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Monogenic functions in a three-dimensional harmonic semi-simple algebra

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Dedicated to memory of Professor Promarz M. Tamrazov

We obtained a constructive description of monogenic functions taking values in the three-dimensional commutative harmonic semi-simple algebra by means of holomorphic functions of the complex variable. We proved that the mentioned monogenic functions have the Gateaux derivatives of all orders.

1. Introduction. Analytic function methods in the complex plane for plane potential fields inspire searching analogous effective methods for spatial potential fields.

Apparently, W. Hamilton (1843) made the first attempts to construct an algebra associated with the three-dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u(x,y,z)=0 \tag{1}$$

in that sense that components of hypercomlex functions satisfy Eq. (1) but the Hamilton's quaternions form a noncommutative algebra.

C. Segre [1] constructed an algebra of commutative quaternions that can be considered as a two-dimensional commutative semi-simple algebra over the field of complex numbers. For functions taking values in the Segre algebra some analogues of results of the classic theory of analytic functions of complex variable are established (see, for example, [2, 3]).

Commutative associative algebras in which there exist three linearly independent elements e_1, e_2, e_3 satisfying the equality

$$e_1^2 + e_2^2 + e_3^2 = 0 \tag{2}$$

are considered in the papers [4 — 9]. Such algebras are called *harmonic* (cf. [4, 7, 8]). We say also that such a triad $\{e_1, e_2, e_3\}$ is *harmonic*.

I. P. Mel'nichenko [6] noticed that functions differentiable doubly in the sense of Gateaux form the largest class of functions Φ satisfying identically the equalities

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi(\zeta) = \Phi''(\zeta)\,(e_1^2 + e_2^2 + e_3^2) = 0\,, \tag{3}$$

where Φ'' is the Gateaux second derivative of the function Φ, and he proved that there exists a three-dimensional harmonic algebra over the field of complex numbers only.

All three-dimensional harmonic algebras with unit are found in the paper [7], and all harmonic bases in these algebras are described in the monograph [8].

S. A. Plaksa and V. S. Shpakivskyi [9] obtained a constructive description of monogenic (i.e. continuous and differentiable in the sense of Gateaux) functions taking values in the three-dimensional harmonic algebra with two-dimensional radical by means of holomorphic functions of the complex variable. Moreover, the infinite differentiability in the sense of Gateaux of the mentioned monogenic functions is proved in [9]. Similar results are established in the paper [10] for monogenic functions taking values in the three-dimensional harmonic algebra with one-dimensional radical.

Below, we consider monogenic functions taking values in the three-dimensional harmonic semi-simple algebra and obtain results similar to the mentioned results from the papers [9, 10].

2. Preliminaries. Let $\mathbb{A}_1$ be a three-dimensional commutative associative Banach algebra over the field of complex numbers $\mathbb{C}$ and let $\{I_1, I_2, I_3\}$ be a basis of the algebra $\mathbb{A}_1$ with the multiplication table

$${I_k}^2 = I_k, \;\; I_k I_j = 0, \qquad k, j = 1, 2, 3, \;\; k \neq j\,.$$

Here $1 = I_1 + I_2 + I_3$.

Algebra $\mathbb{A}_1$ is harmonic (see [8, p. 38]) because there exist harmonic bases $\{e_1, e_2, e_3\}$ in $\mathbb{A}_1$. All harmonic bases in $\mathbb{A}_1$ are described in Theorem 1.10 [8]. In particular, a basis $\{e_1, e_2, e_3\}$ is harmonic if decompositions of its elements with respect to the basis $\{I_1, I_2, I_3\}$ are of the form

$$\begin{aligned} e_1 &= I_1 + I_2 + I_3, \\ e_2 &= n_1 I_1 + n_2 I_2 + n_3 I_3, \\ e_3 &= m_1 I_1 + m_2 I_2 + m_3 I_3, \end{aligned} \tag{4}$$

where n_k and m_k for $k = 1, 2, 3$ are complex numbers satisfying the relations

$$\begin{aligned} &1 + n_1^2 + m_1^2 = 0, \quad 1 + n_2^2 + m_2^2 = 0, \quad 1 + n_3^2 + m_3^2 = 0, \\ &n_1(m_2 - m_3) + n_2(m_3 - m_1) + n_3(m_1 - m_2) \neq 0. \end{aligned}$$

Let $E_3 := \{\zeta = xe_1 + ye_2 + ze_3 : \; x, y, z \in \mathbb{R}\}$ be a linear span in $\mathbb{A}_1$ over the field of real numbers $\mathbb{R}$. In what follows, $\zeta = xe_1 + ye_2 + ze_3$ and $x, y, z \in \mathbb{R}$.

Let Ω be a domain in E_3. We say that a continuous function $\Phi : \Omega \to \mathbb{A}_1$ is *monogenic* in Ω if Φ is differentiable in the sense of Gateaux in every point of Ω, i.e. if for every $\zeta \in \Omega$ there exists an element $\Phi'(\zeta) \in \mathbb{A}_1$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3.$$

$\Phi'(\zeta)$ is the Gateaux derivative of the function Φ in the point ζ.

Consider the decomposition of a function $\Phi : \Omega \to \mathbb{A}_1$ with the respect to the basis $\{e_1, e_2, e_3\}$:

$$\Phi(\zeta) = \sum_{j=1}^{3} U_j(x, y, z) e_j \,. \tag{5}$$

If the functions U_j are $\mathbb{R}$-differentiable in $\Omega_{\mathbb{R}} := \{(x, y, z) : xe_1 + ye_2 + ze_3 \in \Omega\}$ for $j = 1, 2, 3$, i.e.

$$U_j(x + \triangle x, y + \triangle y, z + \triangle z) - U_j(x, y, z) = \frac{\partial U_j}{\partial x}\,\Delta x + \frac{\partial U_j}{\partial y}\,\Delta y + \frac{\partial U_j}{\partial z}\,\Delta z +$$

$$+\,o\left(\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}\right), \quad (\Delta x)^2+(\Delta y)^2+(\Delta z)^2\to 0\,,$$

then it is follows from Theorem 1.3 [8] that the function Φ is monogenic in the domain Ω if and only if the following Cauchy – Riemann conditions are satisfied in Ω:

$$\frac{\partial\Phi}{\partial y}=\frac{\partial\Phi}{\partial x}\,e_2, \quad \frac{\partial\Phi}{\partial z}=\frac{\partial\Phi}{\partial x}\,e_3.$$

It will be shown below that the components U_1, U_2, U_3 of the decomposition (5) of a monogenic function $\Phi:\Omega\to\mathbb{A}_1$ are infinitely differentiable in the domain $\Omega_{\mathbb{R}}$.

The algebra $\mathbb{A}_1$ has three maximal ideals

$$\mathfrak{I}_k := \{\zeta=\sum_{j=1,\,j\neq k}^{3}\alpha_{kj}I_j, \;\; \alpha_{kj}\in\mathbb{C}\}, \;\; k=1,2,3.$$

The radical of algebra $\mathbb{A}_1$ consists only of the zero element. Thus, $\mathbb{A}_1$ is a semi-simple algebra (see [11, p.133]).

Consider three linear functionals $f_k:\mathbb{A}_1\to\mathbb{C}$ for $k=1,2,3$ such that

$$f_k(I_k)=1, \quad f_k(I_j)=0, \quad j=1,2,3, \;\; k\neq j. \tag{6}$$

It follows from (6) that the maximal ideal $\mathfrak{I}_k$ is the kernel of functional f_k for $k=1,2,3$. It is well known [11, p.135] that f_k are multiplicative functionals for all $k=1,2,3$.

From equations (4) and (6) we obtain the following relations:

$$f_k(\zeta)=f_k(xe_1+ye_2+ze_3)=x+n_ky+m_kz:=\xi_k, \;\; k=1,2,3.$$

It follows from the equality

$$\zeta^{-1}=\frac{1}{\xi_1}I_1+\frac{1}{\xi_2}I_2+\frac{1}{\xi_3}I_3\,. \tag{7}$$

that the element $\zeta=x+ye_2+ze_3\in E_3$ is invertible in $\mathbb{A}_1$ if and only if $\xi_k\neq 0$ for $k=1,2,3$.

It follows from the equality (7) that noninvertible elements form three straight line in E_3:

$$L_k:\{te_k^*: e_k^*:=(\mathrm{Re}\,n_k\mathrm{Im}\,m_k-\mathrm{Im}\,n_k\mathrm{Re}\,m_k)e_1-\mathrm{Im}\,m_ke_2+$$

$$+\mathrm{Im}\,n_ke_3\,, t\in\mathbb{R}\}, \;\; k=1,2,3.$$

The straight lines L_1, L_2 and L_3 have at least one common point 0 but two of them may coincide. For example, for the harmonic basis

$$e_1 = 1, \quad e_2 = iI_1, \quad e_3 = iI_2 - iI_3. \tag{8}$$

we have the equality $L_1 = \{te_3 : \ t \in \mathbb{R}\}$, $L_2 = L_3 = \{te_2 : \ t \in \mathbb{R}\}$.

3. An auxiliary affirmations. We say that a domain $\Omega \subset E_3$ is *convex in the direction of the straight line* L if Ω contains every segment which is parallel to L and connects two points ζ_1 , $\zeta_2 \in \Omega$.

Lemma 1. *Let a domain $\Omega \subset E_3$ be convex in the direction of the straight line L_k for some $k \in \{1, 2, 3\}$ and $\Phi : \Omega \to \mathbb{A}_1$ be a monogenic function in Ω. If $\zeta_1, \zeta_2 \in \Omega$ and $\zeta_2 - \zeta_1 \in L_k$ then*

$$\Phi(\zeta_2) - \Phi(\zeta_1) \in \mathfrak{I}_k. \tag{9}$$

The relations (9) is proved in a such way as in the proof of Lemma 2.1 [12] where you must take Ω, L_k, f_k instead of Ω_ζ, L, f, respectively.

Note that the condition of convexity of Ω in the direction of the line L_k is essential for the truth of Lemma 1. We show it in an example, where we construct both a domain Ω which is not convex in the direction of L_1 and a monogenic function $\Phi : \Omega \to \mathbb{A}_1$ for which the relation (9) is not satisfied for some $\zeta_1, \zeta_2 \in \Omega$ such that $\zeta_2 - \zeta_1 \in L_1$.

Example 1. Consider the harmonic basis (8). In this case $L_1 = \{te_3 : t \in \mathbb{R}\}$ and $\xi_1 = x + iy$. Consider a domain Ω which is the union of sets

$$\Omega_1 := \{xe_1 + ye_2 + ze_3 \in E_3 : |\xi_1| < 2, 0 < z < 2, -\pi/4 < \arg \xi_1 < 3\pi/2\},$$

$$\Omega_2 := \{xe_1 + ye_2 + ze_3 \in E_3 : |\xi_1| < 2, \ 2 \le z \le 4, \ \pi/2 < \arg \xi_1 < 3\pi/2\},$$

$$\Omega_3 := \{xe_1 + ye_2 + ze_3 \in E_3 : |\xi_1| < 2, \ 4 < z < 6, \ \pi/2 < \arg \xi_1 < 9\pi/4\}.$$

and is constructed similarly to the domain Ω_ζ in Example 2.5 [12]. It is evident that the domain $\Omega \subset E_3$ is not convex in the direction of the straight line L_1.

In the domain $\{\xi_1 \in \mathbb{C} : |\xi_1| < 2, -\pi/4 < \arg \xi_1 < 3\pi/2\}$ of the complex plane let us consider a holomorphic branch $H_1(\xi_1) := \ln |\xi_1| + i \arg \xi_1$ of analytic function $\mathrm{Ln}\, \xi_1$ for which $H_1(1) = 0$. In the domain $\{\xi_1 \in \mathbb{C} : |\xi_1| < 2, \ \pi/2 < \arg \xi_1 < 9\pi/4\}$ let us consider a holomorphic branch $H_2(\xi_1) := \ln |\xi_1| + i \arg \xi_1$ of function $\mathrm{Ln}\, \xi_1$ for which $H_2(1) = 2\pi i$.

Consider the extension Φ_1 of function H_1 into the set $\Omega_1 \cup \Omega_2$ and the extension Φ_2 of function H_2 into the set $\Omega_2 \cup \Omega_3$ constructed with using the following formulas:

$$\Phi_1(\zeta) = H_1(\xi_1) I_1\,, \quad \Phi_2(\zeta) = H_2(\xi_1) I_1\,,$$

where $\zeta = xe_1 + ye_2 + ze_3$.

Inasmuch as $\Phi_1(\zeta) \equiv \Phi_2(\zeta)$ everywhere in Ω_2, the function

$$\Phi(\zeta) := \begin{cases} \Phi_1(\zeta) & \text{for } \zeta \in \Omega_1 \cup \Omega_2, \\ \Phi_2(\zeta) & \text{for } \zeta \in \Omega_3 \end{cases}$$

is monogenic in the domain Ω. At the same time, for the points $\zeta_1 = e_1 + e_3$ and $\zeta_2 = e_1 + 5\,e_3$ we have $\zeta_2 - \zeta_1 \in L_1$ but

$$\Phi(\zeta_2) - \Phi(\zeta_1) = (H_2(1) - H_1(1)) I_1 = 2\pi i I_1 \;\notin \mathfrak{I}_1\,,$$

i.e. the relation (9) is not fulfilled.

Now, let a domain $\Omega \subset E_3$ be convex in the direction of the straight line L_k for all $k \in \{1,2,3\}$ and $D_k := f_k(\Omega)$ for $k = 1,2,3$.

Let A_k be the linear operator which assigns a holomorphic function $F_k:\ D_k \to \mathbb{C}$ to every monogenic function $\Phi : \Omega \to \mathbb{A}_1$ by the formula

$$F_k(\xi_k) = f_k(\Phi(\zeta)), \tag{10}$$

where $\zeta = xe_1 + ye_2 + ze_3$ and $\xi_k = f_k(\zeta)$ for $k = 1,2,3$. It follows from Lemma 1 that the value $F_k(\xi_k)$ does not depend on a choice of a point ζ for which $f_k(\zeta) = \xi_k$ for all $k \in \{1,2,3\}$.

Similar operators A which map monogenic functions taking values in certain commutative algebras onto holomorphic functions of the complex variable are explicitly constructed in the papers [9, 13, 14]. Furthermore, principal extensions of holomorphic functions of the complex variable are used there as *generalized inverse* operators $A^{(-1)}$ satisfying the equality $AA^{(-1)}A = A$. It was also established for every monogenic function Φ that values of the monogenic function $\Phi - A^{(-1)}A\Phi$ belong to a certain maximal ideal $\mathfrak{I}$ of given algebra. Finally, after describing all monogenic functions taking values in the ideal $\mathfrak{I}$, constructive descriptions of monogenic functions taking values in the mentioned algebras by means of holomorphic functions of the complex variable are obtained in the papers [9, 14].

Note that principal extensions of holomorphic functions of the complex variable into domains of the linear span $E_3 \subset \mathbb{A}_1$ are explicitly constructed in Theorem 1.11 [8].

But operators generalized inverse to the operators A_k for $k = 1, 2, 3$ can not be expressed in the form of principal extensions of holomorphic functions of the complex variable. Indeed, in the general case, the mentioned principal extensions are not defined in the domain Ω where monogenic functions $\Phi : \Omega \to \mathbb{A}_1$ are given.

We proceed to constructing operators which are generalized inverse to the operators A_k for $k = 1, 2, 3$.

Let B_k be the operator which assigns a function $\Phi_k : \Omega \to \mathbb{A}_1$ to every holomorphic function $F_k : D_k \to \mathbb{C}$ by the formula

$$\Phi_k(\zeta) = F_k(\xi_k) I_k, \qquad \xi_k = f_k(\zeta)\,, \quad \forall\, \zeta \in \Omega\,. \tag{11}$$

Lemma 2. *Let a domain $\Omega \subset E_3$ be convex in the direction of the straight line L_k for some $k \in \{1, 2, 3\}$ and the function $F_k : D_k \to \mathbb{C}$ be holomorphic in the domain D_k. Then the function (11) is monogenic in the domain Ω, and the Gateaux n-th derivatives $\Phi_k^{(n)}$ are monogenic functions in Ω for any n.*

Proof. Let $h := h_1 e_1 + h_2 e_2 + h_3 e_3 \in E_3$ be an arbitrary nonzero element. Denote $\eta := f_k(h_1 e_1 + h_2 e_2 + h_3 e_3) = h_1 + n_k h_2 + m_k h_3$, where n_k and m_k are the coefficients of the decomposition (4). It is follows from this denotation and the decomposition (4) that $\eta I_k = h I_k$.

We find the limit

$$\lim_{\varepsilon \to 0+0} \frac{\Phi_k(\zeta + \varepsilon h) - \Phi_k(\zeta)}{\varepsilon} = I_k \lim_{\varepsilon \to 0+0} \frac{F_k(\xi_k + \varepsilon \eta) - F_k(\xi_k)}{\varepsilon} =$$

$$= \eta I_k \lim_{\varepsilon \to 0+0} \frac{F_k(\xi_k + \varepsilon \eta) - F_k(\xi_k)}{\varepsilon \eta} = h I_k F_k'(\xi_k) = h \Phi_k'(\zeta),$$

where $\Phi_k'(\zeta) = F_k'(\xi_k) I_k$.

Thus, the function (11) is monogenic in the domain Ω. In a similar way we establish that the Gateaux n-th derivatives $\Phi_k^{(n)}$ are monogenic functions in Ω for any n. The lemma is proved.

It follows from Lemma 2 that the operator B_k is generalized inverse to the operator A_k for all $k = 1, 2, 3$.

3. A constructive description of monogenic functions taking values in the algebra $\mathbb{A}_1$. The following analogue of Theorem 1 [9] (see also Theorem 2.4 [8]) holds true for monogenic functions $\Phi:\Omega\rightarrow\mathbb{A}_1$.

Theorem 1. *Let a domain $\Omega\subset E_3$ be convex in the direction of the straight line L_k for all $k\in\{1,2,3\}$. Then every monogenic function $\Phi:\Omega\rightarrow\mathbb{A}_1$ can be expressed in the form*

$$\Phi(\zeta)=B_kA_k\Phi(\zeta)+\Phi_{0k}(\zeta),\qquad k=1,2,3\,,$$

where $\Phi_{0k}(\zeta)$ is a monogenic in Ω function taking values in the ideal $\mathfrak{I}_k$.

Proof. Consider the function $\Phi_{01}=\Phi-B_kA_k\Phi$ which is monogenic in Ω due to Lemma 2. Taking into account the equalities (10), (11), (6), we obtain

$$f_1(\Phi_{01}(\zeta))=f_1(\Phi(\zeta)-B_kA_k\Phi(\zeta))=f_1(\Phi(\zeta))-f_1(B_kA_k\Phi(\zeta))=$$

$$=F_1(\xi_1)-F_1(\xi_1)=0.$$

Thus, $\Phi_{01}(\zeta)\in\mathfrak{I}_1$. The theorem is proved.

The following theorem describes all monogenic functions taking values in the ideals $\mathfrak{I}_k\,,\ k=1,2,3$.

Theorem 2. *Let a domain $\Omega\subset E_3$ be convex in the direction of the straight line L_k for all $k\in\{1,2,3\}$. Then every monogenic function $\Phi_{0k}:\Omega\rightarrow\mathfrak{I}_k$ can be expressed in the form*

$$\Phi_{0k}(\zeta)=\sum_{j=1,\,j\neq k}^{3}F_j(\xi_j)I_j\qquad\forall\,\zeta\in\Omega\,,\ \ k=1,2,3,$$

where $\xi_j=f_j(\zeta)$ and F_j is a function holomorphic in the domain D_j.

Proof. Inasmuch as Φ_{0k} is a monogenic function taking values in the ideal $\mathfrak{I}_k$,

$$\Phi_{0k}(\zeta)=\sum_{j=1,\,j\neq k}^{3}V_j(x,y,z)I_j,\tag{12}$$

where $V_j:\Omega_{\mathbb{R}}\rightarrow\mathbb{C}$.

Acting onto the equality (12) by the operator A_j with $j=1,2,3,\,j\neq k$, and taking into account the equalities (10), (6), we obtain the equality $A_j\Phi_{0k}=V_j$. At the same time, $A_j\Phi_{0k}$ is a function F_j holomorphic in the domain D_j due to the definition of the operator A_j . The theorem is proved.

It follows from Theorem 1, Theorem 2 and the equality (11) that in an arbitrary domain Ω convex in the direction of the straight line L_k for all $k \in \{1, 2, 3\}$, every monogenic function $\Phi(\zeta)$ can be explicitly constructed with using three holomorphic functions in the form:

$$\Phi(\zeta) = F_1(\xi_1)I_1 + F_2(\xi_2)I_2 + F_3(\xi_3)I_3, \tag{13}$$

where $\xi_j = f_j(\zeta)$ and F_j is a function holomorphic in the domain D_j for $j = 1, 2, 3$.

A similar result is established for analytic functions of a bicomplex variable in any domain of the Segre algebra without an assumption about convexity of domain in the direction of any straight lines (see, for example, [2, 3]). In contrast to it, the condition of convexity of Ω in the direction of the straight line L_k for all $k \in \{1, 2, 3\}$ is essential for monogenic functions $\Phi : \Omega \to \mathbb{A}_1$ to be represented in the form (13) as it follows from Example 1.

The following statement follows from the equality (13) because its right-hand part is a monogenic function in the domain $\Delta := \{\zeta = xe_1+ye_2+ze_3 : f_k(\zeta) \in D_k,\ k = 1, 2, 3\}$.

Theorem 3. *Let a domain $\Omega \subset E_3$ be convex in the direction of the straight line L_k for all $k \in \{1, 2, 3\}$ and a function $\Phi : \Omega \to \mathbb{A}_1$ be monogenic in Ω. Then Φ can be continued to a function monogenic in the domain Δ.*

The following statement is true for monogenic functions in an arbitrary domain Ω.

Theorem 4. *For every monogenic function $\Phi : \Omega \to \mathbb{A}_1$ in an arbitrary domain Ω, the Gateaux n-th derivatives $\Phi^{(n)}$ are monogenic functions in Ω for any n.*

Proof. Consider a ball $\mho \subset \Omega$ with the center in an arbitrary point $\zeta_0 = x_0e_1 + y_0e_2 + z_0e_3 \in \Omega$. Inasmuch as $\mho$ is a convex set, in the neighbourhood $\mho$ of the point ζ_0 we have the equality (13). Now, the statement of theorem follows from Lemma 2.

Now, we can state that every monogenic function $\Phi : \Omega \to \mathbb{A}_1$ satisfies the equalities (3) in Ω due to Theorem 4 and the equality (2), i.e. the components U_1, U_2, U_3 from (5) satisfy the three-dimensional Laplace equation in the domain $\Omega_{\mathbb{R}}$.

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