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On the approaches to the derivation of the Boltzmann equation with hard sphere collisions

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In the paper the possible approaches to the rigorous derivation of the Boltzmann kinetic equation with hard sphere collisions from underlying dynamics are considered. In particular, a formalism for the description of the evolution of infinitely many hard spheres within the framework of marginal observables in the Boltzmann–Grad scaling limit is developed. Moreover, we develop one more approach to the description of the kinetic evolution of hard spheres within the framework of a one-particle distribution function governed by the generalized Enskog equation and we establish its scaling asymptotic behavior.

В роботі розглянуто можливі підходи до строгого виведення кінетичного рівняння Больцмана з динаміки твердих куль із пружним розсіянням. Зокрема розвинуто формалізм опису еволюції нескінченної кількості твердих куль в рамках маргінальних спостережуваних в скейлінґовій границі Больцмана–Ґреда. Крім того розвинуто ще один підхід до опису кінетичної еволюції твердих куль в термінах одночастинкової функції розподілу, яка визначається узагальненим рівнянням Енскоґа, і встановлено її скейлінґову асимптотику.

1 Introduction

As is known the collective behavior of many-particle systems can be effectively described within the framework of a one-particle marginal distribution function governed by the kinetic equation in a suitable scaling limit of underlying dynamics [1]-[3]. At present the considerable advance in the rigorous derivation of the Boltzmann kinetic equation of a system of hard spheres in the scaling Boltzmann–Grad limit [4] is observed [5]-[12]. Nowadays this result was extended on many-particle systems with short-range interaction potentials [13],[14].

Another approach to the description of the many-particle evolution is given within the framework of marginal observables governed by the dual BBGKY hierarchy [15]. The main goal of this paper consists in the description of the kinetic evolution of many hard spheres in terms of the evolution of observables, i.e. the Heisenberg picture of the evolution.

We note that in case of quantum many-particle systems the possible approaches to the derivation of quantum kinetic equations from underlying dynamics were considered in reviews [16],[17].

We briefly outline the structure of the paper and the main results. In section 2 we formulate necessary preliminary facts about evolution equations of a hard sphere system and review recent rigorous results on the derivation of the Boltzmann kinetic equation from underlying dynamics. In section 3 we develop an approach to the description of the kinetic evolution of infinitely many hard spheres within the framework of the evolution of marginal observables. For this purpose we establish the Boltzmann–Grad asymptotic behavior of a solution of the Cauchy problem of the dual BBGKY hierarchy for marginal observables of hard spheres. The constructed scaling limit is governed by the set of recurrence evolution equations, namely by the dual Boltzmann hierarchy. Moreover, the links of the dual Boltzmann hierarchy for the limit marginal observables with the Boltzmann kinetic equation is established in this section. In section 4 we develop one more approach to the description of the kinetic evolution of hard spheres. We prove that the Boltzmann–Grad scaling limit of a solution of the Cauchy problem of the generalized Enskog kinetic equation is governed by the Boltzmann equation and the property on the propagation of a chaos is established. Finally, in section 5 we conclude with some observations and perspectives for future research.

2 Evolution equations of many hard spheres

It is well known that a description of many-particle systems is formulated in terms of two sets of objects: observables and states. The functional of the mean value of observables defines a duality between observables and states and as a consequence there exist two approaches to the description of the evolution. Usually the evolution of many-particle systems is described within the framework of the evolution of states by the BBGKY hierarchy for marginal distribution functions. An equivalent approach to the description of the evolution of many-particle systems is given in terms of marginal observables governed by the dual BBGKY hierarchy.

2.1 The BBGKY hierarchy for hard spheres

We consider a system of identical particles of a unit mass with the diameter $\sigma > 0$, interacting as hard spheres with elastic collisions. Every particle is characterized by the phase coordinates: $(q_i, p_i) \equiv x_i \in \mathbb{R}^3 \times \mathbb{R}^3$, $i \geq 1$.

Let $L^1_\alpha = \oplus_{n=0}^{\infty} \alpha^n L^1_n$ be the space of sequences $f = (f_0, f_1, \ldots, f_n, \ldots)$ of integrable functions $f_n(x_1, \ldots, x_n)$ defined on the phase space of n hard spheres, that are symmetric with respect to permutations of the arguments $x_1, \ldots, x_n$, equal to zero on the set of the forbidden configurations: $\mathbb{W}_n \equiv \{(q_1, \ldots, q_n) \in \mathbb{R}^{3n} \big| |q_i - q_j| < \sigma$ for at least one pair $(i,j) : i \neq j \in (1, \ldots, n)\}$ and equipped with the norm: $\|f\|_{L^1_\alpha} = \sum_{n=0}^{\infty} \alpha^n \|f_n\|_{L^1_n} = \sum_{n=0}^{\infty} \alpha^n \int dx_1 \ldots dx_n |f_n(x_1, \ldots, x_n)|$, where $\alpha > 1$ is a real number. We denote by $L^1_0 \subset L^1_\alpha$ the everywhere dense set in L^1_α of finite sequences of continuously differentiable functions with compact supports.

The evolution of all possible states of a system of a non-fixed, i.e. arbitrary but finite, number of hard spheres is described by the sequence $F(t) = (1, F_1(t, x_1), \ldots, F_s(t, x_1, \ldots, x_s), \ldots) \in L^1_\alpha$ of the marginal (s-particle) distribution functions $F_s(t, x_1, \ldots, x_s)$, $s \geq 1$, governed by the Cauchy problem of the BBGKY hierarchy [1]:

$$\frac{\partial}{\partial t} F_s(t) = \big(\sum_{j=1}^{s} \mathcal{L}^*(j) + \epsilon^2 \sum_{j_1 < j_2 = 1}^{s} \mathcal{L}^*_{\mathrm{int}}(j_1, j_2) \big) F_s(t) + \qquad (1)$$

$$+ \epsilon^2 \sum_{i=1}^{s} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{s+1} \mathcal{L}^*_{\mathrm{int}}(i, s+1) F_{s+1}(t),$$

$$F_s(t)_{|t=0} = F_s^{0,\epsilon}, \quad s \geq 1. \qquad (2)$$

In the hierarchy of evolution equations (1) represented in a dimensionless form the coefficient $\epsilon > 0$ is a scaling parameter (the ratio of the diameter $\sigma > 0$ to the mean free path of hard spheres) and, if $t \geq 0$, the operators

$\mathcal{L}^*(j)$ and $\mathcal{L}^*_{\mathrm{int}}(j_1,j_2)$ are defined on the subspace $L^1_{n,0}\subset L^1_n$ by the formulas:

$$\mathcal{L}^*(j)f_n \doteq -\langle p_j,\frac{\partial}{\partial q_j}\rangle f_n, \tag{3}$$

$$\mathcal{L}^*_{\mathrm{int}}(j_1,j_2)f_n \doteq \int\limits_{\mathbb{S}^2_+} d\eta\langle\eta,(p_{j_1}-p_{j_2})\rangle\big(f_n(x_1,\ldots,q_{j_1},p^*_{j_1},\ldots, \tag{4}$$

$$q_{j_2},p^*_{j_2},\ldots,x_n)\delta(q_{j_1}-q_{j_2}+\epsilon\eta)-f_n(x_1,\ldots,x_n)\delta(q_{j_1}-q_{j_2}-\epsilon\eta)\big),$$

where the following notations are used: the symbol $\langle\cdot,\cdot\rangle$ means a scalar product, δ is the Dirac measure, $\mathbb{S}^2_+\doteq\{\eta\in\mathbb{R}^3\big|\,|\eta|=1,\,\langle\eta,(p_{j_1}-p_{j_2})\rangle\geq 0\}$ and the momenta $p^*_{j_1},p^*_{j_2}$ are determined by the expressions:

$$\begin{aligned} p^*_{j_1} &= p_{j_1}-\eta\,\langle\eta,(p_{j_1}-p_j)\rangle\,, \\ p^*_{j_2} &= p_{j_2}+\eta\,\langle\eta,(p_{j_1}-p_{j_2})\rangle\,. \end{aligned} \tag{5}$$

The adjoint Liouville operator $\mathcal{L}^*_s=\sum_{i=1}^s\mathcal{L}^*(i)+\epsilon^2\sum_{i<j=1}^s\mathcal{L}^*_{\mathrm{int}}(i,j)$ is an infinitesimal generator of the group of operators of s hard spheres: $S^*_s(t)\equiv S^*_s(t,1,\ldots,s)$, which is adjoint to the group of operators $S_s(t)$ defined almost everywhere on the phase space $\mathbb{R}^{3s}\times(\mathbb{R}^{3s}\setminus\mathbb{W}_s)$ as the shift operator of phase space coordinates along the phase space trajectories of s hard spheres [1]. The adjoint group of operators $S^*_s(t)$ coincides with the group of operators of hard spheres $S_s(-t)$ [1].

In case of $t\leq 0$ a generator of the BBGKY hierarchy with hard sphere collisions is determined by the corresponding operator [1].

If $F(0)=(1,F^{0,\epsilon}_1,\ldots,F^{0,\epsilon}_n,\ldots)\in L^1_\alpha$ and $\alpha>e$, then for $t\in\mathbb{R}$ a unique non-perturbative solution of the Cauchy problem of the BBGKY hierarchy with hard sphere collisions (1),(2) exists and it is represented by the sequence [18]:

$$\begin{aligned} &F_s(t,x_1,\ldots,x_s)= \\ &=\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1}\ldots dx_{s+n}\mathfrak{A}_{1+n}(-t,\{Y\},X\setminus Y)F^{0,\epsilon}_{s+n}, \\ &s\geq 1, \end{aligned} \tag{6}$$

where the generating operator of the n term of series (6) is the $(n+1)th$-

order cumulant of adjoint groups of operators of hard spheres:

$$\mathfrak{A}_{1+n}(-t,\{Y\},X\setminus Y)= \\ =\sum_{\mathrm{P}:\,(\{Y\},X\setminus Y)=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}}S_{|\theta(X_i)|}(-t,\theta(X_i)), \tag{7}$$

and the following notations are used: $\{Y\}$ is a set consisting of one element $Y\equiv(1,\ldots,s)$, i.e. $|\{Y\}|=1$, $\sum_{\mathrm{P}}$ is a sum over all possible partitions P of the set $(\{Y\},X\setminus Y)\equiv(\{Y\},s+1,\ldots,s+n)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i\in(\{Y\},X\setminus Y)$, the mapping θ is the declusterization mapping defined by the formula: $\theta(\{Y\},X\setminus Y)=X$.

For initial data $F(0)\in L^1_{\alpha,0}\subset L^1_\alpha$ sequence (6) is a strong solution of the Cauchy problem (1),(2) and for arbitrary initial data from the space L^1_α it is a weak solution.

We remark, as a result of the application to cumulants (7) of analogs of the Duhamel equations, solution series (6) reduces to the iteration series of the BBGKY hierarchy (1).

In order to describe the evolution of infinitely many particles we must construct the solutions for initial data from more general Banach spaces. In the capacity of such Banach space in [1]-[3],[6] it was used the space L^∞_ξ of sequences of continuous functions defined on the phase space of hard spheres with the norm:

$$\|f\|_{L^\infty_\xi}=\sup_{n\geq0}\xi^{-n}\sup_{x_1,\ldots,x_n}|f_n(x_1,\ldots,x_n)|\exp(\frac{\beta}{2}\sum_{i=1}^n p_i^2),$$

where $\beta>0$ and $\xi>0$ are some parameters.

If $F(0)\in L^\infty_\xi$, every term of series (6) exists and this series converges uniformly on each compact almost everywhere for finite time interval. Sequence (6) is a weak unique solution of the Cauchy problem of the BBGKY hierarchy (1),(2).

2.2 On the Boltzmann–Grad asymptotic behavior

To consider the conventional approach to the derivation of the Boltzmann kinetic equation with hard sphere collisions from underlying dynamics [7] (see also [1] and references cited therein) we represent a solution of the Cauchy problem of the BBGKY hierarchy for hard spheres in the form of the perturbation (iteration) series:

$$F_s(t,x_1,\ldots,x_s) = \tag{8}$$
$$= \sum_{n=0}^{\infty} \epsilon^{2n} \int\limits_0^t dt_1 \ldots \int\limits_0^{t_{n-1}} dt_n \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1}\ldots dx_{s+n}\, S_s(-t+t_1)$$
$$\times \sum_{i_1=1}^{s} \mathcal{L}^*_{\mathrm{int}}(i_1,s+1) S_{s+1}(-t_1+t_2)\ldots$$
$$\times S_{s+n-1}(-t_n+t_n) \sum_{i_n=1}^{s+n-1} \mathcal{L}^*_{\mathrm{int}}(i_n,s+n) S_{s+n}(-t_n) F^{0,\epsilon}_{s+n}, \quad s\geq 1,$$

where the notations from formulas (3) are used.

If $F(0) \in L^{\infty}_{\xi}$, every term of series (8) exists and the iteration series converges uniformly on each compact almost everywhere for $t < t_0(\beta,\xi)$. Sequence (8) is a unique weak solution of the Cauchy problem of the BBGKY hierarchy (1) [19].

The Boltzmann–Grad asymptotic behavior of perturbative solution (8) is described by the following statement [7]:

Theorem 1 *If for initial data* $F(0) = (1, F^{0,\epsilon}_1, \ldots, F^{0,\epsilon}_n, \ldots) \in L^{\infty}_{\xi}$ *uniformly on every compact set in the phase space* $\mathbb{R}^{3n}\times(\mathbb{R}^{3n}\setminus\mathbb{W}_n)$ *it holds:* $\lim_{\epsilon\to 0} |\epsilon^{2n} F^{0,\epsilon}_n(x_1,\ldots,x_n) - f^0_n(x_1,\ldots,x_n)| = 0$, *then for any finite time interval the function* $\epsilon^{2s}F_s(t,x_1,\ldots,x_s)$ *defined by series (8) converges in the Boltzmann–Grad limit uniformly with respect to configuration variables from any compact set and in a weak sense with respect to momentum variables to the limit marginal distribution function* $f_s(t,x_1,\ldots,x_s)$ *given by the series:*

$$f_s(t,x_1,\ldots,x_s) = \tag{9}$$
$$= \sum_{n=0}^{\infty} \int\limits_0^t dt_1 \ldots \int\limits_0^{t_{n-1}} dt_n \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1}\ldots dx_{s+n} \prod_{i_1=1}^{s} S_1(-t+t_1,i_1)$$
$$\times \sum_{k_1=1}^{s} \mathcal{L}^{0,*}_{\mathrm{int}}(k_1,s+1) \prod_{j_1=1}^{s+1} S_1(-t_1+t_2,j_1)\ldots$$
$$\times \prod_{i_n=1}^{s+n-1} S_1(-t_n+t_n,i_n) \sum_{k_n=1}^{s+n-1} \mathcal{L}^{0,*}_{\mathrm{int}}(k_n,s+n) \prod_{j_n=1}^{s+n} S_1(-t_n,j_n) f^0_{s+n}.$$

In series (9) it was introduced the operator:

$$(\mathcal{L}_{\mathrm{int}}^{0,*}(j_1,j_2)f_n)(x_1,\ldots,x_n) \doteq \int_{\mathbb{S}_+^2} d\eta\langle\eta,(p_{j_1}-p_{j_2})\rangle\big(f_n(x_1,\ldots \\ q_{j_1},p_{j_1}^*,\ldots q_{j_2},p_{j_2}^*,\ldots,x_n)-f_n(x_1,\ldots,x_n)\big)\delta(q_{j_1}-q_{j_2}), \tag{10}$$

and the momenta $p_{j_1}^*, p_{j_2}^*$ are determined by expressions (5).

If $f(0)=(f_0,f_1^0,\ldots,f_n^0,\ldots)\in L_\xi^\infty$, every term of series (9) exists and this series converges uniformly on each compact almost everywhere for $t<t_0(\beta,\xi)$.

We note that for $t\geq 0$ sequence (9) is a weak solution of the Cauchy problem of the limit BBGKY hierarchy known as the Boltzmann hierarchy with hard sphere collisions:

$$\frac{\partial}{\partial t}f_s(t)=\sum_{j=1}^{n}\mathcal{L}^*(j)f_s(t)+\sum_{i=1}^{s}\int_{\mathbb{R}^3\times\mathbb{R}^3} dx_{s+1}\mathcal{L}_{\mathrm{int}}^{0,*}(i,s+1)f_{s+1}(t), \tag{11}$$

$$f_s(t)_{|t=0}=f_s^0, \quad s\geq 1, \tag{12}$$

where the operator $\mathcal{L}_{\mathrm{int}}^{0,*}(i,s+1)$ is defined by formula (10).

We remark that the same statement takes place concerning the Boltzmann–Grad behavior of non-perturbative solution (6) of the Cauchy problem of the BBGKY hierarchy for hard spheres.

To derive the Boltzmann kinetic equation [12] we will consider initial data (2) satisfying the chaos condition [1], which means the absence of correlations at initial time (statistically independent hard spheres)

$$F_s^{0,\epsilon}(x_1,\ldots,x_s)=\prod_{i=1}^{s}F_1^{0,\epsilon}(x_i)\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s}, \quad s\geq 1, \tag{13}$$

where $\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s}$ is a characteristic function of the set $\mathbb{R}^{3s}\setminus\mathbb{W}_s$ of allowed configurations. Such assumption about initial data is intrinsic for the kinetic description of a gas, because in this case all possible states are characterized only by a one-particle marginal distribution function.

Since the initial limit marginal distribution functions satisfy a chaos property too

$$f_s^0(x_1,\ldots,x_s)=\prod_{i=1}^{s}f_1^0(x_i), \quad s\geq 2, \tag{14}$$

perturbative solution (9) of the Cauchy problem of the Botzmann hierarchy (11),(12) has the following property (the propagation of initial chaos):

$$f_s(t,x_1,\ldots,x_s)=\prod_{i=1}^{s}f_1(t,x_i),\quad s\geq 2,$$

where for $t\geq 0$ the limit one-particle distribution function is defined by series (9) in case of $s=1$ and initial data (14). This limit one-particle distribution function is governed by the Boltzmann kinetic equation with hard sphere collisions [1]:

$$\frac{\partial}{\partial t}f_1(t,x_1)=-\langle p_1,\frac{\partial}{\partial q_1}\rangle f_1(t,x_1)+\int\limits_{\mathbb{R}^3\times\mathbb{S}^2_+}dp_2\,d\eta\,\langle\eta,(p_1-p_2)\rangle$$
$$\times\big(f_1(t,q_1,p_1^*)f_1(t,q_1,p_2^*)-f_1(t,x_1)f_1(t,q_1,p_2)\big),$$

where the momenta p_1^* and p_2^* are defined by expressions (5).

2.3 The dual BBGKY hierarchy for hard spheres

Let C_γ be the space of sequences $b=(b_0,b_1,\ldots,b_n,\ldots)$ of continuous functions $b_n\in C_n$ equipped with the norm: $\|b\|_{C_\gamma}=\max_{n\geq 0}\frac{\gamma^n}{n!}\|b_n\|_{C_n}=\max_{n\geq 0}\frac{\gamma^n}{n!}\sup_{x_1,\ldots,x_n}|b_n(x_1,\ldots,x_n)|$, and $C_\gamma^0\subset C_\gamma$ is the subspace of finite sequences of infinitely differentiable functions with compact supports.

If $t\geq 0$, the evolution of marginal observables of a system of a non-fixed number of hard spheres is described by the Cauchy problem of the dual BBGKY hierarchy [15]:

$$\frac{\partial}{\partial t}B_s(t)=\big(\sum_{j=1}^{s}\mathcal{L}(j)+\epsilon^2\sum_{j_1<j_2=1}^{s}\mathcal{L}_{\mathrm{int}}(j_1,j_2)\big)B_s(t)+ \tag{15}$$
$$+\epsilon^2\sum_{j_1\neq j_2=1}^{s}\mathcal{L}_{\mathrm{int}}(j_1,j_2)B_{s-1}(t,x_1,\ldots,x_{j_1-1},x_{j_1+1},\ldots,x_s),$$

$$B_s(t,x_1,\ldots,x_s)\mid_{t=0}=B_s^{\epsilon,0}(x_1,\ldots,x_s),\quad s\geq 1. \tag{16}$$

In recurrence evolution equations (15), as above, the coefficient $\epsilon>0$ is a scaling parameter (the ratio of the diameter $\sigma>0$ to the mean free path of hard spheres) and the dimensionless operators $\mathcal{L}(j)$ and $\mathcal{L}_{\mathrm{int}}(j_1,j_2)$ are defined on the subspace C_s^0 by the formulas:

$$\mathcal{L}(j)b_n \doteq \langle p_j, \frac{\partial}{\partial q_j}\rangle b_n, \tag{17}$$

$$\mathcal{L}_{\mathrm{int}}(j_1,j_2)b_n \doteq \int\limits_{\mathbb{S}^2_+} d\eta \langle \eta,(p_{j_1}-p_{j_2})\rangle \big(b_n(x_1,\ldots,q_{j_1},p^*_{j_1},\ldots \tag{18}$$
$$q_{j_2},p^*_{j_2},\ldots,x_n)-b_n(x_1,\ldots,x_n)\big)\delta(q_{j_1}-q_{j_2}+\epsilon\eta),$$

where the momenta $p^*_{j_1},p^*_{j_2}$ are determined by expressions (5) and $\mathbb{S}^2_+ \doteq \{\eta\in\mathbb{R}^3 \big| \,|\eta|=1,\, \langle\eta,(p_{j_2}-p_{j_2})\rangle \geq 0\}$. Operators (3) and (4) are adjoint operators to operators (17) and (18), respectively. If $t\leq 0$, a generator of the dual BBGKY hierarchy is determined by corresponding operator [15].

Let $Y\equiv(1,\ldots,s), Z\equiv(j_1,\ldots,j_n)\subset Y$ and $\{Y\setminus Z\}$ is the set consisting from one element $Y\setminus Z=(1,\ldots,j_1-1,j_1+1,\ldots,j_n-1,j_n+1,\ldots,s)$.

The solution $B(t)=(B_0,B_1(t,x_1),\ldots,B_s(t,x_1,\ldots,x_s),\ldots)$ of the Cauchy problem (15),(16) is determined by the sequence:

$$B_s(t,x_1,\ldots,x_s)=\sum_{n=0}^{s}\frac{1}{n!}\sum_{j_1\neq\ldots\neq j_n=1}^{s}\mathfrak{A}_{1+n}\big(t,\{Y\setminus Z\},Z\big)B^{\epsilon,0}_{s-n}(x_1, \tag{19}$$
$$\ldots x_{j_1-1},x_{j_1+1},\ldots,x_{j_n-1},x_{j_n+1},\ldots,x_s), \quad s\geq 1,$$

where the generating operator of n term of this expansion is the $(1+n)th$-order cumulant of the groups of operators of hard spheres defined by the formula:

$$\mathfrak{A}_{1+n}(t,\{Y\setminus Z\},Z)\doteq \tag{20}$$
$$\doteq \sum_{\mathrm{P}:\,(\{Y\setminus Z\},Z)=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}}S_{|\theta(X_i)|}(t,\theta(X_i)),$$

and notations accepted in (7) are used.

If $\gamma<e^{-1}$, then for $B(0)=(B_0,B_1^{\epsilon,0},\ldots,B_s^{\epsilon,0},\ldots)\in C^0_\gamma\subset C_\gamma$ of finite sequences of infinitely differentiable functions with compact supports it is a classical solution and for arbitrary initial data $B(0)\in C_\gamma$ it is a generalized solution.

We remark that expansion (19) can be also represented in the form of the perturbation (iteration) series of the dual BBGKY hierarchy (15) [15] as a result of applying of analogs of the Duhamel equations to cumulants of the groups of operators of hard spheres (20).

The one component sequences of marginal observables correspond to observables of certain structure, namely the marginal observable $b^{(1)} = (0, b_1^{(1)}(x_1), 0, \ldots)$ corresponds to the additive-type observable, and the marginal observable $b^{(k)} = (0, \ldots, 0, b_k^{(k)}(x_1, \ldots, x_k), 0, \ldots)$ corresponds to the k-ary-type observable [15]. If in capacity of initial data (16) we consider the additive-type marginal observable, then the structure of solution expansion (19) is simplified and attains the form

$$B_s^{(1)}(t, x_1, \ldots, x_s) = \mathfrak{A}_s(t, 1, \ldots, s) \sum_{j=1}^{s} B_1^{(1),\epsilon}(0, x_j), \quad s \geq 1. \tag{21}$$

We note that the mean value of the marginal observable $B(t) \in C_\gamma$ at $t \in \mathbb{R}$ in the initial marginal state $F(0) = (1, F_1^{\epsilon,0}, \ldots, F_n^{\epsilon,0}, \ldots) \in L^1 = \bigoplus_{n=0}^{\infty} L_n^1$ is defined by the following functional:

$$\begin{aligned} &\big\langle B(t) \big| F(0) \big\rangle = \\ &= \sum_{s=0}^{\infty} \frac{1}{s!} \int\limits_{(\mathbb{R}^3 \times \mathbb{R}^3)^s} dx_1 \ldots dx_s \, B_s(t, x_1, \ldots, x_s) F_s^{\epsilon,0}(x_1, \ldots, x_s). \end{aligned} \tag{22}$$

Owing to the estimate: $\|B(t)\|_{C_\gamma} \leq e^2(1-\gamma e)^{-1}\|B(0)\|_{C_\gamma}$, functional (22) exists under the condition that: $\gamma < e^{-1}$. In case of $F(0) \in L_\xi^\infty$ the existence of mean value functional (22) is proved in the one-dimensional space in paper [20].

In particular, functional (22) of mean values of the additive-type marginal observables $B^{(1)}(0) = (0, B_1^{(1),\epsilon}(0, x_1), 0, \ldots)$ takes the form:

$$\begin{aligned} &\big\langle B^{(1)}(t) \big| F(0) \big\rangle = \big\langle B^{(1)}(0) \big| F(t) \big\rangle = \\ &= \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 \, B_1^{(1),\epsilon}(0, x_1) F_1(t, x_1), \end{aligned}$$

In the general case for mean values of marginal observables the following equality is true:

$$\big\langle B(t) \big| F(0) \big\rangle = \big\langle B(0) \big| F(t) \big\rangle, \tag{23}$$

where the sequence $F(t)$ is given by formula (6). This equality signify the equivalence of two pictures of the description of the evolution of hard spheres by means of the BBGKY hierarchy (1) and the dual BBGKY hierarchy (15).

2.4 The generalized Enskog kinetic equation

In paper [21] it was established that, if initial state of a hard sphere system is specified by a one-particle distribution function on allowed configurations, then at arbitrary moment of time the evolution of states governed by the BBGKY hierarchy can be completely described within the framework of the one-particle marginal distribution function $F_1(t)$ governed by the generalized Enskog kinetic equation. In this case the all possible correlations, creating by hard sphere dynamics, are described in terms of the marginal functionals of the state $F_s\big(t \mid F_1(t)\big)$, $s \geq 2$.

In case of $t \geq 0$ the one-particle distribution function is governed by the Cauchy problem of the following generalized Enskog equation [21]:

$$\begin{aligned}
&\frac{\partial}{\partial t}F_1(t,x_1) = -\langle p_1, \frac{\partial}{\partial q_1}\rangle F_1(t,x_1) + \qquad (24)\\
&+\epsilon^2 \int\limits_{\mathbb{R}^3\times\mathbb{S}^2_+} dp_2 d\eta \,\langle \eta, (p_1 - p_2)\rangle \\
&\times\big(F_2(t, q_1, p_1^*, q_1 - \epsilon\eta, p_2^* \mid F_1(t)) - F_2(t, x_1, q_1 + \epsilon\eta, p_2 \mid F_1(t))\big),
\end{aligned}$$

$$F_1(t,x_1)|_{t=0} = F_1^{\epsilon,0}(x_1), \qquad (25)$$

where $\epsilon > 0$ is a scaling parameter, the momenta p_1^*, p_2^* are determined by expressions (5), $\mathbb{S}^2_+ \doteq \{\eta \in \mathbb{R}^3 \big| \, |\eta| = 1, \, \langle \eta, (p_1 - p_2)\rangle \geq 0\}$ and the collision integral in kinetic equation (24) is represented in terms of the marginal functionals of the state $F_s(t, x_1, \ldots, x_s \mid F_1(t))$, $s \geq 2$, in case of $s = 2$:

$$\begin{aligned}
&F_s(t, x_1, \ldots, x_s \mid F_1(t)) \doteq \qquad (26)\\
&\sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n}\, \mathfrak{V}_{1+n}(t, \{Y\}, X \setminus Y) \prod_{i=1}^{s+n} F_1(t, x_i).
\end{aligned}$$

In series expansion (26) the following notations are used: $Y \equiv (1, \ldots, s)$, $X \equiv (1, \ldots, s+n)$, and the $(n+1)th$-order generating evolution operator $\mathfrak{V}_{1+n}(t)$, $n\geq 0$, is defined by the expansion:

$$\mathfrak{V}_{1+n}(t,\{Y\},X\setminus Y)\doteq \tag{27}$$
$$\doteq\sum_{k=0}^{n}(-1)^k\sum_{m_1=1}^{n}\ldots\sum_{m_k=1}^{n-m_1-\ldots-m_{k-1}}\frac{n!}{(n-m_1-\ldots-m_k)!}$$
$$\times\widehat{\mathfrak{A}}_{1+n-m_1-\ldots-m_k}(t)\prod_{j=1}^{k}\sum_{k_2^j=0}^{m_j}\ldots\sum_{k^j_{n-m_1-\ldots-m_j+s}=0}^{k^j_{n-m_1-\ldots-m_j+s-1}}$$
$$\prod_{i_j=1}^{s+n-m_1-\ldots-m_j}\frac{1}{(k^j_{n-m_1-\ldots-m_j+s+1-i_j}-k^j_{n-m_1-\ldots-m_j+s+2-i_j})!}$$
$$\times\widehat{\mathfrak{A}}_{1+k^j_{n-m_1-\ldots-m_j+s+1-i_j}-k^j_{n-m_1-\ldots-m_j+s+2-i_j}}(t,i_j,$$
$$s+n-m_1-\ldots-m_j+1+k^j_{s+n-m_1-\ldots-m_j+2-i_j},\ldots,$$
$$s+n-m_1-\ldots-m_j+k^j_{s+n-m_1-\ldots-m_j+1-i_j}),$$

where it means that: $k_1^j\equiv m_j$ and $k^j_{n-m_1-\ldots-m_j+s+1}\equiv 0$. In expression (27) we denote by the evolution operator $\widehat{\mathfrak{A}}_{1+n-m_1-\ldots-m_k}(t)\equiv\widehat{\mathfrak{A}}_{1+n-m_1-\ldots-m_k}(t,\{Y\},s+1,\ldots,s+n-m_1-\ldots-m_k)$ the $(n-m_1-\ldots-m_k)th$-order scattering cumulant, namely

$$\widehat{\mathfrak{A}}_{1+n}(t,\{Y\},X\setminus Y)\doteq$$
$$\doteq\mathfrak{A}_{1+n}(-t,\{Y\},X\setminus Y)\mathcal{X}_{\mathbb{R}^{3(s+n)}\setminus\mathbb{W}_{s+n}}\prod_{i=1}^{s+n}\mathfrak{A}_1(t,i),\quad n\geq 1,$$

where the operator $\mathfrak{A}_{1+n}(-t)$ is $(1+n)th$-order cumulant (7) of the adjoint groups of operators of hard spheres. We give several simplest examples of generating evolution operators (27):

$$\mathfrak{V}_1(t,\{Y\})=\widehat{\mathfrak{A}}_1(t,\{Y\})\doteq S_s(-t,1,\ldots,s)\mathfrak{I}_s(Y)\prod_{i=1}^{s}S_1(t,i),$$
$$\mathfrak{V}_2(t,\{Y\},s+1)=\widehat{\mathfrak{A}}_2(t,\{Y\},s+1)-\widehat{\mathfrak{A}}_1(t,\{Y\})\sum_{i_1=1}^{s}\widehat{\mathfrak{A}}_2(t,i_1,s+1).$$

If $\|F_1(t)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}<e^{-(3s+2)}$, series of marginal functional of the state (26) converges in the norm of the space L^1_s for arbitrary $t\in\mathbb{R}$, and thus,

the collision integral series in kinetic equation (24) converges under the condition that: $\|F_1(t)\|_{L^1(\mathbb{R}\times\mathbb{R})} < e^{-8}$ [21].

A solution of the Cauchy problem (24),(25) is represented by the series:

$$F_1(t,x_1) = \tag{28}$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_2\dots dx_{n+1}\,\mathfrak{A}_{1+n}(-t)\prod_{i=1}^{n+1}F_1^{\epsilon,0}(x_i)\mathcal{X}_{\mathbb{R}^{3(n+1)}\setminus\mathbb{W}_{n+1}},$$

where the generating operator $\mathfrak{A}_{1+n}(-t) \equiv \mathfrak{A}_{1+n}(-t,1,\dots,n+1)$ is the $(1+n)th$-order cumulant (7) of adjoint groups of operators of hard spheres. If initial one-particle distribution function $F_1^{\epsilon,0}$ is a continuously differentiable integrable function with compact support, then function (28) is a strong solution of initial-value problem (24),(25) and for the arbitrary integrable function $F_1^{\epsilon,0}$ it is a weak solution [21].

If initial one-particle marginal distribution function satisfies the following condition:

$$|F_1^{\epsilon,0}(x_1)| \leq ce^{-\frac{\beta}{2}p_1^2}, \tag{29}$$

where $\beta > 0$ is a parameter, $c < \infty$ is some constant, then every term of series (28) exists, series (28) converges uniformly on each compact almost everywhere with respect to x_1 for finite time interval and function (28) is a unique weak solution of the generalized Enskog kinetic equation (24).

The proof of the last statement is based on analogs of the Duhamel equations for cumulants of groups of operators (7) and estimates established for the iteration series of the BBGKY hierarchy for hard spheres [1].

We point out the relationship of the description of the evolution of many hard spheres in terms of the marginal observables and by the one-particle marginal distribution function governed by the generalized Enskog equation (24).

For mean value functional (22) the following equality holds:

$$\big\langle B(t)\big|F^c(0)\big\rangle = \big\langle B(0)\big|F(t\mid F_1(t))\big\rangle, \tag{30}$$

where $F^c(0) = \big(1, F_1^{0,\epsilon}(x_1),\dots,\prod_{i=1}^n F_1^{0,\epsilon}(x_i)\mathcal{X}_{\mathbb{R}^{3n}\setminus\mathbb{W}_n}\big)$ is a sequence of initial marginal distribution functions and $F(t\mid F_1(t)) = \big(1, F_1(t), F_2(t\mid F_1(t)),\dots,F_s(t\mid F_1(t))\big)$ is the sequence which consists from solution expansion (28) and marginal functionals of the state (26).

In particular case of the s-ary initial marginal observable $B^{(s)}(0) = (0,\ldots,0,B_s^{(s),\epsilon}(0,x_1,\ldots,x_s),0,\ldots)$, $s\geq 2$, established equality (30) takes the form

$$\big\langle B^{(s)}(t)\big|F^c(0)\big\rangle = \big\langle B^{(s)}(0)\big|F(t\mid F_1(t))\big\rangle = \\ = \frac{1}{s!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1\ldots dx_s\, B_s^{(s),\epsilon}(0,x_1,\ldots,x_s)F_s(t,x_1,\ldots,x_s\mid F_1(t)),$$

where the marginal functionals of the state $F_s(t\mid F_1(t))$ are determined by series (26).

Correspondingly, in case of the additive-type marginal observables $B^{(1)}(0) = (0,B_1^{(1),\epsilon}(0,x_1),0,\ldots)$ equality (30) takes the form

$$\big\langle B^{(1)}(t)\big|F^c(0)\big\rangle = \big\langle B^{(1)}(0)\big|F(t\mid F_1(t))\big\rangle = \\ = \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx_1\, B_1^{(1),\epsilon}(0,x_1)F_1(t,x_1),$$

where the one-particle marginal distribution function $F_1(t)$ is determined by series (28). Therefore for the additive-type marginal observables the generalized Enskog kinetic equation (24) is dual to the dual BBGKY hierarchy for hard spheres (15) with respect to bilinear form (22).

Thus, if the initial state is completely specified by the one-particle distribution function on allowed configurations, then the evolution of hard spheres governed by the dual BBGKY hierarchy (15) for marginal observables of hard spheres can be completely described in terms of the generalized Enskog kinetic equation (24) and by the sequence of marginal functionals of the state (26).

3 The kinetic evolution within the framework of marginal observables

In this section we consider the problem of the rigorous description of the kinetic evolution within the framework of many-particle dynamics of observables by giving an example of the Boltzmann–Grad asymptotic behavior of a solution of the dual BBGKY hierarchy with hard sphere collisions. Moreover, we establish the links of the dual Boltzmann hierarchy for the limit marginal observables with the Boltzmann kinetic equation.

3.1 The Boltzmann–Grad asymptotic behavior of the dual BBGKY hierarchy

The Boltzmann–Grad scaling limit of non-perturbative solution (19) of the Cauchy problem of the dual BBGKY hierarchy (15),(16) is described by the following statement.

Theorem 2 *Let for* $B_n^{\epsilon,0} \in C_n$, $n \geq 1$, *it holds:* $\mathrm{w}^*\!-\lim_{\epsilon\to 0}(\epsilon^{-2n}B_n^{\epsilon,0} - b_n^0) = 0$, *then for arbitrary finite time interval there exists the Boltzmann–Grad limit of solution (19) of the Cauchy problem of the dual BBGKY hierarchy (15),(16) in the sense of the* $*$*-weak convergence on the space* C_s

$$\mathrm{w}^*\!-\lim_{\epsilon\to 0}\big(\epsilon^{-2s}B_s^{\epsilon}(t) - b_s(t)\big) = 0, \tag{31}$$

which is determined by the following expansion:

$$b_s(t) = \sum_{n=0}^{s-1}\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n S_s^0(t-t_1) \sum_{i_1\neq j_1=1}^{s} \mathcal{L}_{\mathrm{int}}^0(i_1,j_1) \tag{32}$$
$$\times S_{s-1}^0(t_1-t_2)\ldots S_{s-n+1}^0(t_{n-1}-t_n) \sum_{\substack{i_n\neq j_n=1,\\ i_n,j_n\neq (j_1,\ldots,j_{n-1})}}^{s} \mathcal{L}_{\mathrm{int}}^0(i_n,j_n)$$
$$\times S_{s-n}^0(t_n) b_{s-n}^0((x_1,\ldots,x_s)\setminus(x_{j_1},\ldots,x_{j_n})), \quad s\geq 1.$$

In expansion (32) for groups of operators of noninteracting particles the following notations are used:

$$S_{s-n+1}^0(t_{n-1}-t_n) \equiv S_{s-n+1}^0(t_{n-1}-t_n, Y\setminus(j_1,\ldots,j_{n-1})) =$$
$$= \prod_{j\in Y\setminus(j_1,\ldots,j_{n-1})} S_1(t_{n-1}-t_n, j),$$

and we denote by $\mathcal{L}_{\mathrm{int}}^0(j_1,j_2)$ the operator:

$$(\mathcal{L}_{\mathrm{int}}^0(j_1,j_2)b_n)(x_1,\ldots,x_n) \doteq \int_{\mathbb{S}_+^2} d\eta \langle\eta,(p_{j_1}-p_{j_2})\rangle\big(b_n(x_1,\ldots \tag{33}$$
$$q_{j_1},p_{j_1}^*,\ldots q_{j_2},p_{j_2}^*,\ldots,x_n) - b_n(x_1,\ldots,x_n)\big)\delta(q_{j_1}-q_{j_2}),$$

where $\mathbb{S}_+^2 \doteq \{\eta\in\mathbb{R}^3 \big| \, |\eta|=1, \langle\eta,(p_{j_1}-p_{j_2})\rangle \geq 0\}$ and the momenta $p_{j_1}^*, p_{j_2}^*$ are determined by expressions (5).

Before to consider the proof scheme of the theorem we give some comments.

If $b^0 \in C_\gamma$, then the sequence $b(t) = (b_0, b_1(t), \ldots, b_s(t), \ldots)$ of limit marginal observables (32) is a generalized global solution of the Cauchy problem of the dual Boltzmann hierarchy with hard sphere collisions

$$\frac{d}{dt} b_s(t, x_1, \ldots, x_s) = \sum_{j=1}^{s} \mathcal{L}(j)\, b_s(t, x_1, \ldots, x_s) + \tag{34}$$

$$+ \sum_{j_1 \neq j_2 = 1}^{s} \mathcal{L}^0_{\mathrm{int}}(j_1, j_2) b_{s-1}(t, x_1, \ldots, x_{j_1 - 1}, x_{j_1 + 1}, \ldots, x_s),$$

$$b_s(t, x_1, \ldots, x_s) \mid_{t=0} = b^0_s(x_1, \ldots, x_s), \quad s \geq 1. \tag{35}$$

It should be noted that equations set (34) has the structure of recurrence evolution equations. We give several examples of the evolution equations of the dual Boltzmann hierarchy (34)

$$\frac{\partial}{\partial t} b_1(t, x_1) = \langle p_1, \frac{\partial}{\partial q_1} \rangle\, b_1(t, x_1),$$

$$\frac{\partial}{\partial t} b_2(t, x_1, x_2) = \sum_{j=1}^{2} \langle p_j, \frac{\partial}{\partial q_j} \rangle\, b_2(t, x_1, x_2) + \int_{\mathbb{S}^2_+} d\eta \langle \eta, (p_1 - p_2) \rangle \times$$

$$\big(b_1(t, q_1, p^*_1) - b_1(t, x_1) + b_1(t, q_2, p^*_2) - b_1(t, x_2)\big) \delta(q_1 - q_2).$$

The proof of the limit theorem for the dual BBGKY hierarchy is based on formulas for cumulants of asymptotically perturbed groups of operators of hard spheres.

For arbitrary finite time interval the asymptotically perturbed group of operators of hard spheres has the following scaling limit in the sense of the $*$-weak convergence on the space C_s:

$$\mathrm{w}^*\!-\lim_{\epsilon \to 0} \big(S_s(t) b_s - \prod_{j=1}^{s} S_1(t, j) b_s\big) = 0. \tag{36}$$

Taking into account analogs of the Duhamel equations for cumulants of asymptotically perturbed groups of operators, in view of formula (36) we have

$$\mathrm{w}^*-\lim_{\epsilon\to 0}\Big(\epsilon^{-2n}\frac{1}{n!}\mathfrak{A}_{1+n}\big(t,\{Y\setminus X\},j_1,\ldots,j_n\big)\,b_{s-n}-$$
$$-\int_0^t dt_1\ldots\int_0^{t_{n-1}}dt_n\,S_s^0(t-t_1)\sum_{\substack{i_1=1,\\ i_1\neq j_1}}^{s}\mathcal{L}^0_{\mathrm{int}}(i_1,j_1)\,S_{s-1}(t_1-t_2)\ldots$$
$$S^0_{s-n+1}(t_{n-1}-t_n)\sum_{\substack{i_n=1,\\ i_n\neq(j_1,\ldots,j_n)}}^{s}\mathcal{L}^0_{\mathrm{int}}(i_n,j_n)S^0_{s-n}(t_n)\,b_{s-n}\Big)=0,$$

where we used notations accepted in formula (32) and $b_{s-n}\equiv b_{s-n}((x_1,\ldots,x_s)\setminus(x_{j_1},\ldots,x_{j_n}))$. As a result of this equality we establish the validity of statement (31) for solution expansion (19) of the dual BBGKY hierarchy with hard sphere collisions (15).

We consider the Boltzmann–Grad limit of a particular case of marginal observables, namely the additive-type marginal observables. As it was noted above in this case solution (19) of the dual BBGKY hierarchy (15) is represented by formula (21).

If for the additive-type marginal observable $B_1^{(1),\epsilon}(0)$ the following condition is satisfied:

$$\mathrm{w}^*-\lim_{\epsilon\to 0}\big(\epsilon^{-2}B_1^{(1),\epsilon}(0)-b_1^{(1)}(0)\big)=0,$$

then, according to statement (31), for additive-type marginal observable (21) we have

$$\mathrm{w}^*-\lim_{\epsilon\to 0}\big(\epsilon^{-2s}B_s^{(1),\epsilon}(t)-b_s^{(1)}(t)\big)=0,$$

where the limit marginal observable $b_s^{(1)}(t)$ is determined as a special case of expansion (32):

$$b_s^{(1)}(t,x_1,\ldots,x_s)=\int_0^t dt_1\ldots\int_0^{t_{s-2}}dt_{s-1}S_s^0(t-t_1)\sum_{i_1\neq j_1=1}^{s}\mathcal{L}^0_{\mathrm{int}}(i_1,j_1)\quad(37)$$
$$\times S^0_{s-1}(t_1-t_2)\ldots S^0_2(t_{s-2}-t_{s-1})\sum_{\substack{i_{s-1}\neq j_{s-1}=1,\\ i_{s-1},j_{s-1}\neq(j_1,\ldots,j_{s-2})}}^{s}\mathcal{L}^0_{\mathrm{int}}(i_{s-1},j_{s-1})$$
$$\times S_1^0(t_{s-1})\,b_1^{(1)}(0,(x_1,\ldots,x_s)\setminus(x_{j_1},\ldots,x_{j_{s-1}})),\quad s\geq 1.$$

We make several examples of the limit additive-type marginal observable expansions (37):

$$b_1^{(1)}(t,x_1) = S_1(t,1)\, b_1^{(1)}(0,x_1),$$
$$b_2^{(1)}(t,x_1,x_2) = \int\limits_0^t dt_1 \prod_{i=1}^{2} S_1(t-t_1,i)\, \mathcal{L}_{\mathrm{int}}^0(1,2) \sum_{j=1}^{2} S_1(t_1,j)\, b_1^{(1)}(0,x_j).$$

Thus, in the Boltzmann–Grad scaling limit the kinetic evolution of hard spheres is described in terms of limit marginal observables (32) governed by the dual Boltzmann hierarchy (34). Similar approach to the description of the mean field asymptotic behavior of quantum many-particle systems was developed in [22].

3.2 The derivation of the Boltzmann kinetic equation

We consider links of the constructed Boltzmann–Grad asymptotic behavior of marginal observables with the nonlinear Boltzmann equation. Furthermore, the relations between the evolution of observables and the description of the kinetic evolution of states in terms of a one-particle marginal distribution function are discussed.

For the additive-type marginal observables the Boltzmann–Grad scaling limit gives an equivalent approach to the description of the kinetic evolution of hard spheres in terms of the Cauchy problem of the Boltzmann equation with respect to the Cauchy problem of the dual Boltzmann hierarchy (34),(35). In case of the k-ary marginal observable a solution of the dual Boltzmann hierarchy (34) is equivalent to the property of the propagation of initial chaos for the k-particle marginal distribution function in the sense of equality (23).

If $b(t) \in C_\gamma$ and $f_1^0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, then under the condition that: $\|f_1^0\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} < \gamma$, there exists the Boltzmann–Grad limit of mean value functional (22) which is determined by the series

$$\big\langle b(t) \big| f^{(c)} \big\rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1 \dots dx_s\, b_s(t,x_1,\dots,x_s) \prod_{i=1}^{s} f_1^0(x_i).$$

Consequently for the limit additive-type marginal observables (37) the following equality is true:

$$\begin{aligned}
&\big\langle b^{(1)}(t)\big|f^{(c)}\big\rangle = \\
&= \sum_{s=0}^{\infty}\frac{1}{s!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1\ldots dx_s\, b_s^{(1)}(t,x_1,\ldots,x_s)\prod_{i=1}^{s} f_1^0(x_i) = \\
&= \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx_1\, b_1^{(1)}(0,x_1) f_1(t,x_1),
\end{aligned}$$

where the function $b_s^{(1)}(t)$ is given by expansion (37) and the limit marginal distribution function $f_1(t,x_1)$ is represented by the series:

$$\begin{aligned}
&f_1(t,x_1) = \qquad (38)\\
&= \sum_{n=0}^{\infty}\int\limits_0^t dt_1\ldots\int\limits_0^{t_{n-1}} dt_n \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_2\ldots dx_{n+1} S_1(-t+t_1,1)\\
&\times\mathcal{L}_{\mathrm{int}}^{0,*}(1,2)\prod_{j_1=1}^{2} S_1(-t_1+t_2,j_1)\ldots\prod_{i_n=1}^{n} S_1(-t_n+t_n,i_n)\\
&\times\sum_{k_n=1}^{n}\mathcal{L}_{\mathrm{int}}^{0,*}(k_n,n+1)\prod_{j_n=1}^{n+1} S_1(-t_n,j_n)\prod_{i=1}^{n+1} f_1^0(x_i).
\end{aligned}$$

In series (38) the operator (10) adjoint to operator (33) in the sense of functional (22) is used.

If the function f_1^0 is continuous, every term of series (38) exists and this series converges uniformly on each compact almost everywhere for finite time interval.

For $t \geq 0$ limit marginal distribution function (38) is a weak solution of the Cauchy problem of the Boltzmann kinetic equation with hard sphere collisions:

$$\begin{aligned}
\frac{\partial}{\partial t} f_1(t,x_1) = -\langle p_1,\frac{\partial}{\partial q_1}\rangle f_1(t,x_1) + \int\limits_{\mathbb{R}^3\times\mathbb{S}_+^2} dp_2\, d\eta\, \langle\eta,(p_1-p_2)\rangle \qquad (39)\\
\times\big(f_1(t,q_1,p_1^*)f_1(t,q_1,p_2^*) - f_1(t,x_1)f_1(t,q_1,p_2)\big),
\end{aligned}$$

$$f_1(t,x_1)_{|t=0} = f_1^0(x_1), \qquad (40)$$

where the momenta p_1^* and p_2^* are determined by expressions (5).

Thus, we establish that the dual Boltzmann hierarchy with hard sphere collisions (34) for additive-type marginal observables and initial states specified by one-particle marginal distribution function (14) describes the evolution of a hard sphere system just as the Boltzmann kinetic equation with hard sphere collisions (39).

3.3 On the propagation of initial chaos

We prove that within the framework of the Heisenberg picture of the evolution of a hard sphere system a chaos property of states is fulfilled.

The property of the propagation of initial chaos is a consequence of the validity of the following equality for the mean value functionals of the limit k-ary marginal observables in case of $k \geq 2$:

$$\begin{aligned}
&\big\langle b^{(k)}(t)\big|f^{(c)}\big\rangle = \\
&= \sum_{s=0}^{\infty} \frac{1}{s!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1 \ldots dx_s \, b_s^{(k)}(t, x_1, \ldots, x_s) \prod_{i=1}^{s} f_1^0(x_i) \\
&= \frac{1}{k!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^k} dx_1 \ldots dx_k \, b_k^{(k)}(0, x_1, \ldots, x_k) \prod_{i=1}^{k} f_1(t, x_i), \\
&k \geq 2,
\end{aligned} \tag{41}$$

where the limit one-particle marginal distribution function $f_1(t, x_i)$ is defined by series (38) and therefore it is governed by the Cauchy problem of the Boltzmann kinetic equation with hard sphere collisions (39),(40).

Thus, in the Boltzmann–Grad scaling limit an equivalent approach to the description of the kinetic evolution of hard spheres in terms of the Cauchy problem of the Boltzmann kinetic equation (39),(40) is given by the Cauchy problem of the dual Boltzmann hierarchy (34),(35) for the additive-type marginal observables. In case of the k-ary marginal observables a solution of the dual Boltzmann hierarchy (34) is equivalent to a chaos property for the k-particle marginal distribution function in the sense of equality (41) or in other words the Boltzmann–Grad scaling dynamics does not create correlations.

In case of quantum many-particle systems the relationship of the evolution of marginal observables and quantum kinetic equations was considered in paper [22].

4 The Boltzmann–Grad asymptotic behavior of the generalized Enskog equation

In this section we consider an approach to the rigorous derivation of the Boltzmann equation with hard sphere collisions from the generalized Enskog kinetic equation.

4.1 The Boltzmann–Grad limit theorem

For a solution of the generalized Enskog kinetic equation (24) the following Boltzmann–Grad scaling limit theorem is true [23].

Theorem 3 *If the initial one-particle marginal distribution function* $F_1^{\epsilon,0}$ *is satisfied condition (29) and there exists the limit in the sense of a weak convergence:* $\mathrm{w-}\lim_{\epsilon\rightarrow 0}(\epsilon^2 F_1^{\epsilon,0}(x_1) - f_1^0(x_1)) = 0$, *then for finite time interval there exists the Boltzmann–Grad limit of solution (28) of the Cauchy problem of the generalized Enskog equation in the same sense:*

$$\mathrm{w-}\lim_{\epsilon\rightarrow 0}\big(\epsilon^2 F_1(t,x_1) - f_1(t,x_1)\big) = 0, \tag{42}$$

where the limit one-particle marginal distribution function is defined by uniformly convergent on arbitrary compact set series (38).

If f_1^0 *satisfies condition (29), then for* $t \geq 0$ *the limit one-particle distribution function represented by series (38) is a weak solution of the Cauchy problem of the Boltzmann kinetic equation with hard sphere collisions (39),(40).*

The proof of this theorem is based on formulas of an asymptotically perturbed cumulants of groups of operators (7). Namely, in the sense of a weak convergence the equality holds:

$$\mathrm{w-}\lim_{\epsilon\rightarrow 0}\Big(\epsilon^{-2n}\frac{1}{n!}\mathfrak{A}_{1+n}(-t,1,\ldots,n+1)f_{1+n} -$$
$$-\int_0^t dt_1\ldots\int_0^{t_{n-1}} dt_n S_1(-t+t_1,1)\mathcal{L}_{\mathrm{int}}^{0,*}(1,2)\prod_{j_1=1}^{2} S_1(-t_1+t_2,j_1)\ldots$$
$$\prod_{i_n=1}^{n} S_1(-t_{n-1}+t_n,i_n)\sum_{k_n=1}^{n}\mathcal{L}_{\mathrm{int}}^{0,*}(k_n,n+1)\prod_{j_n=1}^{n+1} S_1(-t_n,j_n)f_{1+n}\Big) = 0,$$

where notations accepted in formula (38) are used.

Thus, the Boltzmann–Grad scaling limit of solution (28) of the generalized Enskog equation is governed by the Boltzmann kinetic equation with hard sphere collisions (39).

We note that one of the advantage of the developed approach to the derivation of the Boltzmann equation is the possibility to construct of the higher-order corrections to the Boltzmann–Grad evolution of many-particle systems with hard sphere collisions.

4.2 A scaling limit of marginal functionals of the state

As we note above the all possible correlations of a hard sphere system are described by marginal functionals of the state (26).

Taking into consideration that there exists limit (42) of a solution of the generalized Enskog kinetic equation (24), for marginal functionals of the state (26) the following statement holds.

Theorem 4 *Under the conditions of the limit theorem for the generalized Enskog kinetic equation for finite time interval there exists the following Boltzmann–Grad limit of marginal functionals of the state (26) in the sense of a weak convergence on the space of bounded functions:*

$$\mathrm{w-}\lim_{\epsilon\to 0}\big(\epsilon^{2s}F_s\big(t,x_1,\ldots,x_s\mid F_1(t)\big)-\prod_{j=1}^{s}f_1(t,x_j)\big)=0,$$

where the limit one-particle distribution function $f_1(t)$ is determined by series (38).

The proof of this limit theorem is based on the formulas for asymptotically perturbed generating evolution operators (27) of marginal functionals of the state (26):

$$\mathrm{w-}\lim_{\epsilon\to 0}\big(\mathfrak{V}_1(t,\{Y\})f_s-If_s\big)=0,$$
$$\mathrm{w-}\lim_{\epsilon\to 0}\epsilon^{-2n}\mathfrak{V}_{1+n}(t,\{Y\},X\setminus Y)f_{s+n}=0,\quad n\geq 1,$$

where the limits exist in the sense of a weak convergence.

Thus, the Boltzmann–Grad scaling limit of marginal functionals of the state (26) are products of solution (38) of the Boltzmann equation with hard sphere collisions (39) that means the propagation of initial chaos.

4.3 Remark: a one-dimensional hard sphere system

We consider the Boltzmann–Grad asymptotic behavior of a solution of the generalized Enskog equation in the one-dimensional space. In this case the dimensionless collision integral $\mathcal{I}_{GEE}$ has the form [24]:

$$\begin{aligned}
&\mathcal{I}_{GEE} = \int_0^{\infty} dP\, P\big(F_2(t, q_1, p_1 - P, q_1 - \epsilon, p_1 \mid F_1(t))\\
&-F_2(t, q_1, p_1, q_1 - \epsilon, p_1 + P \mid F_1(t) + F_2(t, q_1, p_1 + P, q_1 + \epsilon, p_1 \mid F_1(t))\\
&-F_2(t, q_1, p_1, q_1 + \epsilon, p_1 - P \mid F_1(t))\big),
\end{aligned}$$

where $\epsilon > 0$ is a scaling parameter (the ratio of a hard rod length $\sigma > 0$ to its mean free path).

As we can see in the Boltzmann–Grad limit the collision integral of the generalized Enskog equation in the one-dimensional space vanishes, i.e. in other words dynamics of a one-dimensional system of elastically interacting hard spheres is trivial (a free molecular motion or the Knudsen flow).

We remark that in paper [25] it was established that in contrast to a one-dimensional hard rod system with elastic collisions the Boltzmann–Grad asymptotic behavior of inelastically interacting hard rods is not trivial and it is governed by the Boltzmann equation for granular gases.

5 Conclusion

In the paper two new approaches to the description of the kinetic evolution of many-particle systems with hard sphere collisions were developed. In particular, a formalism for the description of the evolution of infinitely many hard spheres within the framework of marginal observables in the Boltzmann–Grad scaling limit is developed. Another approach to the description of the kinetic evolution of hard spheres is based on the generalized Enskog kinetic equation.

One of the advantage of such approaches is the possibility to construct the kinetic equations in scaling limits, involving correlations at initial time which can characterize the condensed states of a hard sphere system.

We emphasize that the approach to the derivation of the Boltzmann equation from underlying dynamics governed by the generalized Enskog kinetic equation enables to construct the higher-order corrections to the Boltzmann–Grad evolution of many-particle systems with hard sphere collisions.

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