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On subharmonic functions in the unit ball growing near a part of the boundary

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Dedicated to memory of Professor Promarz M. Tamrazov

We get an integral estimate for Riesz measures of subharmonic functions in the n-dimensional unit ball, which grow near some subset of the boundary sphere at most as a given function.

1. Introduction. It is well known (see, for example, [1]) that the Riesz measure $\mu = \frac{1}{2\pi}\triangle v$ of any bounded from above subharmonic function $v(z)$ in the unit disk satisfies the following inequality

$$\int_{|\lambda|<1} (1 - |\lambda|)\mu(d\lambda) < \infty. \tag{1}$$

Actually, it is a subharmonic analog of the classical Blaschke condition for zeros of bounded analytic functions.

The estimate (1) has a lot of generalizations for analytic and subharmonic functions growing near the boundary of the unit disk (see [2 — 6]) or its part (see [7 — 10]). In particular, in [7] the corresponding bound was obtained for Riesz measures of subharmonic functions growing polynomially near some compact subset E on the unit circle. Clearly, such bound depends on thinness of E.

In the paper [9] we investigated the case of subharmonic function in the unit disk growing near E as an arbitrary function φ. Instead of (1) we obtained the inequality

$$\int \psi(\rho(\lambda))(1-|\lambda|)\mu(d\lambda) < \infty \tag{2}$$

under some condition connected functions ψ, φ and the set E. We also proved that this conditions are optimal, in a sense.

In the given paper we extend our results to subharmonic functions in the unit ball $B \subset \mathbb{R}^n$, $n > 2$.

2. Main results. Suppose $E = \overline{E} \subset \partial B$, $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically decreasing continues function, $\varphi(t) \to \infty$ as $t \to 0$. For $z \in \overline{B}$ put $\rho(z) = dist(z, E)$, $F(t) = m\{\zeta \in \partial B : \rho(\zeta) < t\}$, where $m(d\zeta)$ is the normalized $(n-1)$-dimensional Lebesgue measure on ∂B. In other words, the usual Lebesgue measure on ∂B is $\sigma_n m(d\zeta)$, where $\sigma_n = \dfrac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the area of unit sphere in $\mathbb{R}^n$.

We prove the following theorem.

Theorem 1. *Let $v(z)$ be a subharmonic function in B, $v \not\equiv -\infty$, and*

$$v(z) \leqslant \varphi(\rho(z)) \tag{3}$$

for all $z \in B$. If

$$\int_0^2 \varphi(s)dF(s) < \infty, \tag{4}$$

then the Riesz measure $\mu = \dfrac{\triangle v}{(n-1)\sigma_n}$ of the function v satisfies the condition

$$\int_B (1-|\lambda|)\mu(d\lambda) < \infty. \tag{5}$$

In the case when condition (4) is invalid, the integral (5) may be divergent. However we control the growth of μ in this case too.

Theorem 2. *Suppose φ, ψ are absolutely continues positive functions on $(0,2)$, $\varphi(t)$ monotonically decreases and $\psi(t)$ monotonically increases, $\varphi(t) \to +\infty$ as $t \to +0$, $\psi(t) \to 0$ as $t \to +0$. Let*

$$\int_0^1 (-\varphi'(t))\psi(t)F(t)dt < \infty. \tag{6}$$

If a subharmonic function $v(z)$ satisfies (3) in B, then the bound

$$\int_B \psi(k\rho(\lambda))(1-|\lambda|)\mu(d\lambda) < \infty, \tag{7}$$

is valid for its Riesz measure μ with the constant $k = k(n)$.

On the other hand, we get

Theorem 3. *Let φ, ψ be the same as above, and, moreover, the function $\varphi(1/t)$ be log-convexity. If*

$$\int_0^a (-\varphi'(t))\psi(t)F(t)dt = \infty, \tag{8}$$

then the Riesz measure μ_0 of the subharmonic function $v_0(x) = \varphi(\rho(x))$ satisfies the condition

$$\int_B \psi(\rho(\lambda))(1-|\lambda|)\mu_0(d\lambda) = \infty.$$

Remark. *The function $v_0(x)$ is subharmonic. Indeed, the function $-\log\rho(x) = \sup_{\zeta\in E}\{-\log|x-\zeta|\}$ is subharmonic in $\mathbb{R}^n\setminus E$. The superposition of convex and subharmonic is a subharmonic function as well.*

Example 1. Let $m(E) > 0$. In this case $\lim_{t\to 0} F(t) = m(E)$. For small $\varepsilon > 0$ we have

$$m(E)\int_0^\varepsilon (-\varphi'(t))\psi(t)dt \leqslant \int_0^\varepsilon (-\varphi'(t))\psi(t)F(t)dt \leqslant \\ \leqslant 2m(E)\int_0^\varepsilon (-\varphi'(t))\psi(t)dt.$$

Hence the condition (6) has the form

$$\int_0^1 (-\varphi'(t))\psi(t)dt < \infty.$$

In particular, one can put $\varphi(t) = t^{-q}$, $\psi(t) = t^{q+\varepsilon}$ for all $\varepsilon > 0$. Inequality (7) is valid with $k = 1$.

Example 2. Let E be a union of N points. Then the set $E_t = \{\zeta \in \partial B : \rho(\zeta) < t\}$ for small t is a union of N hats $\{\zeta \in \partial B : |\zeta - \zeta_j| < t\}$. Hence, for small t we have

$$Nc(n)t^{n-1} \leqslant F(t) = m(E_t) \leqslant NC(n)t^{n-1}.$$

Since condition (6) has the form

$$\int_0^1 (-\varphi'(t))\psi(t)t^{n-1}dt < \infty,$$

we can take $\varphi(t) = t^{-q}$, $\psi(t) = t^{q-n+1+\varepsilon}$ for arbitrary $\varepsilon > 0$.

So, in the case of exponential functions φ and ψ one can see certain relations between growth of these functions and thinness of E.

Definition. *Suppose E is a compact subset of $\mathbb{R}^n$, $N(E,\varepsilon)$ is the minimal number of the balls of radius ε covering E. The upper and lower Minkowski's dimension for the set E are the numbers*

$$\overline{æ}(E) = \overline{\lim}_{\varepsilon\to 0}\frac{\log N(E,\varepsilon)}{\log 1/\varepsilon}, \qquad \underline{æ}(E) = \underline{\lim}_{\varepsilon\to 0}\frac{\log N(E,\varepsilon)}{\log 1/\varepsilon}.$$

Theorem 4 (was proved in [8] for $n = 2$). *Suppose v is a subharmonic function in B such that $v(x) \leqslant \rho^{-q}(x)$ for all $x \in B$. Then for any $\varepsilon > 0$ its Riesz measure μ satisfies the condition*

$$\int_B \rho(\lambda)^{q-n+\overline{æ}(E)+1+\varepsilon}\mu(d\lambda) < \infty.$$

Also, for $v(x) = \rho^{-q}(x)$ we have

$$\int_B \rho(\lambda)^{q-n+\underline{æ}(E)+1-\varepsilon}\mu(d\lambda) = \infty.$$

3. Proofs. Proofs of Theorems 1 — 4 use the next Lemmas.

Lemma 1. *Suppose ν is a finite Borel measure on X, $g(x)$ is a Borel function on X, $\varphi(t)$ is a Borel function on $\mathbb{R}$. Then*

$$\int_X \varphi(g(x))\nu(dx) = \int_{\mathbb{R}} \varphi(s)H(ds),$$

with $H(s) = \nu\{x : g(x) < s\}$.

The lemma immediately reduces to the case of probability measure ν, i.e., such that $\nu(X) = 1$. In this case the Lemma is well-known (see., for example. [11, formula (15.3.1)]).

For $y_0 \in \partial B$ and $t > 0$ put

$$L(y_0, t) = \{y \in \partial B : |y - y_0| < t\}.$$

Lemma 2. *The harmonic measure $\omega_{L(y_0,t)}(x)$ of the set $L(y_0, t)$ with respect to B satisfies the condition*

$$\inf_{x \in B : |x - y_0| = t} \omega_{L(y_0,t)}(x) = C > 0, \qquad t \leqslant t_0.$$

The constants C, t_0 depend only on n.

Remark. *This property of harmonic measure in the case $n = 2$ was observed in* [7].

Proof. For $x \in B$, $|x - y_0| = t$, put $x^* = \dfrac{x}{|x|}$, $s = |x^* - x|$. Note that $s = 1 - |x| \leqslant |y_0 - x| = t$. Let y be an arbitrary point of $L(y_0, t)$. In the triangle with vertexes in x, x^*, y the angle in x^* is acute, hence,

$$|x - y|^2 \leqslant |x^* - y|^2 + |x^* - x|^2 = |x^* - y|^2 + s^2.$$

Moreover, if $y \in L(x^*, 3s)$, then $|x - y|^2 \leqslant 10s^2$. Therefore,

$$\omega_{L(y_0,t)}(x) = \int_{L(y_0,t)} \frac{1 - |x|^2}{|y - x|^n}\, m(dy) \geqslant$$
$$\geqslant s \int_{L(y_0,t) \cap L(x^*,3s)} \frac{m(dy)}{|y - x|^n} \geqslant \frac{1}{10^{n/2}} \cdot \frac{m(L(y_0, t) \cap L(x^*, 3s))}{s^{n-1}}.$$

Next, show that for $t \leqslant t(n)$ we will obtain

$$m(L(y_0, t) \cap L(x^*, 3s)) \geqslant C_1(n) s^{n-1}. \tag{9}$$

Indeed,

$$|x^* - y_0| \leqslant |x^* - x| + |x - y_0| = s + t\,.$$

In the case $|y_0 - x^*| > 2s$ put

$$\widehat{y} = x^* + 2s \frac{y_0 - x^*}{|y_0 - x^*|}\,, \quad y^* = \frac{\widehat{y}}{|\widehat{y}|}\,.$$

Also, note that $|\widehat{y}-x^*|=2s$, $|\widehat{y}-y_0|=|y_0-x^*|-2s$. Consider the right-angled triangles with vertexes in $\left(0,\dfrac{y_0+x^*}{2},x^*\right)$ and $\left(0,\dfrac{y_0+x^*}{2},\widehat{y}\right)$. We have

$$\left|\frac{y_0+x^*}{2}\right|^2=1-\left|\frac{y_0-x^*}{2}\right|^2,$$

$$\begin{aligned}|\widehat{y}|^2=\left|\frac{y_0+x^*}{2}\right|^2+\left|\frac{|y_0-x^*|}{2}-2s\right|^2&=\\ =1-\left[\left(\frac{|y_0-x^*|}{2}\right)^2-\left(\frac{|y_0-x^*|}{2}-2s\right)^2\right]&=\\ &=1-2s(|y_0-x^*|-2s).\end{aligned}$$

Let $t<\dfrac{1}{8}$. Then $|y_0-x^*|\leqslant 2t<\dfrac{1}{4}$. We get

$$|y^*-\widehat{y}|=1-|\widehat{y}|=1-\sqrt{1-2s(|y_0-x^*|-2s)}<2s(|y_0-x^*|-2s)<\frac{s}{2}.$$

We claim that $L(y^*,\frac{s}{2})\subset L(y_0,t)\cap L(x^*,3s)$. Indeed, for any y such that $|y-y^*|<\dfrac{s}{2}$ we have

$$\begin{aligned}|y-y_0|\leqslant\ & |y-y^*|+|y^*-\widehat{y}|+|\widehat{y}-y_0|<\ && \tfrac{s}{2}+\tfrac{s}{2}+t+s-2s=t,\\ |y-x^*|\leqslant\ & |y-y^*|+|y^*-\widehat{y}|+|\widehat{y}-x^*|<\ && \tfrac{s}{2}+\tfrac{s}{2}+2s=3s.\end{aligned}$$

Thus in this case $L(y^*,\frac{s}{2})\subset L(y_0,t)\cap L(x^*,3s)$.

Investigate the case $|y_0-x^*|\leqslant 2s$. Consider the triangle with vertexes in x, x^*, y_0. The angle α in x^* is acute. Hence we have

$$t^2=|y_0-x|^2\leqslant|y_0-x^*|^2+|x^*-x|^2\leqslant 5s^2,\qquad s\geqslant\frac{t}{\sqrt{5}}.$$

On other hand, for small t the angle α is close to $\dfrac{\pi}{2}$, hence we can suppose $\cos\alpha\leqslant\dfrac{1}{8}$ and

$$t^2=|x^*-y_0|^2+s^2-2\cos\alpha|x^*-y_0|s\geqslant|x^*-y_0|^2+\frac{s^2}{2}.$$

Therefore, $|x^* - y_0| \leqslant \dfrac{3t}{\sqrt{10}}$. Consider $\widehat{y} = \dfrac{x^* + y_0}{2}$. We have $|\widehat{y} - x^*| = |\widehat{y} - y_0| = \dfrac{|x^* - y_0|}{2} \leqslant \dfrac{3t}{2\sqrt{10}}$. Put $y^* = \dfrac{\widehat{y}}{|\widehat{y}|}$. Consider the rectangular triangle with vertexes in 0, $\widehat{y}$, y_0. We get

$$|y^* - \widehat{y}| = 1 - |\widehat{y}| = 1 - \sqrt{1 - \frac{|x^* - y_0|^2}{4}} \leqslant \frac{|x^* - y_0|^2}{4} \leqslant \frac{9t^2}{40}.$$

Now, if $|y - y^*| < \frac{s}{2}$, then we have

$$|y - y_0| \leqslant |y - y^*| + |y^* - \widehat{y}| + |\widehat{y} - y_0| < \frac{s}{2} + \frac{9t^2}{40} + \frac{3t}{2\sqrt{10}} < t$$

for small t and

$$|y - x^*| \leqslant |y - y^*| + |y^* - \widehat{y}| + |\widehat{y} - x^*| < \\ < \frac{s}{2} + \frac{9t^2}{40} + \frac{3t}{2\sqrt{10}} \leqslant \frac{s}{2} + \frac{9s^2}{8} + \frac{3s}{2\sqrt{2}} < 3s\,.$$

Therefore in this case $L(y^*, \frac{s}{2}) \subset L(y_0, t) \cap L(x^*, 3s)$ too.

If we project $L(y^*, \frac{s}{2})$ on the hyperplane l that tangent to B in the point y^*, then for all $y, y' \in L(y^*, \frac{s}{2})$ and small $s \leqslant t$ we get

$$|y - y'| \leqslant \frac{3}{2}|Pr_l y - Pr_l y'|.$$

Hence $Pr_l L(y^*, \frac{s}{2})$ contains an $(n-1)$-dimensional ball B' with radius $\dfrac{s}{3}$. Thus, for small t we have

$$m\left(L\left(y^*, \frac{s}{2}\right)\right) \geqslant m(B') \geqslant \left(\frac{s}{3}\right)^{n-1}.$$

This implies (9). The proof is complete.

Let

$$G(z, \lambda) = \frac{1}{|z - \lambda|^{n-2}} - h(z, \lambda), \qquad z \in \overline{\Omega},\ \lambda \in \Omega, \tag{10}$$

be the Green function for the Laplace operator in $\Omega \subset \mathbb{R}^n$, $h(z, \lambda)$ be harmonic in $z \in \Omega$ and continues in $z \in \overline{\Omega}$ such that $h(\zeta, \lambda) = \dfrac{1}{|\zeta - \lambda|^{n-2}}$ for $\zeta \in \partial\Omega$. Note that $G(z, \lambda) = G(\lambda, z)$, $\forall z, \lambda \in \Omega$ (see [1]).

The connected component of the set $\{z \in B : \rho(z) > t\}$ that contains the point 0 we denote by Ω_t.

Lemma 3. There are $t_0 = t_0(n) > 0$ *and* $\beta = \beta(n) \in (1, +\infty)$ *such that*

$$G_{\Omega_t}(0, \lambda) \geqslant \frac{n-2}{2}(1 - |\lambda|), \qquad \forall t \leqslant t_0,\ \forall \lambda \in \Omega_{\beta t}.$$

Proof. Clearly,

$$1 + (n-2)(1-s) \leqslant s^{-(n-2)} \leqslant 1 + (n-1)(1-s), \tag{11}$$

for all $s \in (1 - t_0, 1)$. The left inequality is true for all $0 < s < 1$. For $\lambda \in \partial\Omega_t$ we have $t = \rho(\lambda) \geqslant 1 - |\lambda|$. Hence for $|\lambda| \geqslant 1 - t$ with $t \leqslant t_0$ we get

$$1 + (n-2)(1 - |\lambda|) \leqslant \frac{1}{|\lambda|^{n-2}} \leqslant \frac{1}{(1-t)^{n-2}} \leqslant 1 + (n-1)t, \tag{12}$$

the left inequality is true for all $|\lambda| < 1$.

Suppose $\lambda \in \partial\Omega_t$. If $|\lambda| = 1$, then $h(0, \lambda) = 1$. If $|\lambda| < 1$ then $\rho(\lambda) = t$. Hence for some $\zeta \in \partial B$ we have $|\zeta - \lambda| = t$. Using lemma 2, we get $\omega_{L(\zeta,t)}(\lambda) \geqslant C$. If $E_t = \{\zeta \in \partial B : \rho(\zeta) < t\}$ then for such λ we have

$$\omega_{E_t}(\lambda) \geqslant \omega_{L(\zeta,t)}(\lambda) \geqslant C.$$

Thus for each $\lambda \in \partial\Omega_t$ we get

$$h(0, \lambda) \leqslant 1 + \frac{(n-1)t}{C}\,\omega_{E_t}(\lambda)\,.$$

By maximum principle, this inequality holds for all $\lambda \in \Omega_t$. If the inequality

$$\omega_{E_t}(\lambda) \leqslant \frac{C(n-2)}{2(n-1)} \cdot \frac{1 - |\lambda|}{t}, \tag{13}$$

holds for some $\beta < \infty$ and all $\lambda \in \Omega_{\beta t}$, then for such λ we have

$$G_{\Omega_t}(0, \lambda) = \frac{1}{|\lambda|^{n-2}} - h(0, \lambda) \geqslant$$
$$\geqslant 1 + (n-2)(1 - |\lambda|) - \left[1 + \frac{n-2}{2}(1 - |\lambda|)\right] = \frac{n-2}{2}(1 - |\lambda|).$$

and the proof will be completed.

We have

$$\omega_{E_t}(\lambda) = \int_{\zeta\in E_t} \frac{1-|\lambda|^2}{|\zeta-\lambda|^n}\, m(d\zeta) =$$

$$= (1-|\lambda|^2)\int_{\zeta\in E_t} \frac{m(d\zeta)}{[(1-|\lambda|)^2+2(1-\cos\gamma)|\lambda|]^{n/2}},$$

with the angle γ between the vectors ζ and $\dfrac{\lambda}{|\lambda|}$.

For $t<\frac{1}{4}$ we have $\frac{1}{2}<|\lambda|<1$, hence we get

$$\omega_{E_t}(\lambda) \leqslant (1-|\lambda|)2^{1+n/2}\int_{\zeta\in E_t} \frac{m(d\zeta)}{[2(1-\cos\gamma)]^{n/2}}.$$

Find a low bound of the angle γ. Take $\zeta'\in E$ such that $|\zeta'-\zeta|<t$. We have

$$\gamma \geqslant 2\sin\frac{\gamma}{2} = (2-2\cos\gamma)^{\frac{1}{2}} = \left|\frac{\lambda}{|\lambda|}-\zeta\right|,$$

$$\left|\frac{\lambda}{|\lambda|}-\zeta\right| \geqslant |\lambda-\zeta'| - \left|\lambda-\frac{\lambda}{|\lambda|}\right| - |\zeta'-\zeta| \geqslant \beta t-(1-|\lambda|)-t \geqslant (\beta-2)\,t.$$

If $\beta'=\beta-2>0$, we get $\gamma\geqslant\beta' t$. To prove (13) it is sufficient to check that the integrals

$$\int_{\zeta\in E_t} \frac{m(d\zeta)}{[2(1-\cos\gamma)]^{n/2}} \leqslant \int_{\zeta:\gamma\geqslant\beta' t} \frac{m(d\zeta)}{[2(1-\cos\gamma)]^{n/2}}$$

are less than $\dfrac{(n-2)C}{2^{2+n/2}(n-1)t}$ for a suitable β.

Take the spherical coordinate system $\theta_1,\ldots,\theta_{n-1}$ on ∂B such that $\gamma=\theta_1\in(0,\pi)$, $\theta_2,\ldots,\theta_{n-2}\in(0,\pi)$, $\theta_{n-1}\in[0,2\pi]$. Using inequations $\sin\theta_1\leqslant 2\sin\dfrac{\theta_1}{2}$, $0\leqslant\sin\theta_i\leqslant 1$, $\theta_i\in[0,\pi]$, $i=2,\ldots,n-2$, we get

$$\int_{\zeta:\gamma\geqslant\beta' t} \frac{m(d\zeta)}{[2(1-\cos\gamma)]^{n/2}} =$$

$$= \frac{1}{\sigma_n}\int\cdots\int_{\theta_1\geqslant\beta' t} \frac{\sin^{n-2}\theta_1\sin^{n-3}\theta_2\ldots\sin\theta_{n-2}}{(2\sin\frac{\theta_1}{2})^n}\, d\theta_1\ldots d\theta_{n-1} \leqslant$$

$$\leqslant \frac{\pi^{n-2}}{2\sigma_n}\int_{\beta' t}^{\pi} \frac{d\theta_1}{(\sin\frac{\theta_1}{2})^2} \leqslant \frac{\pi^{n-2}}{2\sigma_n}\pi^2\int_{\beta' t}^{\pi}\frac{d\theta_1}{\theta_1^2} < \frac{\pi^n}{2\sigma_n\beta' t}.$$

Thus, for sufficiently large β we obtain the required estimate. The proof is complete.

Lemma 4. *For all $x \in B$ and $\tau \in [0,1]$ we have $\rho(x) \leqslant 2\rho(\tau x)$.*

Proof. The ball with the center at the point τx and radius $1 - \tau|x|$ is contained in B and touches it at the point $\frac{x}{|x|}$. Hence for each point $\zeta \in \partial B \backslash \frac{x}{|x|}$ we have $|\zeta - \tau x| > 1 - \tau|x|$. Therefore, $\rho(\tau x) \geqslant 1 - \tau|x|$. This implies that

$$\rho(x) \leqslant \rho(\tau x) + |\tau x - x| \leqslant \rho(\tau x) + 1 - \tau|x| \leqslant 2\rho(\tau x).$$

The proof is complete.

The proof of Theorem 1. Using Lemma 1 with the measure $m(d\zeta)$, we get

$$\int_{\partial B} \varphi(\rho(y)) m(dy) = \int_0^\infty \varphi(s) dF(s) < \infty.$$

Hence the function $\varphi(\rho(y))$ is integrable on ∂B. Consider the harmonic function

$$U(x) = \int_{\partial B} \frac{1 - |x|^2}{|y - x|^n} \varphi(\rho(y))\, m(dy).$$

For each $\zeta \in \partial B \backslash E$ we have

$$\varliminf_{x \to \zeta} U(x) \geqslant \varphi(\rho(\zeta)).$$

Therefore,

$$\varlimsup_{x \to \zeta} (v(x) - U(x)) \leqslant \varlimsup_{x \to \zeta} \varphi(\rho(x)) - \varphi(\rho(\zeta)) = 0. \tag{14}$$

Let Ω_t be the connected component of the set $\{x \in B : \rho(x) > t\}$ containing 0 and $z \in \partial\Omega_t \backslash \partial B$. Then $\rho(z) = t$ and for some point $\zeta \in E$ we have $|z - \zeta| = t$. Also, we have $v(z) \leqslant \varphi(\rho(z)) = \varphi(t)$. Using Lemma 2, we get $\omega_{L(\zeta,t)}(z) \geqslant C$. Since the inequality $\varphi(\rho(y)) \geqslant \varphi(t)$ holds for $y \in L(\zeta, t)$, it follows that

$$U(z) = \int_{\partial B} \frac{1 - |z|^2}{|z - y|^n} \varphi(\rho(y)) m(dy) \geqslant$$
$$\geqslant \varphi(t) \int_{L(\zeta,t)} \frac{1 - |z|^2}{|z - y|^n} m(dy) \geqslant v(z)\omega_{L(\zeta,t)}(z) \geqslant C v(z).$$

Consequently,

$$\overline{\lim_{x\to z}}\left[v(x)-\frac{U(x)}{C}\right]\leqslant v(z)-\frac{U(z)}{C}\leqslant 0.$$

If we combine this inequality with (14) and the maximum module principle, we obtain that the function $\max\{1,C^{-1}\}U(x)$ is the harmonic majorant for v in Ω_t. Hence the Green representation is true for $v(x)$. So, we have

$$v(x)=u_t(x)-\int_{\partial\Omega_t}G_{\Omega_t}(x,y)\mu(dy),\quad x\in\Omega_t,$$

with the Riesz measure μ for v and the least harmonic majorant $u_t(x)$ for v in Ω_t (see. [1]).

First consider the case $v(0)\neq-\infty$. Using Lemma 3 for $t\leqslant t_0$, we get

$$\int_{\Omega_{\beta t}}(1-|\lambda|)\mu(d\lambda)\leqslant\frac{2}{n-2}\int_{\Omega_t}G_{\Omega_t}(0,\lambda)\mu(d\lambda)=\frac{2}{n-2}\left(u_t(0)-v(0)\right).\quad(15)$$

Since

$$u_t(0)\leqslant\max\{1,C^{-1}\}U(0)=\max\{1,C^{-1}\}\int_{\partial B}\varphi(\rho(y))m(dy)<\infty,$$

we see that the right-hand side of inequality (15) is bounded uniformly for $t\in(0,1)$.

Note that $\bigcup_{t\in(0,1)}\Omega_{\beta t}=B$. Therefore, (15) implies (5).

If $v(0)=-\infty$, we can replace the function $v(x)$ by the function $v_1(x)$ that equals $v(x)$ for $|x|\geqslant\frac{1}{2}$ and harmonic in the ball $|x|<\frac{1}{2}$ with the values $v(x)$ on the sphere $|x|=\frac{1}{2}$. According to [12, Cor. 3.2.5], the function $v_1(x)$ is subharmonic in B. Clearly, $v_1(0)\neq-\infty$. Since the Riesz measure μ_1 for the function v_1 is equal to the measure μ for $|x|>\frac{1}{2}$, the difference between integrals

$$\int_{\Omega_t}(1-|\lambda|)\mu(d\lambda)\qquad\text{and}\qquad\int_{\Omega_t}(1-|\lambda|)\mu_1(d\lambda)$$

is bounded. If the first integral is uniformly bounded for $t\to 0$, then the second integral is uniformly bounded too. Finally note that the condition (4) holds for the function $v_1(z)\leqslant\varphi_1(\rho(z))$, with $\varphi_1(t)=$ $=\max\{\varphi(t),\varphi(\frac{1}{2})\}$. The proof in complete.

The proof of Theorem 2. Consider the harmonic function

$$V_t(x) = \int_{\partial B} \frac{1-|x|^2}{|x-y|^n} \min\{\varphi(\rho(y)), \varphi(t)\} m(dy), \quad x \in B\,. \tag{16}$$

The function $V_t(x)$ is continues in $\overline{B}$. Note that

$$\lim_{x \to y \in \partial B} V_t(x) = \varphi(t) \quad \text{for} \quad \rho(y) < t$$

and

$$\lim_{x \to y \in \partial B} V_t(x) = \varphi(\rho(y)) \quad \text{for} \quad \rho(y) \geq t.$$

Let Ω_t be the same as in the proof of the previous theorem, $z \in \partial\Omega_t \cap B$. arguing as in the proof of the previous theorem, we get $\omega_{L(\zeta,t)}(z) \geqslant C$ for some $\zeta \in E$, where C is the constant from Lemma 2. Since $\rho(y) < t$ for $y \in L(\zeta, t)$, we get

$$V_t(z) \geqslant \int_{L(\zeta,t)} \frac{1-|z|^2}{|z-y|^n} \varphi(t) m(dy) = \varphi(t)\omega_{L(\zeta,t)}(z) \geqslant C\varphi(t) \geqslant Cv(z).$$

For $z \in \partial\Omega_t \cap \partial B$ we have $V_t(z) = \varphi(\rho(z))$, therefore,

$$\overline{\lim_{x \to z}}\, v(x) \leqslant \overline{\lim_{x \to z}}\, \varphi(\rho(x)) = \varphi(\rho(z)) = V_t(z).$$

By the maximum principle,

$$v(x) \leqslant \max\{1, C^{-1}\}\, V_t(x)$$

for all $x \in \Omega_t$. Applying the Green formula for $v(x)$ in Ω_t, we get the inequality

$$\int_{\Omega_t} G_{\Omega_t}(0, \lambda)\mu(d\lambda) \leqslant \max\{1, C^{-1}\}\, V_t(0) - v(0)\,. \tag{17}$$

Arguing as in the proof of the previous theorem, we may suppose $v(0) \neq$ $\neq -\infty$.

Furthermore,

$$V_t(0) = \int_{\{y \in \partial B : \rho(y) < t\}} \varphi(t) m(dy) + \int_{\{y \in \partial B : \rho(y) \geqslant t\}} \varphi(\rho(y)) m(dy).$$

Applying Lemma 1 with $g(y)=\rho(y)$ and

$$H(s)=m\{y\in\partial B:\rho(y)<s\}-m\{y\in\partial B:\rho(y)<t\}=F(s)-F(t),$$

we get

$$V_t(0)=\varphi(t)F(t)+\int_t^2\varphi(s)dF(s)=\varphi(2)+\int_t^2(-\varphi'(s))F(s)ds. \tag{18}$$

Note that $F(2)=1$. By Lemma 4, if $x\in B$ and $\rho(x)>2t$, then the whole segment $[0,x]$ is is contained in the set $\{x:\rho(x)>t\}$, hence, $\{x:\rho(x)>2t\}\subset\Omega_t$. Let $\beta>2$. By Lemma 3,

$$\int_{\{\lambda\in B:\rho(\lambda)>2\beta t\}}(1-|\lambda|)\mu(d\lambda)\leqslant\int_{\Omega_{\beta t}}(1-|\lambda|)\mu(d\lambda)\leqslant$$
$$\leqslant\frac{2}{n-2}\int_{\Omega_{\beta t}}G_{\Omega_t}(0,\lambda)\mu(d\lambda)\leqslant\frac{2}{n-2}\int_{\Omega_t}G_{\Omega_t}(0,\lambda)\mu(d\lambda).$$

Combining the latter inequality with (17), (18), we get

$$\int_{\{\lambda\in B:\rho(\lambda)>2\beta t\}}(1-|\lambda|)\mu(d\lambda)\leqslant$$
$$\leqslant\mathrm{const}+\max\{1,C^{-1}\}\frac{2}{n-2}\int_t^2(-\varphi'(s))F(s)ds. \tag{19}$$

Put $k=\dfrac{1}{2\beta}<1$. Apply Lemma 1 to the restriction of the measure $(1-|\lambda|)\mu(d\lambda)$ on the set $\{\lambda\in B:\rho(\lambda)>\varepsilon\}$. We get

$$\int_{\{\lambda\in B:\rho(\lambda)>\varepsilon\}}\psi(k\rho(\lambda))(1-|\lambda|)\mu(d\lambda)=\int_\varepsilon^2\psi(kt)d\widetilde{H}(t), \tag{20}$$

with

$$\widetilde{H}(t)=\int_{\{\lambda\in B:\varepsilon<\rho(\lambda)<t\}}(1-|\lambda|)\mu(d\lambda)=$$
$$=\int_{\{\lambda\in B:\rho(\lambda)>\varepsilon\}}(1-|\lambda|)\mu(d\lambda)-\int_{\{\lambda\in B:\rho(\lambda)\geqslant t\}}(1-|\lambda|)\mu(d\lambda).$$

Taking into account that $\rho(\lambda) < 2$ for all $\lambda \in B$ and integrating by parts, we have

$$\int_{\varepsilon}^{2} \psi(kt)d\widetilde{H}(t) = -\int_{\varepsilon}^{2} \psi(kt)d\left(\int_{\{\lambda:\rho(\lambda)\geqslant t\}} (1-|\lambda|)\mu(d\lambda)\right) = \\ = \psi(k\varepsilon)\int_{\{\lambda:\rho(\lambda)\geqslant\varepsilon\}} (1-|\lambda|)\mu(d\lambda) + \\ + k\int_{\varepsilon}^{2} \psi'(kt)\left(\int_{\{\lambda:\rho(\lambda)\geqslant t\}} (1-|\lambda|)\mu(d\lambda)\right) dt. \quad (21)$$

Note that the set $\{t \in [0,1];\ \mu\{\lambda : \rho(\lambda) = t\} > 0\}$ is at most countable. Hence we can replace $\{\lambda : \rho(\lambda) \geqslant t\}$ by $\{\lambda : \rho(\lambda) > t\}$ in the previous formula. Moreover, we may suppose that $\mu\{\lambda : \rho(\lambda) = \varepsilon\} = 0$, therefore we replace $\{\lambda : \rho(\lambda) \geqslant \varepsilon\}$ by $\{\lambda : \rho(\lambda) > \varepsilon\}$. Let us check that the integral

$$\int_{\varepsilon}^{2} \psi'(kt)\left(\int_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)\mu(d\lambda)\right) dt = \\ = \frac{1}{k}\int_{k\varepsilon}^{k2} \psi'(t)\left(\int_{\{\lambda:\rho(\lambda)>2\beta t\}} (1-|\lambda|)\mu(d\lambda)\right) dt \quad (22)$$

is bounded from above uniformly in $\varepsilon > 0$. Indeed, by (19), integral (22) is bounded from above by

$$\text{const}\int_{k\varepsilon}^{2k} \psi'(t)dt + \max\{1, C^{-1}\}\frac{2}{n-2}\int_{k\varepsilon}^{2k} \psi'(t)\int_{t}^{2} (-\varphi'(s))F(s)\,ds\,dt.$$

Note that

$$\int_{k\varepsilon}^{2k} \psi'(t)dt = \psi(2k) - \psi(k\varepsilon) \to \psi(2k) \qquad \text{при} \quad \varepsilon \to 0,$$

and

$$\int_{k\varepsilon}^{2k} \psi'(t)\int_{t}^{2} (-\varphi'(s))F(s)\,ds\,dt = \psi(2k)\int_{2k}^{2} (-\varphi'(s))F(s)\,ds - \\ - \psi(k\varepsilon)\int_{k\varepsilon}^{2} (-\varphi'(s))F(s)\,ds + \int_{k\varepsilon}^{2k} \psi(t)(-\varphi'(t))F(t)\,dt.$$

The first integral from the right-hand side does not depends on ε, the second one is negative. Taking into account the condition (6), we get that the latter integral is bounded from above uniformly on ε. Therefore, the same is valid for integral (22).

Hence for each $\eta > 0$, for all sufficiently small ε, and for all $\delta < \varepsilon$ we have

$$(\psi(k\varepsilon) - \psi(k\delta)) \int_{\{\lambda:\rho(\lambda)>\varepsilon\}} (1-|\lambda|)\mu(d\lambda) \leqslant \\ \leqslant k \int_{\delta}^{\varepsilon} \psi'(kt) \int_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)\mu(d\lambda) dt < \eta.$$

Here we use that the value $\int_{\{\lambda:\rho(\lambda)>t\}}(1-|\lambda|)\mu(d\lambda)$ is monotonically decreases with the growth of t.

Put $\delta \to 0$. We obtain that the summand

$$\psi(k\varepsilon) \int_{\{\lambda:\rho(\lambda)>\varepsilon\}} (1-|\lambda|)\mu(d\lambda)$$

is arbitrarily small. Therefore the integral (21) is uniformly bounded. Hence the integral (7) is finite. The proof is complete.

The proof of Theorem 3. Let Ω_t be the same as in the previous proofs. According to the Green representation for the function $v_0(z)$ in Ω_t, we get

$$\varphi(1) = \varphi(\rho(0)) = \widetilde{u_t}(0) - \int_{\Omega_t} G_{\Omega_t}(0,\lambda)\mu_0(d\lambda), \tag{23}$$

with the least harmonic majorant $\widetilde{u_t}(z)$ for $v_0(z)$ in Ω_t. Let $V_t(x)$ be harmonic function defining by equality (16). By the maximum principle, $V_t(x) \leqslant \varphi(t)$ in B. Since $v_0(x) = \varphi(t)$ on the $\partial\Omega_t \cap B$ and $V_t(\zeta) = v_0(\zeta)$ for $\zeta \in \partial B$ such that $\rho(\zeta) \geqslant t$, we see that $v_0(x) \geqslant V_t(x)$ in $\partial\Omega_t$. Hence $\widetilde{u_t}(x) \geqslant V_t(x)$ in Ω_t. According to (18), we get

$$\widetilde{u_t}(0) \geqslant V_t(0) = \varphi(2) + \int_t^2 (-\varphi'(s))F(s)ds. \tag{24}$$

Combining (23) and (24), we obtain

$$\int_t^2 (-\varphi'(s))F(s)ds \leqslant \varphi(1) - \varphi(2) + \int_{\Omega_t} G_{\Omega_t}(0,\lambda)\mu_0(d\lambda),$$

where G_{Ω_t} is the Green function on Ω_t. Note that

$$G_{\Omega_t}(0,\lambda) = \frac{1}{|\lambda|^{n-2}} - h_t(0,\lambda),$$

where $h_t(0,\lambda) \geq 1$ is the solution of the Dirichlet's problem in Ω_t with the value $|\lambda|^{2-n}$ on $\partial\Omega_t$. Using (11), we get

$$G_{\Omega_t}(0,\lambda) \leqslant (n-1)(1-|\lambda|), \qquad |\lambda| \geqslant 1-t_0.$$

Thus, we get

$$\int_t^2 (-\varphi'(s))F(s)ds \leqslant \leqslant \varphi(1) - \varphi(2) + (n-1)\int_{\Omega_t \setminus \{\lambda:|\lambda|\leqslant 1-t_0\}} (1-|\lambda|)\mu_0(d\lambda) + \int_{\{\lambda:|\lambda|<1-t_0\}} \frac{1}{|\lambda|^{n-2}}\,\mu_0(d\lambda).$$

By the Green representation in the ball $B' = \{\lambda : |\lambda| < 1-t_0\}$ we have

$$\varphi(1) = v_0(0) = \widehat{u}(0) - \int_{B'} (|\lambda|^{2-n} - (1-t_0)^{2-n})\mu_0(d\lambda),$$

where $\widehat{u}(z)$ is the least harmonic majorant for $v_0(z)$ in B', and $|\lambda|^{2-n} - (1-t_0)^{2-n}$ is the Green function for B' at the point $\zeta = 0$. Since $\widehat{u}(z) \leq \varphi(t_0)$, the integral

$$\int_{B'} |\lambda|^{2-n}\mu_0(d\lambda)$$

is finite. Hence we obtain for $t < t_0$

$$\int_t^2 (-\varphi'(s))F(s)ds \leqslant \text{const} + (n-1)\int_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)\mu_0(d\lambda). \qquad (25)$$

On the other hand, using equations (20) and (21) with $k=1$ and rejecting nonnegative summand, we get for all small ε

$$\int_{\{\lambda\in B:\rho(\lambda)>\varepsilon\}} \psi(\rho(\lambda))(1-|\lambda|)\mu_0(d\lambda) \geqslant \geqslant \int_\varepsilon^2 \psi'(t)\left(\int_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)\mu_0(d\lambda)\right)dt.$$

By inequality (25), we get

$$\int_{\{\lambda\in B:\rho(\lambda)>\varepsilon\}}\psi(\rho(\lambda))(1-|\lambda|)\mu_0(d\lambda)\geqslant$$
$$\geqslant \mathrm{const}+(n-1)^{-1}\int_\varepsilon^2\psi'(t)\int_t^2(-\varphi'(s))F(s)\,ds\,dt.$$

Finally, we claim that the expression

$$\int_\varepsilon^2\psi'(t)\int_t^2(-\varphi'(s))F(s)\,ds\,dt=$$
$$=\int_\varepsilon^2\psi(t)(-\varphi'(t))F(t)\,dt-\psi(\varepsilon)\int_\varepsilon^2(-\varphi'(t))F(t)\,dt \tag{26}$$

unbounded as $\varepsilon\to 0$.

Indeed, in the converse case, the integral

$$\int_0^2\psi'(t)\int_t^2(-\varphi'(s))F(s)\,ds\,dt$$

is finite. Hence for all sufficiently small ε and for all $\delta<\varepsilon$ we have

$$1>\int_\delta^\varepsilon\psi'(t)\int_t^2(-\varphi'(s))F(s)\,ds\,dt\geqslant(\psi(\varepsilon)-\psi(\delta))\int_\varepsilon^2(-\varphi'(s))F(s)\,ds.$$

Passing to a limit as $\delta\to 0$, we get the inequality

$$\psi(\varepsilon)\int_\varepsilon^2(-\varphi'(s))F(s)\,ds<1.$$

Therefore, we see that the integral

$$\int_0^2\psi(t)(-\varphi'(t))F(t)\,dt$$

is finite. This contradiction concludes the proof.

The proof of Theorem 4. Using Theorems 2 and 3 (conditions (6) and (8), respectively) we get that it is sufficiently to prove convergence of the integral

$$\int_0^1 t^rF(t)\,dt \tag{27}$$

for $r > \overline{æ}(E) - n$ and its divergence for $r < \underline{æ}(E) - n$.

If $r + n > \overline{æ}(E)$, then take $\delta < r + n - \overline{æ}(E)$. By the definition, there is a covering of the set E by at most $t^{\delta-(r+n)}$ sets $L(\zeta_j, t)$, $\zeta_j \in E$. Clearly, the sets $L(\zeta_j, 3t)$ overlap the set $E_t = \{\zeta \in \partial B : \rho(\zeta) \leqslant t\}$. Since $m(L(\zeta_j, 3t)) \le C(n)\,(3t)^{n-1}$, we get

$$F(t) = m(E_t) \leqslant C(n)\, t^{\delta-(r+n)}(3t)^{n-1}.$$

Hence integral (27) converges.

Conversely, let $r + n < \underline{æ}(E)$. Consider a finite covering of the set E by sets $L(\zeta_j, t/2)$, $j = 1, \dots, n$, $\zeta_j \in E$. Rejecting sequentially some of the points ζ_j, we may suppose that there is a set $A \subset \{\zeta_1, \zeta_2, \dots, \zeta_n\}$ such that $|\zeta_k - \zeta_j| \ge \dfrac{t}{2}$ for all $\zeta_k, \zeta_j \in A$ and

$$\bigcup_{\zeta_j \in A} L(\zeta_j, t) \supset \bigcup_{j=1}^{n} L(\zeta_j, t/2) \supset E.$$

Therefore the number of points in A is at least $N(E, t)$. By definition of $\underline{æ}(E)$, we have for sufficiently small t

$$N(E, t) \geqslant t^{-(r+n)}.$$

On the other hand, the sets $L(\zeta_j, t/4)$, $\zeta_j \in A$, are mutually disjoint. Hence,

$$F(t) = m(E_t) \geqslant \sum_{k=1}^{N} m(L(\zeta_k', t/4)) = NC(n)\left(\frac{t}{4}\right)^{n-1} \geqslant C(n)4^{1-n}t^{-r-1}$$

for small t. Consequently, the integral (27) diverges. The proof is complete.

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