X}_n^{\langle q \rangle}(0) \right\}_{q=1}^{Q_n^*(0)} =$$

$$= \left\{ \mathbf{X}_n^{\langle q \rangle}(0) \right\}_{q=1}^{Q_n^*(0)} \in D^{\langle n \rangle}\left( \left\{ S_m^{\langle n \rangle} \right\}_{m=1}^{M_n} \right) \subset U_n \quad (49)$$

so that the value

$$\min_{w \in \left\{ \overline{q+1, Q_n^*(0)} \right\}} \rho_{\mathbb{R}^{M_n}} \left( \overline{\mathbf{X}}_n^{\langle q \rangle}(0), \overline{\mathbf{X}}_n^{\langle w \rangle}(0) \right) = \min_{w \in \left\{ \overline{q+1, Q_n^*(0)} \right\}} \left\| \overline{\mathbf{X}}_n^{\langle q \rangle}(0) - \overline{\mathbf{X}}_n^{\langle w \rangle}(0) \right\| =$$

$$= \min_{w \in \left\{ \overline{q+1, Q_n^*(0)} \right\}} \sqrt{ \sum_{m=1}^{M_n} \left( x_{nm}^{\langle \overline{j}_k(q, 0) \rangle} - x_{nm}^{\langle \overline{j}_k(w, 0) \rangle} \right)^2 } \quad (50)$$

with the re-sorted subset

$$\left\{ \overline{j}_k(q, 0) \right\}_{m=1}^{M_n} \cap \left\{ j_k(q, 0) \right\}_{m=1}^{M_n} = \left\{ j_k(q, 0) \right\}_{m=1}^{M_n} \subset J \quad (51)$$

at (48) is reached at $w = q + 1$ for each $q = \overline{1, Q_n^*(0) - 1}$ by $Q_n^*(0) < Q_n(0)$, $n = \overline{1, N}$. Importantly, one ought to be aware of that the result of sorting in (49) depends on selection of the initial points

$$\overline{\mathbf{X}}_n^{\langle 1 \rangle}(0) \in \left\{ \mathbf{X}_n^{\langle q \rangle}(0) \right\}_{q=1}^{Q_n^*(0)} \quad \forall\, n = \overline{1, N}.$$

In the case of completely mixed strategies, the sorting is needless:

$$\overline{\mathbf{X}}_n^{\langle q \rangle}(0) = \mathbf{X}_n^{\langle q \rangle}(0) \quad \forall\, q = \overline{1, Q_n^*(0)} \text{ and } \forall\, n = \overline{1, N} \text{ by } Q_1^*(0) = Q_1(0). \quad (52)$$

For $\delta$-neighboring situations, let the sets

$$\left\{ \left\{ \overline{\mathbf{X}}_n^{\langle q \rangle}(\delta) \right\}_{q=1}^{Q_n^*(\delta)} \right\}_{n=1}^{N}$$

regard built and found with identifications (32) and turning to these sets' description in (47) — (52). Just right along with (40), the solution (28) of FNCG (25) can be said that it approximates a genuine solution in UHINCG (3) only if

$$\max_{q \in \left\{ \overline{1, Q_n^*(1) - 1} \right\}} \rho_{\mathbb{R}^{M_n}} \left( \overline{\mathbf{X}}_n^{\langle q \rangle}(1), \overline{\mathbf{X}}_n^{\langle q+1 \rangle}(1) \right) \leqslant \max_{q \in \left\{ \overline{1, Q_n^*(0) - 1} \right\}} \rho_{\mathbb{R}^{M_n}} \left( \overline{\mathbf{X}}_n^{\langle q \rangle}(0), \overline{\mathbf{X}}_n^{\langle q+1 \rangle}(0) \right) \quad \forall\, n = \overline{1, N}. \quad (53)$$

Inequalities (53) imply that density of the pure strategies from the support at each player in 1-neighboring situation shall not decrease. This comes along also with the convention expressed in (34) for choosing the sampling numbers (31).

**Definition 3.** The solution (28) of FNCG (25) is called weakly consistent or weakly 1-consistent for being the approximate solution of UHINCG (3) if the inequalities (39), (40), (44), (45), (53) hold. Every strategy and its support in the weakly 1-consistent solution (situation) are called weakly consistent or weakly 1-consistent.

Weak 1-consistency is the most primitive method in ranking the finite approximation accurateness for UHINCG (3). It invokes minimal number of $\delta$-neighboring situations, approximating the initial UHINCG (3). But inequalities (40) have been strengthened already into (41). Inequalities (53) might be strengthened as well involving FNCG with $(-1)$-neighboring situation.

## REINFORCEMENT OF WEAK CONSISTENCY

**Definition 4.** The weakly consistent solution (28) of FNCG (25) is called consistent or 1-consistent for being the approximate solution of UHINCG (3) if the inequalities (41) and

$$\max_{q\in\left\{\overline{1,\, Q_n^*(0)-1}\right\}} \rho_{\mathbb{R}^{M_n}}\left(\overline{\mathbf{X}}_n^{\langle q\rangle}(0),\, \overline{\mathbf{X}}_n^{\langle q+1\rangle}(0)\right) \leqslant \max_{q\in\left\{\overline{1,\, Q_n^*(-1)-1}\right\}} \rho_{\mathbb{R}^{M_n}}\left(\overline{\mathbf{X}}_n^{\langle q\rangle}(-1),\, \overline{\mathbf{X}}_n^{\langle q+1\rangle}(-1)\right) \quad \forall\, n=\overline{1,\, N} \tag{54}$$

hold. Every strategy and its support in the 1-consistent solution (situation) are called consistent or 1-consistent.

The reinforced weak 1-consistency with (41) and (54) presumes that after the minimal decrement of the sampling points number the support cardinality cannot increase and the support pure strategies configuration cannot become denser. These presumptions complete the requirement of that the 1-consistent situation (28) in FNCG (25) within its 1-neighborhood shall maintain monotonic-like properties of the supports and the corresponding payoffs.

Hypothetical similarity between 1-consistent solution (28) of FNCG (25) and the same type solution of UHINCG (3) is provided with that the solution (28) is relatively independent under some tolerance upon the sampling points (13) in $\delta$-neighboring situations by $\delta \in \{-1, 1\}$. The tolerance is expressed by leftside terms of inequalities (39), (40) and (41), (44), (45), (53), (54), where minimal increment of the sampling points number can only cause lesser differentiation of the players' payoffs, and of hypersurfaces, approximating their supports piecewise-linearly, and increment of cardinalities of these supports, and denser configuration of pure strategies from these supports.

In controlling the weak 1-consistency, there are $5N$ inequalities to be checked. Controlling the 1-consistency requires additionally $2N$ inequalities to be checked. Below is an opportunity to avoid superfluous computations in controlling the weak 1-consistency.

**Theorem 3.** If 1-neighboring situation in relation to the situation (28) in FNCG (25) is completely mixed, then for checking the weak 1-consistency of the solution (28) it is sufficient to check $3N$ inequalities (39), (44), (45).

**Proof.** Inasmuch as the 1-neighboring situation is completely mixed then the inequalities

$$Q_n^*(1) = Q_n(1) = \prod_{m=1}^{M_n}\left(S_m^{\langle n\rangle}+2\right) > \prod_{m=1}^{M_n}\left(S_m^{\langle n\rangle}+1\right) = Q_n(0) \geqslant Q_n^*(0) \quad \forall\, n=\overline{1,\, N}$$

give us all the inequalities (40), even strictly. Through the convention (52) there is the set

$$\left\{\overline{\mathbf{X}}_n^{\langle q\rangle}(1)\right\}_{q=1}^{Q_n^*(1)} = \left\{\mathbf{X}_n^{\langle q\rangle}(1)\right\}_{q=1}^{Q_n(1)} \quad \forall\, n=\overline{1,\, N}$$

such that

$$\max_{q\in\left\{\overline{1,\, Q_n^*(1)-1}\right\}} \rho_{\mathbb{R}^{M_n}}\left(\overline{\mathbf{X}}_n^{\langle q\rangle}(1),\, \overline{\mathbf{X}}_n^{\langle q+1\rangle}(1)\right) = \max_{m=\overline{1,\, M_n}}\left\{\max_{d_m=\overline{1,\, S_m^{\langle n\rangle}+1}}\left(x_{nm}^{\langle d_m+1\rangle}(1) - x_{nm}^{\langle d_m\rangle}(1)\right)\right\} \quad \forall\, n=\overline{1,\, N}. \tag{55}$$

Due to (34), from (55) and by (27) have:

$$\max_{q\in\left\{\overline{1,\, Q_n^*(1)-1}\right\}} \rho_{\mathbb{R}^{M_n}}\left(\overline{\mathbf{X}}_n^{\langle q\rangle}(1),\, \overline{\mathbf{X}}_n^{\langle q+1\rangle}(1)\right) \leqslant \max_{m=\overline{1,\, M_n}}\left\{\max_{d_m=\overline{1,\, S_m^{\langle n\rangle}}}\left(x_{nm}^{\langle d_m+1\rangle} - x_{nm}^{\langle d_m\rangle}\right)\right\} \leqslant$$

$$\leqslant \max_{q\in\left\{\overline{1,\, Q_n^*(0)-1}\right\}} \sqrt{\sum_{m=1}^{M_n}\left(x_{nm}^{\langle \bar{j}_k(q+1,\,0)\rangle} - x_{nm}^{\langle \bar{j}_k(q,\,0)\rangle}\right)^2} = \max_{q\in\left\{\overline{1,\, Q_n^*(0)-1}\right\}} \rho_{\mathbb{R}^{M_n}}\left(\overline{\mathbf{X}}_n^{\langle q\rangle}(0),\, \overline{\mathbf{X}}_n^{\langle q+1\rangle}(0)\right) \quad \forall\, n=\overline{1,\, N}$$

giving us all the inequalities (53). The theorem has been proved.

The case of the completely mixed solution facilitates in controlling its consistency. This is confirmed with the assertion below.

**Theorem 4.** If weakly 1-consistent solution is completely mixed then it is consistent.

**Proof.** For the weakly consistent strategy of the $n$-th player in the completely mixed solution (28), the inequalities

$$Q_n^*(0) = Q_n(0) = \prod_{m=1}^{M_n}\left(S_m^{\langle n\rangle}+1\right) > \prod_{m=1}^{M_n} S_m^{\langle n\rangle} = Q_n(-1) \geqslant Q_n^*(-1) \quad \forall\, n=\overline{1,\, N}$$

give us all the inequalities

$$Q_n^*(0) = \left|\mathfrak{U}_n(0)\right| > \left|\mathfrak{U}_n(-1)\right| = Q_n^*(-1) \quad \forall\, n=\overline{1,\, N}\,. \tag{56}$$

The inequalities (56) with (40) give the inequalities (41) true. Then, once again, the convention (52) leads to that

$$\max_{q\in\left\{\overline{1,\, Q_n^*(0)-1}\right\}} \rho_{\mathbb{R}^{M_n}}\left(\overline{\mathbf{X}}_n^{\langle q\rangle}(0),\, \overline{\mathbf{X}}_n^{\langle q+1\rangle}(0)\right) = \max_{m=\overline{1,\, M_n}}\left\{\max_{d_m=\overline{1,\, S_m^{\langle n\rangle}}}\left(x_{nm}^{\langle d_m+1\rangle} - x_{nm}^{\langle d_m\rangle}\right)\right\} \quad \forall\, n=\overline{1,\, N}\,. \tag{57}$$

And due to (34), from (57) and by (27) have:

$$\max_{q\in\left\{\overline{1,\, Q_n^*(0)-1}\right\}} \rho_{\mathbb{R}^{M_n}}\left(\overline{\mathbf{X}}_n^{\langle q\rangle}(0),\, \overline{\mathbf{X}}_n^{\langle q+1\rangle}(0)\right) \leqslant \max_{m=\overline{1,\, M_n}}\left\{\max_{d_m=\overline{1,\, S_m^{\langle n\rangle}-1}}\left(x_{nm}^{\langle d_m+1\rangle}(-1) - x_{nm}^{\langle d_m\rangle}(-1)\right)\right\} \leqslant$$

$$\leqslant \max_{q\in\left\{\overline{1,\, Q_n^*(-1)-1}\right\}} \sqrt{\sum_{m=1}^{M_n}\left(x_{nm}^{\langle \bar{j}_k(q+1,\,-1)\rangle} - x_{nm}^{\langle \bar{j}_k(q,\,-1)\rangle}\right)^2} = \max_{q\in\left\{\overline{1,\, Q_n^*(-1)-1}\right\}} \rho_{\mathbb{R}^{M_n}}\left(\overline{\mathbf{X}}_n^{\langle q\rangle}(-1),\, \overline{\mathbf{X}}_n^{\langle q+1\rangle}(-1)\right) \quad \forall\, n=\overline{1,\, N}$$

giving us all the inequalities (54), what confirms the 1-consistency. The theorem has been proved.

Consistency by either Definition 3 or Definition 4 of the solution (28), approximating the unknown solution of UHINCG (3), ranks accurateness of the approximation for the narrowest neighborhood of the sampling numbers (10) for the sampling points (13). For that, there has been used 1-neighborhood of the situation (28) in FNCG (25). Apparently, the approximation accurateness rank conception in the form of (weak) 1-consistency is easily generalized to the form of (weak) $\lambda$-consistency by $\lambda \in \mathbb{N}$, what is going to make possible to ascertain whether the tolerable dependence of the solution upon the sampling points extends over $\lambda$-neighborhoods.

## APPROXIMATING A GENUINE SOLUTION IN UHINCG (3) WITH $\lambda$-CONSISTENT SOLUTION OF FNCG (25)

**Definition 5.** The situation (28) in FNCG (25) is called weakly $\lambda$-consistent for being the approximate solution of UHINCG (3) if $\forall\, n=\overline{1,\, N}$ the inequalities

$$\left|v_n^*(\mu) - v_n^*(\mu+1)\right| \leqslant \left|v_n^*(\mu-1) - v_n^*(\mu)\right|, \tag{58}$$

$$\left|\mathfrak{U}_n(\mu+1)\right| \geqslant \left|\mathfrak{U}_n(\mu)\right|, \tag{59}$$

$$\max_{U_n}\left|h_n(u_n,\, \mu) - h_n(u_n,\, \mu+1)\right| \leqslant \max_{U_n}\left|h_n(u_n,\, \mu-1) - h_n(u_n,\, \mu)\right|, \tag{60}$$

$$\left\| h_n(u_n, \mu) - h_n(u_n, \mu+1) \right\| \leqslant \left\| h_n(u_n, \mu-1) - h_n(u_n, \mu) \right\| \text{ in } \mathbb{L}_2(U_n), \tag{61}$$

and

$$\max_{q \in \left\{\overline{1, Q_n^*(\mu+1)-1}\right\}} \rho_{\mathbb{R}^{M_n}} \left( \overline{\mathbf{X}}_n^{\langle q \rangle}(\mu+1), \overline{\mathbf{X}}_n^{\langle q+1 \rangle}(\mu+1) \right) \leqslant \max_{q \in \left\{\overline{1, Q_n^*(\mu)-1}\right\}} \rho_{\mathbb{R}^{M_n}} \left( \overline{\mathbf{X}}_n^{\langle q \rangle}(\mu), \overline{\mathbf{X}}_n^{\langle q+1 \rangle}(\mu) \right) \tag{62}$$

are true $\forall\, \mu = \overline{1-\lambda, \lambda-1}$ by $\lambda \in \mathbb{N}$. Every strategy and its support in the weakly $\lambda$-consistent solution (situation) are called weakly $\lambda$-consistent.

**Definition 6.** The weakly $\lambda$-consistent solution (28) of FNCG (25) is called $\lambda$-consistent for being the approximate solution of UHINCG (3) if $\forall\, n = \overline{1, N}$ the inequalities

$$\left| \mathfrak{U}_n(\mu) \right| \geqslant \left| \mathfrak{U}_n(\mu-1) \right| \tag{63}$$

and

$$\max_{q \in \left\{\overline{1, Q_n^*(\mu)-1}\right\}} \rho_{\mathbb{R}^{M_n}} \left( \overline{\mathbf{X}}_n^{\langle q \rangle}(\mu), \overline{\mathbf{X}}_n^{\langle q+1 \rangle}(\mu) \right) \leqslant \max_{q \in \left\{\overline{1, Q_n^*(\mu-1)-1}\right\}} \rho_{\mathbb{R}^{M_n}} \left( \overline{\mathbf{X}}_n^{\langle q \rangle}(\mu-1), \overline{\mathbf{X}}_n^{\langle q+1 \rangle}(\mu-1) \right) \tag{64}$$

are true $\forall\, \mu = \overline{1-\lambda, \lambda-1}$ by $\lambda \in \mathbb{N}$. Every strategy and its support in the $\lambda$-consistent solution (situation) are called $\lambda$-consistent.

For checking $\lambda$-consistency of the weakly $\lambda$-consistent solution, it is not of necessity to check all $4N\lambda - 2N$ inequalities (63) and (64). It is sufficient to check $2N$ inequalities ever.

**Theorem 5.** If for some $\lambda \in \mathbb{N}$ the inequalities

$$\left| \mathfrak{U}_n(1-\lambda) \right| \geqslant \left| \mathfrak{U}_n(-\lambda) \right| \tag{65}$$

and

$$\max_{q \in \left\{\overline{1, Q_n^*(1-\lambda)-1}\right\}} \rho_{\mathbb{R}^{M_n}} \left( \overline{\mathbf{X}}_n^{\langle q \rangle}(1-\lambda), \overline{\mathbf{X}}_n^{\langle q+1 \rangle}(1-\lambda) \right) \leqslant \max_{q \in \left\{\overline{1, Q_n^*(-\lambda)-1}\right\}} \rho_{\mathbb{R}^{M_n}} \left( \overline{\mathbf{X}}_n^{\langle q \rangle}(-\lambda), \overline{\mathbf{X}}_n^{\langle q+1 \rangle}(-\lambda) \right) \tag{66}$$

are true $\forall\, n = \overline{1, N}$ then the weakly $\lambda$-consistent solution (28) of FNCG (25) is $\lambda$-consistent.

**Proof.** Inasmuch as the inequalities (59) and (62) are true $\forall\, \mu = \overline{1-\lambda, \lambda-1}$ then, having added the inequalities (65) and (66) to them, there are true the inequalities (63) and (64) $\forall\, \mu = \overline{1-\lambda, \lambda-1}$ by that known $\lambda$. The theorem has been proved.

Approximation of UHINCG (3) in $\lambda$-consistency under the generalizing Definition 5 and Definition 6 prescribes the monotonic-like properties of the supports and the corresponding payoffs within $\lambda$-neighborhood of the situation (28) in FNCG (25). The greater $\lambda$ the higher the approximation accurateness rank is. Apparently, weak $(\lambda-1)$-consistency follows weak $\lambda$-consistency and $(\lambda-1)$-consistency follows $\lambda$-consistency for $\lambda \in \mathbb{N} \setminus \{1\}$. However, for ranking the approximation accurateness in $\lambda$-consistency there are needed $2\lambda$ $\delta$-neighboring situations (35) at $\delta \in \left\{\overline{-\lambda, \lambda}\right\} \setminus \{0\}$ to be found. So, for trying to approximate the initial UHINCG (3), it takes $2\lambda+1$ FNCG (30) at $\delta = \overline{-\lambda, \lambda}$.

## DISCUSSION AND CONCLUSIVE REMARKS

UHINCG (3) is approximated as FNCG (25) through sampling PPF and simplification of FNCG (24) if only the situation (28) has a rank of the approximation accurateness. This rank has been conceived as consistency of the supports and the corresponding payoffs within $\lambda$-neighborhood of the situation (28) in FNCG (25). Principally, there are three steps to get the approximated UHINCG (3). The sampling goes first. After having reshaped the sampled PPF as matrices $\left\{\mathbf{P}_n\left(0\right)\right\}_{n=1}^{N}$ of the format (14) into matrices $\left\{\mathbf{G}_n\left(0\right)\right\}_{n=1}^{N}$ of the format (20), the solution (28) of FNCG (25) is supposed to be searched faster. Finally, the solution (28) is ascertained whether it is (weakly) $\lambda$-consistent. In other words, it is ascertained how much this finite solution is tolerably dependent (or relatively independent) upon the sampling points (13) for watching hypothetical similarity between (28) and the same type solution of UHINCG.

The idea of approximating UHINCG finitely lies in that every player's strategies in the finite solution shall be sufficiently close when the sampling slightly varies. The sufficient closeness is dissociated into four components. Expectedly, it concerns the supports' cardinalities, what has been expressed with (59) and (63) in Definition 5 and Definition 6. Also the sufficient closeness is understood in the sense of the most widespread metrics, what has been expressed with (60) and (61) in Definition 5 for measuring the corresponding distances between hypersurfaces, approximating the supports piecewise-linearly. And the sufficient closeness concerns measuring Euclidean distances between differently generated supports' pure strategies, what has been expressed with (62) and (64) in Definition 5 and Definition 6 for watching the supports' configurations through their density property. To the completion, the sufficient closeness presumes that the player's payoffs as the aftermath of the support usage shall differentiate less as the sampling numbers increase, what has been expressed with (58) in Definition 5.

Naturally, before approximating the weak 1-consistency ought to be checked first. The check consecution is recommended to start with checking the inequalities (39) or (40), because there are lesser computations. Then subconsecution of checking the inequalities (44), (45), (53) goes, needing more computational resources for solving the sorting problems (50) at (48) and (51) by $n=\overline{1, N}$ for (49). Namely these consecutions are preferable, because the easiest requirements are checked before the more complicated ones for preventing needless huge computations over non-consistent solutions, being revealed after easier comparisons (39) or (40).

An essential demerit is that existence of limits (38), (42), (46) remains non-proved. If consistency were conceived in the strict form, where all the inequalities in Definition 5 and Definition 6 were with signs "greater than" and "less than" instead of "greater than or equal" and "less than or equal", then it would have been probably a way to prove some convergences in those limits. But requirement with the strictness could have been very rigid. And anyway questions of effective computations for revealing consistent solutions are of higher importance. These questions have been particularly resolved with the proved Theorems 2 — 5. Reshaping the player's payoffs into line array under Theorem 2 impacts favorably on simplification and speedup.

Furthermore, another merit of the finite approximation is an opportunity of obtaining a solution to the conflict object even when UHINCG (3) is solved solely in $\varepsilon$-equilibrium situations or doesn't have solution at all. Besides, the (weak) consistent approximate solution (28), where every player has the finite strategy support, is practiced freer with discrete variates unlike practicing on continuous variates. And, conspicuously, DDIS of PPF allow to sample dimensions depending on oscillation of PPF: quickly oscillating dimensions of PPF are sampled tighter, and slowly oscillating dimensions of PPF are sampled sparser.

According to the isomorphism of UHINCG (3) and compact INCG in Euclidean finite-dimensional subspaces of appropriate dimensions, the stated approximation method is applicable to INCG on compact action spaces. Just normalization to unit hypercubes $\left\{U_n\right\}_{n=1}^{N}$ is obligatory. And further work will be focused on building an efficient sorter for solving the problems (50) at (48) and (51) by $n=\overline{1, N}$ for (49). And there are five important open questions:

1. Shall the $n_1$-th player at $n_1 \in \{\overline{1, N}\}$ use its strategy from the solution of FNCG (25) if the $n_1$-th player's strategy satisfies conditions of (weakly) $\lambda$-consistency, but $\exists\, n_0 \in \{\overline{1, N}\}$ such that the $n_0$-th player's strategy does not satisfy conditions of (weakly) $\lambda$-consistency?

2. Shall the $n_1$-th player at $n_1 \in N_1 \subset \{\overline{1, N}\}$ use its strategy from the solution of FNCG (25) if the $n_1$-th player's strategy satisfies conditions of (weakly) $\lambda$-consistency and $|N_1| = N - 1$, but $\exists\, n_0 \in \{\overline{1, N}\} \setminus N_1$ singly such that the $n_0$-th player's strategy does not satisfy conditions of (weakly) $\lambda$-consistency or is just (weakly) $(\lambda - 1)$-consistent?

3. Shall the $n_1$-th player at $n_1 \in N_1 \subset \{\overline{1, N}\}$ use its strategy from the solution of FNCG (25) if the $n_1$-th player's strategy satisfies conditions of $\lambda$-consistency, but $\exists\, n_0 \in N_0 \subset \{\overline{1, N}\}$ by $N_1 \bigcup N_0 = \{\overline{1, N}\}$ such that the $n_0$-th player's strategy is weakly $\lambda$-consistent?

4. Is it possible to determine (weak) $\lambda$-consistency of the solution of FNCG (25) if it is already known that the $n_1$-th player's strategy at $n_1 \in N_1 \subset \{\overline{1, N}\}$ and $|N_1| \leqslant N - 1$ satisfies conditions of (weakly) $\lambda$-consistency?

5. Are two different situations (28) in FNCG (25) necessarily (weakly) $\lambda$-consistent if one of them is (weakly) $\lambda$-consistent already?

These questions motivate to continue studies of approximating INCG. They may evolve into more complicated fundamental research. It will include $\varepsilon$-equilibrium solutions in pure strategies or in mixed strategies with finite supports, whose relation to game approximation is obvious [5, 9]. Nevertheless, the stated approximation method has a strong connection with practicing finite supports on discrete variates [23], what is a good supplement to finite game approximation theory and praxis.

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# РАСЧЕТ УСТАЛОСТНОЙ ДОЛГОВЕЧНОСТИ ЦИЛИНДРА ПРИ МНОГОСЛОЙНОМ НАРАЩИВАНИИ ЖИДКИМ МЕТАЛЛОМ ПО БОКОВОЙ ПОВЕРХНОСТИ И ПОСЛЕДУЮЩЕМ ЦИКЛИЧЕСКОМ ТЕРМОМЕХАНИЧЕСКОМ НАГРУЖЕНИИ


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Развита методика расчета остаточных напряжений в цилиндре при многослойной наплавке, эксплуатационного термомеханического состояния и долговечности при последующем циклическом термомеханическом нагружении. Получены количественные оценки указанных параметров.


*Ключевые слова: многослойное наращивание, численное моделирование, микроструктура, остаточные напряжения, циклическое нагружение, долговечность.*


# РОЗРАХУНОК ВТОМНОЇ ДОВГОВІЧНОСТІ ЦИЛІНДРА ПРИ БАГАТОШАРОВОМУ НАРОЩУВАННІ РІДКИМ МЕТАЛОМ ПО БОКОВІЙ ПОВЕРХНІ І НАСТУПНОМУ ЦИКЛІЧНОМУ ТЕРМОМЕХАНІЧНОМУ НАВАНТАЖЕННІ


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Розвинуто методику розрахунку залишкових напружень у циліндрі при багатошаровому наплавленні, термомеханічного стану та довговічності при наступному циклічному термомеханічному навантаженні. Отримано кількісні оцінки вказаних параметрів.


*Ключові слова: багатошарове нарощування, чисельне моделювання, мікроструктура, залишкові напруження, циклічне навантаження, довговічність.*