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Integral theorems for monogenic functions in an infinite-dimensional space with a commutative multiplication

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Dedicated to memory of Professor Promarz M. Tamrazov

We establish integral theorems for monogenic functions taking values in an infinite-dimensional commutative Banach algebra associated with spatial potential solenoid fields symmetric with respect to an axis. We establish also integral theorems for monogenic functions taking values in a topological vector space being an expansion of the mentioned algebra. We discuss some open problems.

1. Introduction. A spatial potential solenoid field symmetric with respect to the axis Ox is described in its meridian plane xOr in terms of the axial-symmetric potential φ and the Stokes flow function ψ satisfying the following system of equations

$$r\,\frac{\partial\varphi(x,r)}{\partial x}=\frac{\partial\psi(x,r)}{\partial r}\,,\qquad r\,\frac{\partial\varphi(x,r)}{\partial r}=-\frac{\partial\psi(x,r)}{\partial x}\,. \tag{1}$$

Under the condition that there exist continuous second-order partial derivatives of the functions $\varphi(x,r)$ and $\psi(x,r)$, the system (1) implies the equation

$$r\Delta_2\varphi(x,r)+\frac{\partial\varphi(x,r)}{\partial r}=0 \tag{2}$$

for the axial-symmetric potential and the equation

$$r\Delta_2\psi(x,r) - \frac{\partial\psi(x,r)}{\partial r} = 0 \tag{3}$$

for the Stokes flow function, where $\Delta_2 := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial r^2}$.

An effectiveness of analytic function methods in the complex plane for researching plane potential fields inspires developing analogous methods for spatial fields. The problem to develop such methods for spatial potential solenoid fields was posed by M. A. Lavrentyev [1, pp. 205, 18].

Being the first head of the Department of Complex Analysis and Potential Theory of the Institute of Mathematics of the National Academy of Sciences of Ukraine, Professor P. M. Tamrazov concerned very closely to developing an algebraic-analytic approach to principal equations of mathematical physics. Moreover, the mentioned approach were essentially developed thanking his support.

This approach means a finding of commutative Banach algebra such that differentiable in the sense of Gateaux functions with values in this algebra have components satisfying the given equation with partial derivatives. Such algebras are constructed for the biharmonic equation and the three-dimensional Laplace equation and elliptic equations degenerating on an axis that describe axial-symmetric potential fields (see [2 — 7]).

We proved in the papers [4, 6] that in a domain convex in the direction of the axis Or the functions φ and ψ can be constructed by means components of principal extensions of holomorphic functions of complex variable into a corresponding domain of a special two-dimensional vector manifold in an infinite-dimensional commutative Banach algebra.

In such a way for solutions of the system (1) we obtained integral expressions which were generalized for domains of general form (see [6, 8]). Using integral expressions for solutions of the system (1), in the papers [6, 9 — 12] we developed methods for solving boundary problems for axial-symmetric potentials and Stokes flow functions that have various applications in the mathematical physics. In particular, the developed methods are applicable for solving a boundary problem about a streamline of the ideal incompressible fluid along an axial-symmetric body (see [6, 13]).

In this paper, we establish integral theorems for monogenic functions associated with solutions of the equations (1) — (3) and discuss some open problems for the mentioned functions.

2. An infinite-dimensional commutative Banach algebras associated with spatial potential fields. Let $\mathbb{H} := \{a = \sum\limits_{k=1}^{\infty} a_k e_k \ : \ a_k \in \mathbb{R}, \sum\limits_{k=1}^{\infty} |a_k| < \infty\}$ be a commutative associative Banach algebra over the field of real numbers $\mathbb{R}$ with the norm $\|a\|_{\mathbb{H}} := \sum\limits_{k=1}^{\infty} |a_k|$ and the following multiplication table for elements of the basis $\{e_k\}_{k=1}^{\infty}$:

$$e_n e_1 = e_n, \quad e_m e_n = \frac{1}{2}\left(e_{m+n-1} + (-1)^{n-1} e_{m-n+1}\right) \quad \forall\, m \geq n \geq 1\,.$$

The algebra $\mathbb{H}$ was offered by I. P. Mel'nichenko [14] for describing spatial axial-symmetric potential fields.

As in the papers [4, 6], consider a comlexification

$$\mathbb{H}_{\mathbb{C}} := \mathbb{H} \oplus i\mathbb{H} \equiv \{c = a + ib \ : \ a, b \in \mathbb{H}\}$$

of the algebra $\mathbb{H}$, where i is the imaginary unit of the algebra of complex numbers $\mathbb{C}$. Meanwhile, the norm of element $g := \sum\limits_{k=1}^{\infty} c_k e_{2k-1} \in \mathbb{H}_{\mathbb{C}}$ is given by means the equality $\|g\|_{\mathbb{H}_{\mathbb{C}}} := \sum\limits_{k=1}^{\infty} |c_k|$.

3. Monogenic and analytic functions taking values in the algebra $\mathbb{H}_{\mathbb{C}}$. Below, we shall consider functions given in domains of the plane $\mu := \{\zeta = xe_1 + re_2 \ : \ x, r \in \mathbb{R}\}$ and the linear manifold $\mathcal{M} := \{\zeta = xe_1 + yie_1 + re_2 \ : \ x, y, r \in \mathbb{R}\}$ containing the plane μ and the complex plane $\mathbb{C}$.

We say that a continuous function $\Phi : \mathcal{Q} \to \mathbb{H}_{\mathbb{C}}$ is *monogenic* in a domain $\mathcal{Q} \subset \mathcal{M}$ (or $\mathcal{Q} \subset \mu$) if Φ is differentiable in the sense of Gateaux in every point of $\mathcal{Q}$, i.e. if for every $\zeta \in \mathcal{Q}$ there exists an element $\Phi'(\zeta) \in \mathbb{H}_{\mathbb{C}}$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \tag{4}$$

for all $h \in \mathcal{M}$ (or $h \in \mu$, respectively).

A function $\Phi : \mathcal{Q} \to \mathbb{H}_{\mathbb{C}}$ is *analytic* in a domain $\mathcal{Q} \subset \mathcal{M}$ (or $\mathcal{Q} \subset \mu$) if in a certain neighborhood of every point $\zeta_0 \in \mathcal{Q}$ it can be represented in the form of the sum of convergent power series

$$\Phi(\zeta) = \sum_{k=0}^{\infty} c_k \, (\zeta - \zeta_0)^k, \qquad c_k \in \mathbb{H}_{\mathbb{C}}\,. \tag{5}$$

It is obvious that an analytic function $\Phi : \mathcal{Q} \to \mathbb{H}_{\mathbb{C}}$ is monogenic in the domain $\mathcal{Q}$ and its derivative $\Phi'(\zeta)$ is also monogenic in $\mathcal{Q}$.

Below, we establish sufficient conditions for a monogenic function $\Phi : \mathcal{Q} \to \mathbb{H}_{\mathbb{C}}$ to be analytic in a domain $\mathcal{Q} \subset \mathcal{M}$.

However, *it remains unknown whether every function $\Phi : \mathcal{Q} \to \mathbb{H}_{\mathbb{C}}$ monogenic in a domain $\mathcal{Q}$ is analytic in this domain in both cases $\mathcal{Q} \subset \mathcal{M}$ and $\mathcal{Q} \subset \mu$.*

In what follows, the variables x, y, r are real and $\tilde{\zeta} := xe_1 + yie_1 + re_2$.

Associate with a set $Q \subset \mathbb{R}^3$ the set $Q_{\tilde{\zeta}} := \{\tilde{\zeta} = xe_1 + yie_1 + re_2 : (x, y, r) \in Q\}$ in $\mathcal{M}$.

Now, let $Q_{\tilde{\zeta}}$ be a domain in $\mathcal{M}$. Consider the decomposition

$$\Phi(\tilde{\zeta}) = \sum_{k=1}^{\infty} U_k(x, y, r)\ e_k\,, \tag{6}$$

of a function $\Phi : Q_{\tilde{\zeta}} \to \mathbb{H}_{\mathbb{C}}$ with respect to the basis $\{e_k\}_{k=1}^{\infty}$, where the functions $U_k : Q \to \mathbb{C}$ are differentiable in the domain Q, i.e.

$$U_k(x + \Delta x, y + \Delta y, r + \Delta r) - U_k(x, y, r) =$$

$$= \frac{\partial U_k(x, y, r)}{\partial x}\,\Delta x + \frac{\partial U_k(x, y, r)}{\partial y}\,\Delta y + \frac{\partial U_k(x, y, r)}{\partial r}\,\Delta r +$$

$$+ \, o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta r)^2}\right), \qquad (\Delta x)^2 + (\Delta y)^2 + (\Delta r)^2 \to 0\,,$$

for all $(x, y, r) \in Q$.

A proof of following theorem is similar to the proof of Theorem 4.1 [7].

Theorem 1. *Let a function $\Phi : Q_{\tilde{\zeta}} \to \mathbb{H}_{\mathbb{C}}$ be continuous in a domain $Q_{\tilde{\zeta}} \subset \mathcal{M}$ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (6) *be differentiable in Q. In order that the function Φ be monogenic in the domain $Q_{\tilde{\zeta}}$, it is necessary and sufficient that the conditions*

$$\frac{\partial \Phi}{\partial y} = i\,\frac{\partial \Phi}{\partial x}\,, \qquad \frac{\partial \Phi}{\partial r} = \frac{\partial \Phi}{\partial x}\,e_2\,. \tag{7}$$

be satisfied in $Q_{\tilde{\zeta}}$ and the following relations be fulfilled in Q:

$$\sum_{k=1}^{\infty} \left|\frac{\partial U_k(x, y, r)}{\partial x}\right| < \infty, \tag{8}$$

$$\lim_{\varepsilon\to 0+0}\sum_{k=1}^{\infty}\left|U_k(x+\varepsilon h_1,y+\varepsilon h_2,r+\varepsilon h_3)-U_k(x,y,r)-\frac{\partial U_k(x,y,r)}{\partial x}\,\varepsilon h_1-\right.$$
$$\left.-\frac{\partial U_k(x,y,r)}{\partial y}\,\varepsilon h_2\;-\frac{\partial U_k(x,y,r)}{\partial r}\,\varepsilon h_3\right|\varepsilon^{-1}=0\quad\forall\,h_1,h_2,h_3\in\mathbb{R}\,. \tag{9}$$

4. Integral theorems for monogenic functions taking values in the algebra $\mathbb{H}_{\mathbb{C}}$. In the paper [15] for functions differentiable in the sense of Lorch in an arbitrary convex domain of commutative associative Banach algebra, some properties similar to properties of holomorphic functions of complex variable (in particular, the integral Cauchy theorem and the integral Cauchy formula, the Taylor expansion and the Morera theorem) are established. The convexity of the domain in the mentioned results from [15] is withdrawn by E. K. Blum [16].

Below we establish similar results for monogenic functions $\Phi : Q_{\tilde\zeta}\to\mathbb{H}_{\mathbb{C}}$ given only in a domain $Q_{\tilde\zeta}$ of the linear manifold $\mathcal{M}$ instead of domain of whole algebra $\mathbb{H}_{\mathbb{C}}$. Let us note that *a priori* the differentiability of the function Φ in the sense of Gateaux is a restriction weaker than the differentiability of this function in the sense of Lorch.

In the case where Γ is a Jordan rectifiable curve in $\mathbb{R}^3$ we shall say that $\Gamma_{\tilde\zeta}$ is also a Jordan rectifiable curve. For a continuous function $\Psi : \Gamma_{\tilde\zeta}\to\mathbb{H}_{\mathbb{C}}$ of the form (6), where $(x,y,r)\in\Gamma$ and $U_k:\Gamma\to\mathbb{C}$, we define an integral along the curve $\Gamma_{\tilde\zeta}$ with $d\tilde\zeta := e_1dx+ie_1dy+e_2dr$ by the equality

$$\int\limits_{\Gamma_{\tilde\zeta}}\Psi(\tilde\zeta)d\tilde\zeta:=\sum_{k=1}^{\infty}e_k\int\limits_{\Gamma}U_k(x,y,r)dx+i\sum_{k=1}^{\infty}e_k\int\limits_{\Gamma}U_k(x,y,r)dy+$$
$$+\sum_{k=1}^{\infty}e_2e_k\int\limits_{\Gamma}U_k(x,y,r)dr \tag{10}$$

in the case where the series on the right-hand side of the equality are elements of the algebra $\mathbb{H}_{\mathbb{C}}$.

Theorem 2. *Let $\Phi : Q_{\tilde\zeta}\to\mathbb{H}_{\mathbb{C}}$ be a monogenic function in a domain $Q_{\tilde\zeta}$ and the functions $U_k : Q\to\mathbb{C}$ from the decomposition* (6) *be continuously differentiable in Q. Then for every closed Jordan rectifiable curve $\Gamma_{\tilde\zeta}\subset Q_{\tilde\zeta}$ homotopic to a point in $Q_{\tilde\zeta}$, the following equality holds:*

$$\int\limits_{\Gamma_{\tilde\zeta}}\Phi(\zeta)d\zeta=0\,. \tag{11}$$

Proof. Using the Stokes formula and the equalities (7), we obtain the equality

$$\int\limits_{\partial\triangle_{\tilde{\zeta}}} \Phi(\zeta)d\zeta = 0 \tag{12}$$

for the boundary $\partial\triangle_{\tilde{\zeta}}$ of every triangle $\triangle_{\tilde{\zeta}}$ such that $\overline{\triangle_{\tilde{\zeta}}} \subset Q_{\tilde{\zeta}}$. Now, we can complete the proof similarly to the proof of Theorem 3.2 [16].

For functions $\Phi : Q_{\tilde{\zeta}} \to \mathbb{H}_{\mathbb{C}}$ the following Morera theorem can be established in the usual way.

Theorem 3. *If a function $\Phi : Q_{\tilde{\zeta}} \to \mathbb{H}_{\mathbb{C}}$ is continuous in a domain $Q_{\tilde{\zeta}}$ and satisfies the equality* (12) *for every triangle $\triangle_{\tilde{\zeta}}$ such that $\overline{\triangle_{\tilde{\zeta}}} \subset Q_{\tilde{\zeta}}$, then the function Φ is monogenic in the domain $Q_{\tilde{\zeta}}$.*

For $\tilde{\zeta}$ we shall also use a notation of the form $\tilde{\zeta} = ze_1 + re_2$, where $z := x + iy$. Let $\tau := te_1 + r_2e_2$, where $t \in \mathbb{C}$ and $r_2 \in \mathbb{R}$. Generalizing a resolvent resolution (cf. the equality (2.9) in [6]), we obtain

$$\left(\tau - \tilde{\zeta}\right)^{-1} = \frac{1}{\sqrt{\big(t - z - i(r - r_2)\big)\big(t - z + i(r - r_2)\big)}} \Bigg(e_1 +$$

$$+2\sum_{k=2}^{\infty}\left(\frac{\sqrt{\big(t - z - i(r - r_2)\big)\big(t - z + i(r - r_2)\big)} - (t - z)}{r - r_2}\right)^{k-1} e_k\Bigg), \tag{13}$$

$$r - r_2 \neq 0\,, \quad t \notin s[z - i(r - r_2), z + i(r - r_2)]\,,$$

where $s[z - i(r - r_2), z + i(r - r_2)]$ is the segment connecting the points $z - i(r - r_2)$ and $z + i(r - r_2)$, and $\sqrt{\big(t - z - i(r - r_2)\big)\big(t - z + i(r - r_2)\big)}$ is that continuous branch of the analytic function

$$H(t) = \sqrt{\big(t - z - i(r - r_2)\big)\big(t - z + i(r - r_2)\big)}$$

outside of $s[z - i(r - r_2), z + i(r - r_2)]$ for which $H(z + a) > 0$ for all $a > 0$. Thus, for every $\tilde{\zeta}$ the element $\left(\tau - \tilde{\zeta}\right)^{-1}$ exists for all $\tau \notin S(\tilde{\zeta}) := \{\tau = te_1 + r_2e_2 : \mathrm{Re}\, t = x,\ |\mathrm{Im}\, t - y| \leq |r_2 - r|\}$.

Now, the next theorem can be proved similarly to Theorem 5 [17].

Theorem 4. *Suppose that Q is a domain convex in the direction of the axis Or. Suppose also that $\Phi : Q_{\zeta} \to \mathbb{H}_{\mathbb{C}}$ is a monogenic function in the domain Q_{ζ} and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (6) *are continuously differentiable in Q. Then for every point $\tilde{\zeta} \in Q_{\zeta}$ the following equality is true:*

$$\Phi(\tilde{\zeta}) = \frac{1}{2\pi i} \int\limits_{\Gamma_{\zeta}} \Phi(\tau)\big(\tau - \tilde{\zeta}\,\big)^{-1} d\tau\,, \tag{14}$$

where Γ_{ζ} is an arbitrary closed Jordan rectifiable curve in Ω_{ζ}, which surrounds once the set $S(\tilde{\zeta})$ and is homotopic to the circle $\{\tau = te_1 + r_2e_2 : |t - x - iy| = R,\, r_2 = r\}$ contained completely in Ω_{ζ}.

Using the formula (14), we obtain the Taylor expansion of monogenic function $\Phi : Q_{\zeta} \to \mathbb{H}_{\mathbb{C}}$ in the usual way (see., for example, [18, p. 107]) in the case where the conditions of Theorem 4 are satisfied. Thus, in this case, $\Phi : Q_{\zeta} \to \mathbb{H}_{\mathbb{C}}$ is an analytic function. In addition, in this case, an uniqueness theorem for monogenic functions can also be proved in the same way as for holomorphic functions of the complex variable (cf. [18, p. 110]).

Thus, the following theorem is true:

Theorem 5. *Let $\Phi : Q_{\zeta} \to \mathbb{H}_{\mathbb{C}}$ be a continuous function in a domain Q_{ζ} and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (6) *be continuously differentiable in Q. Then the function Φ is monogenic in Q_{ζ} if and only if one of the following conditions is satisfied:*

(I) *the conditions* (7) *are satisfied in Q_{ζ} and the relations* (8), (9) *are fulfilled in Q;*

(II) *the function Φ satisfies the equality* (12) *for every triangle $\triangle_{\zeta}$ such that $\overline{\triangle_{\zeta}} \subset Q_{\zeta}$;*

(III) *the function Φ is analytic in the domain Q_{ζ}.*

5. Relations between monogenic functions and axial-symmetric potential fields. Let us consider principal extensions of holomorphic functions of complex variable into corresponding domains of the manifold $\mathcal{M}$ and describe its relations to solutions of the system (1).

For a domain $D \subset \mathbb{R}^2$ we consider the congruent domain $D_z := \{z = x + iy : (x, y) \in D\}$ in the complex plane $\mathbb{C}$.

Let D_z be a bounded domain symmetric with respect to the real axis and convex in the direction of the imaginary axis. Therefore, there exists a real function $y(x)$ on the segment $[b_1, b_2]$ such that $\{t = x + y(x) : x \in [b_1, b_2]\} = \{t \in \partial D_z : \mathrm{Im}\, t \geq 0\}$, where by b_1 and b_2 we have denoted the points at which the boundary ∂D_z crosses the real axis.

Using the equality (13), we obtain explicitly the principal extension Φ_F of holomorphic function $F : D_z \to \mathbb{C}$ into the domain $\Omega_{\tilde{\zeta}} := \{\tilde{\zeta} = = xe_1 + yie_1 + re_2 : x \in (b_1, b_2),\ |y| + |r| < y(x)\}$ (cf. [19, p. 165]):

$$\Phi_F(\tilde{\zeta}) := \frac{1}{2\pi i} \int\limits_{\gamma} (te_1 - \tilde{\zeta})^{-1}\, F(t)\, dt \equiv \frac{e_1}{2\pi i} \int\limits_{\gamma} \frac{F(t)}{\sqrt{(t-z-ir)(t-z+ir)}}\, dt +$$

$$+ \frac{1}{\pi i} \sum_{k=2}^{\infty} e_k \int\limits_{\gamma} \frac{F(t)}{\sqrt{(t-z-ir)(t-z+ir)}} \times$$

$$\times \left(\frac{\sqrt{(t-z-ir)(t-z+ir)} - (t-z)}{r} \right)^{k-1} dt =:$$

$$=: \sum_{k=1}^{\infty} V_k(x, y, r)\, e_k \qquad \forall\, \tilde{\zeta} = xe_1 + yie_1 + re_2 \in \Omega_{\tilde{\zeta}} \ : \ r \neq 0, \tag{15}$$

where γ is an arbitrary closed rectifiable Jordan curve in D_z which embraces the segment $s[z - ir, z + ir]$ that is the spectrum of element $\tilde{\zeta}$.

The equality (15) generalizes a representation of the principal extension of function F into the domain $D_\zeta := \{\zeta = xe_1 + ye_2 \ : \ x + iy \in D_z\} \subset \mu$ congruent to D_z that was obtained in the papers [4, 6].

It is follows from Theorem 18 [4] (see also Theorem 2.6 [6]) that the first and the second components of the function (15) generate the solutions φ and ψ of the system (1) in the domain D by the formulas

$$\varphi(x, r) = V_1(x, 0, r)\,, \qquad \psi(x, r) = \frac{r}{2}\, V_2(x, 0, r)\,. \tag{16}$$

Moreover, the functions (16) are solutions of equations (2) and (3) in D, respectively.

In the following theorem we establish a representation of monogenic functions $\Phi : \Omega_{\tilde{\zeta}} \to \mathbb{H}_{\mathbb{C}}$ via principal extensions of holomorphic functions of complex variable.

Theorem 6. *Let $\Phi : \Omega_{\tilde{\zeta}} \to \mathbb{H}_{\mathbb{C}}$ be a monogenic function in $\Omega_{\tilde{\zeta}}$ and the functions $U_k : \Omega \to \mathbb{C}$ from the decomposition* (6) *be continuously* *differentiable in* Ω. *Then* Φ *is expressed in the form*

$$\Phi(\tilde{\zeta}) = \sum_{k=1}^{\infty} \Phi_{U_k}(\tilde{\zeta})\, e_k\,, \tag{17}$$

Proof. Note that we consider D_z as a subset of the domain $\Omega_{\tilde{\zeta}}$. Therefore, it follows from the first of equalities (7) that the functions U_k are holomorphic in D_z for $k = 1, 2, \ldots$. Now, to complete the proof it is enough to substitute the expression (6) into the equality (14), where we set $\Gamma_{\tilde{\zeta}} = \gamma$ and γ is the same as in the equality (15).

It follows from Theorem 6 that every monogenic function $\Phi : \Omega_{\tilde{\zeta}} \to \mathbb{H}_{\mathbb{C}}$ generates a set of solutions of the system (1) in D that correspond to the functions Φ_{U_k} from (17) by the formulas of the form (16).

In the papers [4, 6], for every monogenic function $\Phi : D_\zeta \to \mathbb{H}_{\mathbb{C}}$ given in a domain D_ζ of the plane μ, we obtained a representation of the form

$$\Phi(\zeta) = \Phi_{U_1}(\zeta) + \Phi_0(\zeta) \qquad \forall\, \zeta \in D_\zeta\,,$$

where $\Phi_0 : D_\zeta \to \mathcal{I}_0$ is a monogenic function taking values in the maximal ideal

$$\mathcal{I}_0 := \Bigg\{ \sum_{k=1}^{\infty} c_k e_k \in \mathbb{H}_{\mathbb{C}} : \quad \sum_{k=1}^{\infty} (-1)^k \left(\operatorname{Re} c_{2k-1} - \operatorname{Im} c_{2k}\right) = 0, \\ \sum_{k=1}^{\infty} (-1)^k \left(\operatorname{Re} c_{2k} + \operatorname{Im} c_{2k-1}\right) = 0 \Bigg\}$$

of the algebra $\mathbb{H}_{\mathbb{C}}$.

At the same time, *it remains unknown whether every monogenic function* $\Phi : D_\zeta \to \mathbb{H}_{\mathbb{C}}$ *can be represented in the form* (17).

It remains also unknown a constructive description of monogenic functions $\Phi_0 : D_\zeta \to \mathcal{I}_0$ *by means of holomorphic functions of the complex variable.* (Let us note that constructive descriptions of similar kinds was obtained for monogenic functions taking values in certain finite-dimensional commutative algebras, cf. [7, 20].)

6. Monogenic functions in a topological vector space $\widetilde{\mathbb{H}}_{\mathbb{C}}$ containing the algebra $\mathbb{H}_{\mathbb{C}}$. Let us generalize the relation (16) between solutions of the system (1) and monogenic functions for domains of more general form.

With this purpose, let us insert the algebra $\mathbb{H}_{\mathbb{C}}$ in the topological vector space $\widetilde{\mathbb{H}}_{\mathbb{C}} := \{g = \sum\limits_{k=1}^{\infty} c_k e_k : c_k \in \mathbb{C}\}$ with the topology of coordinate-wise convergence.

Note that $\widetilde{\mathbb{H}}_{\mathbb{C}}$ is not an algebra because the product of elements $g_1, g_2 \in \in \widetilde{\mathbb{H}}_{\mathbb{C}}$ is defined not always. But for each $g = \sum\limits_{k=1}^{\infty} c_k e_k \in \widetilde{\mathbb{H}}_{\mathbb{C}}$ and $\tilde{\zeta} = = ze_1 + re_2$, one can define the product

$$g\tilde{\zeta} \equiv \tilde{\zeta} g := \Big(c_1 z - \frac{c_2}{2}\, r\Big) e_1 + \Big(c_2 z + \Big(c_1 - \frac{c_3}{2}\Big)\, r\Big) e_2 +$$

$$+ \sum_{k=3}^{\infty} \Big(c_k z + \frac{1}{2}\Big(c_{k-1} - c_{k+1}\Big)\, r\Big) e_k\,.$$

We shall consider functions $\Phi : Q_{\tilde{\zeta}} \to \widetilde{\mathbb{H}}_{\mathbb{C}}$ for which the functions $U_k : Q \to \mathbb{C}$ in the decomposition (6) are differentiable in the domain Q. Such a function Φ is continuous in $Q_{\tilde{\zeta}}$ and, therefore, we call Φ a *monogenic* function in $Q_{\tilde{\zeta}}$ if $\Phi'(\zeta) \in \widetilde{\mathbb{H}}_{\mathbb{C}}$ in the equality (4).

The next theorem is similar to Theorem 1, where the necessary and sufficient conditions for a function $\Phi : Q_{\tilde{\zeta}} \to \mathbb{H}_{\mathbb{C}}$ to be monogenic include additional relations (8), (9) conditioned by the norm of absolute convergence in the algebra $\mathbb{H}_{\mathbb{C}}$.

Theorem 7. *Let a function $\Phi : Q_{\tilde{\zeta}} \to \widetilde{\mathbb{H}}_{\mathbb{C}}$ be of the form* (6) *and the functions $U_k : Q \to \mathbb{C}$ be differentiable in Q. In order that the function Φ be monogenic in the domain $Q_{\tilde{\zeta}}$, it is necessary and sufficient that the conditions* (7) *be satisfied in $Q_{\tilde{\zeta}}$.*

For a continuous function $\Psi : \Gamma_{\tilde{\zeta}} \to \widetilde{\mathbb{H}}_{\mathbb{C}}$ of the form (6), we define an integral along a Jordan rectifiable curve $\Gamma_{\tilde{\zeta}}$ by the equality (10) in the case where the series on the right-hand side of this equality are elements of the space $\widetilde{\mathbb{H}}_{\mathbb{C}}$.

In the next theorem, for the sake of simplicity, we suppose that the curve $\Gamma_{\tilde{\zeta}}$ is the piece-smooth edge of a piece-smooth surface. In this case the following statement is a result of the Stokes formula and the equalities (7).

Theorem 8. *Suppose that $\Phi : Q_{\tilde{\zeta}} \to \widetilde{\mathbb{H}}_{\mathbb{C}}$ is a monogenic function in a domain $Q_{\tilde{\zeta}}$ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (6) *are continuously differentiable in Q. Suppose also that Σ is a piece-smooth surface in Q with the piece-smooth edge Γ. Then the equality* (11) *holds.*

Let us define the product $gh \equiv hg$ for each $g = \sum\limits_{k=1}^{\infty} c_k e_k \in \widetilde{\mathbb{H}}_{\mathbb{C}}$ and $h = \sum\limits_{k=1}^{\infty} t_k e_k \in \mathbb{H}_{\mathbb{C}}$ in the case where the sequence $\{c_k\}_{k=1}^{\infty}$ is bounded:

$$gh \equiv hg := \left(c_1 t_1 - \frac{1}{2}\sum_{k=2}^{\infty} (-1)^k\, c_k t_k\right) e_1 +$$

$$+\left(c_2 t_1 + \left(c_1 - \frac{c_3}{2}\right) t_2 - \frac{1}{2}\sum_{k=3}^{\infty}\Big(c_{k-1} - c_{k+1}\Big) t_k\right) e_2 +$$

$$+\sum_{m=3}^{\infty}\left(c_m t_1 + \frac{1}{2}\sum_{k=2}^{m-1}\Big(c_{m-k+1} - (-1)^k\, c_{m+k-1}\Big) t_k + \Big(c_1 - (-1)^m\, c_{2m-1}\Big) t_m -\right.$$

$$\left. -\frac{1}{2}\sum_{k=m+1}^{\infty}\Big(c_{k-m+1} - (-1)^m\, c_{k+m-1}\Big) t_k\right) e_m \,.$$

In the case where Γ is a piece-smooth curve (or Σ is a piece-smooth surface) in $\mathbb{R}^3$ we shall say that $\Gamma_{\tilde\zeta}$ is also a piece-smooth curve (or $\Sigma_{\tilde\zeta}$ is also a piece-smooth surface, respectively).

The next theorem can be proved similarly to Theorem 5 [17].

Theorem 9. *Suppose that Q is a domain convex in the direction of the axis Or. Suppose also that $\Phi : Q_{\tilde\zeta} \to \widetilde{\mathbb{H}}_{\mathbb{C}}$ is a monogenic function in the domain $Q_{\tilde\zeta}$, and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (6) *are continuously differentiable and form an uniformly bounded family in Q. Then for every point $\tilde\zeta \in Q_{\tilde\zeta}$ the equality* (14) *holds, where $\Gamma_{\tilde\zeta}$ is a piece-smooth curve that surrounds once the set $S(\tilde\zeta)$ and, in addition, $\Gamma_{\tilde\zeta}$ and the circle $\{\tau = te_1 + r_2 e_2 : |t - x - iy| = R, r_2 = r\}$ are edges of a piece-smooth surface $\Sigma_{\tilde\zeta}$ contained completely in $\Omega_{\tilde\zeta}$.*

In what follows, D is such a bounded domain in $\mathbb{R}^2$ that the domain D_z is simply connected and symmetric with respect to the real axis but is not convex in the direction of the imaginary axis, generally speaking.

Now, let $\Omega_{\tilde\zeta} \subset \mathcal{M}$ be the domain that contains the segments $s[xe_1 + re_2, x + ir]$, $s[xe_1 + re_2, x - ir]$ for every $\zeta = xe_1 + re_2 \in D_\zeta$.

For every $\tilde\zeta = ze_1 + re_2 \in \Omega_{\tilde\zeta}$ with $r \neq 0$, we fix an arbitrary Jordan rectifiable curve $\gamma[z - ir, z + ir]$ in D_z which connects the points $z - ir$ and

$z+ir$. Let $\sqrt{(t-z-ir)(t-z+ir)}$ be that continuous branch of the analytic function $H(t)=\sqrt{(t-z-ir)(t-z+ir)}$ outside of the cut along $\gamma[z-ir,z+ir]$ for which $H(z+a)>0$ for all $a>\max\limits_{\tau\in\gamma[z-ir,z+ir]}\mathrm{Re}(\tau-z)$.

For every function $F:D_z\to\mathbb{C}$ holomorphic in the domain D_z we obtain the equality (15), where γ is an arbitrary closed rectifiable Jordan curve in D_z which embraces $\gamma[z-ir,z+ir]$.

It is follows from Theorems 3.2 [6] that the first and the second components of the function (15) generate the solutions φ and ψ of the system (1) in the domain D by the formulas (16). In addition, the functions (16) are solutions of equations (2) and (3) in D, respectively.

On the contrary, it is follows from Theorems 3.4, 3.5 [6] that every axial-symmetric potential and every Stokes flow function can be represented in the domain D by the formulas (16).

Note that the functions $V_k:Q\to\mathbb{C}$ from the decomposition (15) are infinitely differentiable but do not form an uniformly bounded family in Q, generally speaking. At the same time, the equality (14) holds for every function (15) and $\Gamma_{\tilde{\zeta}}=\gamma$.

Furthermore, the equality (14) holds for every monogenic function $\Phi:\Omega_{\tilde{\zeta}}\to\widetilde{\mathbb{H}}_{\mathbb{C}}$ in the case where $\Gamma_{\tilde{\zeta}}$ is an arbitrary closed Jordan rectifiable curve, which surrounds the point $\tilde{\zeta}$ and lies in the intersection of the domain $\Omega_{\tilde{\zeta}}$ and the plane $\{\tilde{\zeta}+te_1:t\in\mathbb{C}\}$.

At the same time, it is an open problem: *to describe closed curves* $\Gamma_{\tilde{\zeta}}$ *for which the equality* (14) *holds for every monogenic function* $\Phi:Q_{\tilde{\zeta}}\to\widetilde{\mathbb{H}}_{\mathbb{C}}$.

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