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Poiseuille flow with spherical paraboloid velocity

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Dedicated to memory of Professor Promarz M. Tamrazov

The velocity field of a classical Poiseuille flow is given by a paraboloid. We consider a flow whose velocity field is spherical paraboloid. For any spherical paraboloid velocity we completely determine the viscosity which realizes the flow.

1. Introduction. We assume that the viscous incompressible fluid occupies a vertical tube $D \times \mathbb{R}_z$ in $\mathbb{R}^3$ with $D : x^2 + (y-b)^2 < \rho^2$. The classical Hagen–Poiseuille law is, as is well-known, given by a fluid velocity $(0, 0, \frac{\gamma}{4\mu}(\rho^2 - x^2 - (y-b)^2))$ and a fluid pressure $-\gamma z$, with a viscous constant denoted by μ and a constant γ.

In contrast to the classical case, we consider an unknown viscosity $\mu(x, y) > 0$ for which the fluid velocity $\boldsymbol{u} = (0, 0, u_S)$

$$u_S(x, y) = \frac{\rho^2 - x^2 - (y-b)^2}{1 + x^2 + y^2} \tag{1}$$

and an unknown pressure $p(x, y, z)$ satisfy the steady Navier–Stokes equations

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla \cdot (\mu \mathbb{T}(\boldsymbol{u})) + \nabla p = \mathbf{0} \quad \text{in } D \times \mathbb{R}, \tag{2}$$
$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } D \times \mathbb{R}, \tag{3}$$
$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \partial D \times \mathbb{R}, \tag{4}$$

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where $\mathbb{T}(\boldsymbol{u})$ is a deformation tensor. We give in this paper, a smooth solution to (2) — (4).

Theorem 1. *The pressure $p = -\gamma z$ and the velocity $\boldsymbol{u} = (0, 0, u_S)$ given in* (1) *satisfy the steady Navier–Stokes equations* (2) — (4) *for*

$$\mu(x,y) = \begin{cases} \dfrac{\gamma}{4b}\dfrac{(x^2+y^2+1)^2}{x^2}\Bigg\{(y-\eta') \\ \qquad\qquad + \dfrac{x^2+(y-\eta)(y-\eta')}{2x}\operatorname{Sin}^{-1}F(x,y)\Bigg\} & \text{if } x \neq 0, \\ -\dfrac{\gamma}{12}\left(\dfrac{y^2+1}{y-\eta'}\right)^2(2y+\eta-3\eta') & \text{if } x = 0, \end{cases}$$

where the constants η, η' are

$$\eta = -\frac{A-\sqrt{4+A^2}}{2}, \quad \eta' = -\frac{A+\sqrt{4+A^2}}{2}$$

with $A = (1+\rho^2-b^2)/b$. The function $F(x,y)$ is given by

$$F(x,y) = \begin{cases} \dfrac{2(y-\eta')|x|}{x^2+(y-\eta')^2} & ((x,y) \in R_1), \\ -\dfrac{2(y-\eta)|x|}{x^2+(y-\eta)^2} & ((x,y) \in R_2), \\ -\dfrac{2(y-\eta')|x|}{x^2+(y-\eta')^2} & ((x,y) \in R_3) \end{cases}$$

for

$$R_1 = \{(x,y) \colon x^2+(y-\eta)(y-\eta') < 0,\ x^2-(y-\eta)(y-\eta') > 0\},$$
$$R_2 = \{(x,y) \colon x^2+(y-\eta)(y-\eta') > 0,\ x^2-(y-\eta)(y-\eta') > 0\},$$
$$R_3 = \{(x,y) \colon x^2+(y-\eta)(y-\eta') > 0,\ x^2-(y-\eta)(y-\eta') < 0\}.$$

2. Proof of Theorem 1. By (1) and (2), we see that the pressure p does not depend on x nor y. So we obtain $p = -\gamma z$ and reduce (2) to

$$\mu_x \partial_x u_S + \mu_y \partial_y u_S + \mu \Delta u_S = \gamma. \tag{5}$$

The characteristics $\partial_y u_S dx - \partial_x u_S dy = 0$ of (5) is either the y-axis, or satisfies the total differential

$$\left(\frac{Ay+y^2}{x^2} - \frac{1}{x^2} - 1\right)dx - \frac{A+2y}{x}dy = 0, \qquad x \neq 0, \tag{6}$$

for $A = (1+\rho^2-b^2)/b$. The integral curve of (6) is a one parameter family of characteristics $C_\alpha\colon -(Ay+y^2)/x+1/x-x=\alpha$ which is written as

$$C_\alpha\colon \Big(x+\frac{\alpha}{2}\Big)^2+\Big(y+\frac{A}{2}\Big)^2=r^2, \qquad x\neq 0\,, \tag{7}$$
$$r=r_\alpha=\sqrt{1+\frac{\alpha^2+A^2}{4}}\,.$$

We see that the family of characteristics $\{C_\alpha\}_\alpha$ and y-axis sweep out the fluid domain D, and that every C_α runs through two fixed points $(0,\eta),(0,\eta')$

$$\eta=-\frac{A-\sqrt{4+A^2}}{2}\,,\quad \eta'=-\frac{A+\sqrt{4+A^2}}{2}\,, \tag{8}$$

where η and η' are the roots of $y^2+Ay-1=0$. From this fact, either of the followings two holds : (i) one of $(0,\eta),(0,\eta')$ belongs to D and the other to $(\bar{D})^c$, or (ii) both of them are on ∂D. However, $y=\eta,\,\eta'$ do not simultaneously satisfy $y^2+Ay-1=0$, so we have $(0,\eta),(0,\eta')\notin\partial D$. On the other hand, η is a continuous function of $b,\ \rho$ and $\eta^2-A\eta+1<0$ when $b=1$. Hence, we conclude that $(0,\eta)\in D$ for any $b,\ \rho$ and $(0,\eta')\in(\bar{D})^c$.

If we denote the restriction of viscosity $\mu(x,y)$ on a characteristic curve $C_\alpha\colon y=y(x;\alpha)$ by $\mu_\alpha(x)=\mu_\alpha(x,y(x;\alpha))$, it satisfies

$$\mu'_\alpha(x)+\mu_\alpha(x)\frac{\Delta u_S}{\partial_x u_S}=\frac{\gamma}{\partial_x u_S}\,, \tag{9}$$

where

$$\partial_x u_S=-2b\frac{x(A+2y)}{(x^2+y^2+1)^2}\,,\quad \Delta u_S=4b\frac{A(x^2+y^2-1)-4y}{(x^2+y^2+1)^3}\,.$$

By (7), we parametrize C_α by $\theta\in[0,2\pi)$ as

$$x=-\frac{\alpha}{2}+r\cos\theta,\quad y=-\frac{A}{2}+r\sin\theta. \tag{10}$$

If we use $\dfrac{dx}{d\theta}=-\dfrac{A+2y}{2}$, $x^2+y^2-1=-(\alpha x+Ay)$, $x^2+y^2+1= =-(\alpha x+Ay)+2$, we obtain

$$\frac{\Delta u_S}{\partial_x u_S}\frac{dx}{d\theta}=-2\frac{A(x^2+y^2-1)-4y}{x(A+2y)(x^2+y^2+1)}\frac{dx}{d\theta}=$$
$$=-\frac{A+2y}{x}-2\frac{Ax-\alpha y}{\alpha x+Ay-2}\,. \tag{11}$$

By (10), the first term in the right hand side of (11) is

$$-\frac{A+2y}{x} = -2\frac{r\sin\theta}{r\cos\theta - \alpha/2}\,.$$

For the second term in (11), we prepare

$$\begin{aligned} Ax - \alpha y &= -r\sqrt{A^2+\alpha^2}\sin(\theta-\theta_0)\,, \\ \alpha x + Ay &= r\sqrt{A^2+\alpha^2}\cos(\theta-\theta_0) - \frac{\alpha^2+A^2}{2}\,, \end{aligned}$$

where $\theta_0 \in [0, 2\pi)$ is given by

$$\cos\theta_0 = \frac{\alpha}{\sqrt{A^2+\alpha^2}}\,, \quad \sin\theta_0 = \frac{A}{\sqrt{A^2+\alpha^2}}\,.$$

Hence, we have

$$-2\frac{Ax-\alpha y}{\alpha x + Ay - 2} = \frac{2r\sqrt{A^2+\alpha^2}\sin(\theta-\theta_0)}{r\sqrt{A^2+\alpha^2}\cos(\theta-\theta_0) - 2r^2}\,.$$

We integrate (11) along C_α to obtain

$$\begin{aligned} \int\frac{\Delta u_S}{\partial_x u_S}\,dx &= -2\int\frac{r\sin\theta}{r\cos\theta-\alpha/2}\,d\theta + 2\int\frac{r\sqrt{A^2+\alpha^2}\sin(\theta-\theta_0)}{r\sqrt{A^2+\alpha^2}\cos(\theta-\theta_0)-2r^2}\,d\theta = \\ &= 2\log|r\cos\theta-\alpha/2| - 2\log\left|r\sqrt{A^2+\alpha^2}\cos(\theta-\theta_0)-2r^2\right|. \end{aligned}$$

So we denote

$$M(\theta) = \exp\int\frac{\Delta u_S}{\partial_x u_S}dx = \frac{(r\cos\theta-\alpha/2)^2}{(r\sqrt{A^2+\alpha^2}\cos(\theta-\theta_0)-2r^2)^2}\,.$$

Meanwhile,

$$\partial_x u_S = \frac{4bx}{(x^2+y^2+1)^2}\frac{dx}{d\theta}\,,$$

we have

$$\begin{aligned} \gamma\int\frac{M(\theta)}{\partial_x u_S(\theta)}\,dx &= \gamma\int\frac{(r\cos\theta-\alpha/2)^2}{(r\sqrt{A^2+\alpha^2}\cos(\theta-\theta_0)-2r^2)^2}\frac{(x^2+y^2+1)^2}{4bx}\,d\theta = \\ &= \gamma\int\frac{r\cos\theta-\alpha/2}{4b}\,d\theta = \frac{\gamma}{4b}\left(r\sin\theta - \frac{\alpha}{2}\theta + c_0\right) \end{aligned}$$

for a constant $c_0 = c_0(\alpha)$ independent of θ. Hence, we obtain

$$\begin{aligned}\mu(x, y(x)) &= \frac{\gamma}{M}\int \frac{M}{\partial_x u_S}\,dx = \\ &= \frac{\gamma}{4b}\frac{(r\sqrt{A^2+\alpha^2}\cos(\theta-\theta_0)-2r^2)^2}{(r\cos\theta-\alpha/2)^2}\Big(r\sin\theta-\frac{\alpha}{2}\theta+c_0\Big) = \\ &= \frac{\gamma}{4b}\frac{(x^2+y^2+1)^2}{x^2}\left\{\frac{2y+A}{2}-\frac{\alpha}{2}\mathrm{Sin}^{-1}\frac{2y+A}{2r}+c_0\right\}, \quad (12)\end{aligned}$$

which is a continuous function of $(x, y) \in D\setminus\{x = 0\}$. We claim $\mu_\alpha(x, y)$ to be smooth along each curve C_α near the point $(0, \eta)$, so we have $c_0 = c_0(\alpha)$

$$c_0 = r\sqrt{1-\frac{\alpha^2}{4r^2}} + \frac{\alpha}{2}\mathrm{Sin}^{-1}\frac{\sqrt{4+A^2}}{2r}\,. \quad (13)$$

Applying the formula $\mathrm{Sin}^{-1}\varphi - \mathrm{Sin}^{-1}\psi = \mathrm{Sin}^{-1}(\varphi\sqrt{1-\psi^2} - \psi\sqrt{1-\varphi^2})$, we have

$$\begin{aligned}&\mathrm{Sin}^{-1}\frac{\sqrt{4+A^2}}{2r} - \mathrm{Sin}^{-1}\frac{2y+A}{2r} = \quad (14)\\ &= \mathrm{Sin}^{-1}\frac{\sqrt{4+A^2}\sqrt{4r^2-(2y+A)^2}-(2y+A)\sqrt{4r^2-(4+A^2)}}{4r^2} = \\ &= \mathrm{Sin}^{-1}\frac{\sqrt{4+A^2}\,\left|x^2-(y-\eta)(y-\eta')\right|-(2y+A)\left|x^2+(y-\eta)(y-\eta')\right|}{4r^2|x|}.\end{aligned}$$

In order to remove the modulus sign from (14), we divide xy-plane into the following three regions :

$$\begin{aligned}R_1 &= \{(x, y)\colon x^2+(y-\eta)(y-\eta') < 0,\ x^2-(y-\eta)(y-\eta') > 0\}\,,\\ R_2 &= \{(x, y)\colon x^2+(y-\eta)(y-\eta') > 0,\ x^2-(y-\eta)(y-\eta') > 0\}\,,\\ R_3 &= \{(x, y)\colon x^2+(y-\eta)(y-\eta') > 0,\ x^2-(y-\eta)(y-\eta') < 0\}\,.\end{aligned}$$

(We note $\{(x, y)\colon x^2+(y-\eta)(y-\eta') < 0,\ x^2-(y-\eta)(y-\eta') < 0\} = \emptyset$.) Since $A = -\eta - \eta'$, we have

$$\begin{aligned}&\sqrt{4+A^2}\,\left|x^2-(y-\eta)(y-\eta')\right|-(2y+A)\left|x^2+(y-\eta)(y-\eta')\right| = \\ &= \begin{cases}2(y-\eta')\{x^2+(y-\eta)^2\} & ((x, y) \in R_1),\\ -2(y-\eta)\{x^2+(y-\eta')^2\} & ((x, y) \in R_2),\\ -2(y-\eta')\{x^2+(y-\eta)^2\} & ((x, y) \in R_3)\end{cases} \quad (15)\end{aligned}$$

and

$$4r^2|x| = \{x^2 + (y-\eta)^2\}\{x^2 + (y-\eta')^2\}/|x|\,. \tag{16}$$

By (12) — (16), we arrive at

$$\mu(x,y) = \frac{\gamma}{4b}\frac{(x^2+y^2+1)^2}{x^2}\left\{(y-\eta') + \frac{x^2+(y-\eta)(y-\eta')}{2x}\mathrm{Sin}^{-1}F(x,y)\right\},$$

where

$$F(x,y) = \begin{cases} \dfrac{2(y-\eta')|x|}{x^2+(y-\eta')^2} & ((x,y)\in R_1), \\ -\dfrac{2(y-\eta)|x|}{x^2+(y-\eta)^2} & ((x,y)\in R_2), \\ -\dfrac{2(y-\eta')|x|}{x^2+(y-\eta')^2} & ((x,y)\in R_3). \end{cases} \tag{17}$$

On the other hand, y–axis is also a characteristic curve of (5), on which we denote the restriction of viscosity by $\mu_\infty(y) = \mu(0,y)$. The equation (5) for $\mu_\infty(y)$ reads

$$\mu'_\infty + \mu_\infty\frac{\Delta u_S}{\partial_y u_S} = \frac{\gamma}{\partial_y u_S}\,.$$

where

$$\partial_y u_S(0,y) = -2b\frac{Ay-(1-y^2)}{(y^2+1)^2}\,,$$

$$\Delta u_S(0,y) = 4b\frac{A(y^2-1)-4y}{(y^2+1)^3}\,.$$

We integrate

$$\int\frac{\Delta u_S}{\partial_y u_S}dy = -2\int\left\{\frac{2y}{y^2+1} - \frac{2y+A}{y^2+Ay-1}\right\}dy =$$

$$= \log\left(\frac{(y-\eta)(y-\eta')}{y^2+1}\right)^2$$

to obtain

$$M_\infty(y) := \exp\int\frac{\Delta u_S}{\partial_y u_S}\,dy = \left(\frac{(y-\eta)(y-\eta')}{y^2+1}\right)^2.$$

Therefore,

$$\mu_\infty(y) = \frac{\gamma}{M_\infty(y)} \int \frac{M_\infty}{\partial_y u_S}\, dy =$$
$$= -\frac{\gamma}{2}\left(\frac{y^2+1}{(y-\eta)(y-\eta')}\right)^2 \int \left(\frac{(y-\eta)(y-\eta')}{y^2+1}\right)^2 \frac{(y^2+1)^2}{(y-\eta)(y-\eta')}\, dy =$$
$$= -\frac{\gamma}{2}\left(\frac{y^2+1}{(y-\eta)(y-\eta')}\right)^2 \left(\frac{1}{3}y^3 - \frac{\eta+\eta'}{2}y^2 - y + c_1\right).$$

Here we claim that $\mu_\infty(y)$ is continuous at $y = \eta$, it is necessary to satisfy

$$c_1 = -\left(\frac{1}{3}\eta^3 - \frac{\eta+\eta'}{2}\eta^2 - \eta\right),$$

yielding

$$\frac{1}{3}y^3 - \frac{\eta+\eta'}{2}y^2 - y + c_1 = \frac{1}{6}(y-\eta)^2(2y+\eta-3\eta').$$

We thus obtain

$$\mu_\infty(y) = -\frac{\gamma}{12}\left(\frac{y^2+1}{y-\eta'}\right)^2 (2y+\eta-3\eta').$$

We remark that for the limit of $\mu = \mu_\alpha$ along each C_α to the y–axis satisfies

$$\lim_{x\to 0} \mu_\alpha(x) = -\frac{\gamma}{4b}\frac{(\eta^2+1)^2}{\eta-\eta'} = \lim_{y\to\eta} \mu_\infty(y).$$

We conclude that μ is a continuous function on $(x, y) \in D$, and obtain Theorem 1.

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