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Equicontinuity of plane homeomorphisms with controlled p-module

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Dedicated to the memory of Professor Promarz M. Tamrazov

We study plane homeomorphisms preserving integrally quasiinvariant the weighted p-module and provide conditions ensuring the local Hölder continuity of such mappings with respect to euclidian distances and to their logarithms. The inequality defining the continuity is sharp with respect to the order.

1. Introductory remarks. In this paper, we continue studying the properties of plane mappings with controlled p-module. The main characterization of these mappings relies on an extension of quasiinvariance of p-moduli of appropriate order for quasiconformal and quasiisomertic mappings. This approach involves the inequalities which the growth of moduli of families of curves via

$$\mathcal{M}_p(f\Gamma) \le \int\limits_G Q(z)\rho^p(z)dm(z),$$

where Q is a given real measurable function.

For many questions concerning quasiconformal mappings and their generalizations it would be interesting to have criteria for the Lipschitz or

Hölder continuity or giving more general regularity conditions in a prescribed point or on a given set. Our main results state that for every $1 < p < 2$ and any locally integrable Q with the exponent exceeding $2/(2-p)$ the corresponding mappings is Hölder continuous while for the degree $2/(2-p)$ it is logarithmically Hölder continuous. Other results concern the equicontinuity and normality of the mapping families.

2. Q-homeomorphisms with respect to p-moduli and related estimates. In this section we give all needed definitions and notations.

2.1. A curve γ in $\mathbb{C}$ is a continuous mapping $\gamma : \Delta \to \mathbb{C}$, where D is an interval in $\mathbb{R}$. The locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family Γ in $\mathbb{C}$ of curves γ, a Borel function $\varrho : \mathbb{C} \to [0, \infty]$ is called admissible for Γ (abbr. $\varrho \in \operatorname{adm} \Gamma$) if

$$\int\limits_{\gamma} \varrho \, |dz| \geq 1$$

for any $\gamma \in \Gamma$. The quantity

$$\mathcal{M}_p(\Gamma) = \inf_{\varrho \in \operatorname{adm} \Gamma} \iint\limits_{\mathbb{C}} \varrho^p(z) \, dm(z), \qquad p \geq 1,$$

is called *p-module* of Γ; here m denotes the two-dimensional Lebesgue measure in $\mathbb{C}$. For the properties of p-module, we refer to [1, 2].

Let G be a domain in $\mathbb{C}$ and $Q : G \to [0, \infty]$ be a Lebesgue measurable function. A homeomorphic mapping $w = f(z) : G \to \mathbb{C}$ is called *Q-homeomorphism* with respect to p-module, if

$$\mathcal{M}_p(f\Gamma) \leq \int\limits_{G} Q(z) \rho^p(z) dm(z) \tag{1}$$

for every family Γ of curves located in G and any ρ admissible for Γ.

The study of such mappings was started in [3]; on their differential and geometric properties see [4 — 6]. Such homeomorphisms in $\mathbb{R}^n$ are close to bilipschitz mappings (see e.g. [7, 8]). Note also that right-hand side in (1) can be treated as a weighted p-module; cf. [9, 10].

2.2. Let $\mathcal{E} = (A, C)$ be a condenser. Denote by $\mathcal{C}_0(A)$ the set of all continuous functions $u : A \to \mathbb{R}^1$ with compact support in A, and let $\mathcal{W}_0(\mathcal{E}) = \mathcal{W}_0(A, C)$ be the set of all nonnegative functions $u : A \to \mathbb{R}^1$ satisfying: $u \in \mathcal{C}_0(A)$, $u(z) \geq 1$ for $z \in C$ and u belongs to ACL (absolutely continuous on lines). Put

$$\operatorname{cap}_p \mathcal{E} = \operatorname{cap}_p (A, C) := \inf_{u \in \mathcal{W}_0(\mathcal{E})} \int\limits_A \left(u_x^2 + u_y^2\right)^{p/2} dm(z), \quad p \geq 1.$$

This quantity is called *p-capacity of condenser* $\mathcal{E}$.

It was proven in [11] that for $p > 1$,

$$\operatorname{cap}_p \mathcal{E} = \mathcal{M}_p(\Delta(\partial A, \partial C; A \setminus C)), \tag{2}$$

where $\Delta(\partial A, \partial C; A \setminus C))$ denotes the set of all continuous curves joining the boundaries ∂A and ∂C in $A \setminus C$. The general properties of p-capacities and their relation to the mapping theory are presented in [12, 13]. In particular, when $1 \leq p < 2$,

$$\operatorname{cap}_p \mathcal{E} \geq 2\sqrt{\pi^p} \left(\frac{2-p}{p-1}\right)^{p-1} [mC]^{\frac{2-p}{2}} . \tag{3}$$

For $1 < p \leq 2$, there is the following lower estimate

$$\operatorname{cap}_p \mathcal{E} \geq \gamma \frac{d^p(C)}{(mA)^{p-1}} , \tag{4}$$

where $d(C)$ denotes the diameter of C, and γ is a positive constant depending only on p (see [14]).

2.3. It is well-known that the class of planar quasiconformal mappings is intrinsically connected with homeomorphic solutions to the Beltrami equation

$$f_{\overline{z}} = \mu(z) f_z,$$

where $\mu(z)$ is a measurable function with $|\mu(z)| < 1$ a.e. (called also *complex dilatation*) in a given subdomain G of the complex plane $\mathbb{C}$ and

$$f_{\overline{z}} := \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right), \qquad f_z := \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad (z = x + iy),$$

are the partial derivatives (in general, the distributional derivatives) of f; see e.g. [15, 16, 2]. The quantity

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|},$$

is called *Lavrentiev characteristic (or dilatation)* of a mappings f at the point z (cf. [17, 18]).

Equivalently $K_\mu(z) = \dfrac{|w_z| + |w_{\bar z}|}{|w_z| - |w_{\bar z}|} = \dfrac{(|w_z| + |w_{\bar z}|)^2}{|w_z|^2 - |w_{\bar z}|^2} = \dfrac{|w_z|^2 - |w_{\bar z}|^2}{(|w_z| - |w_{\bar z}|)^2}$. Since $|w_z|^2 - |w_{\bar z}|^2$, $|w_z| + |w_{\bar z}|$ and $|w_z| - |w_{\bar z}|$ are equal to the Jacobian, maximal and minimal stretching of a sense-preserving mapping w, one can recall counterparts of the Lavrentiev dilation with respect to a given p, $p \geq 1$. The quantities

$$K^O_{p,w}(z) = \frac{(|w_z| + |w_{\bar z}|)^p}{|w_z|^2 - |w_{\bar z}|^2} \quad \text{and} \quad K^I_{p,w}(z) = \frac{|w_z|^2 - |w_{\bar z}|^2}{(|w_z| - |w_{\bar z}|)^p}$$

stand for the *p-outer and p-inner dilatations* of a mapping w at z, respectively.

Due to Theorem 4.1 from [19], the majorant $Q(z)$ in (1) can be regarded as $K^I_{p,f}(z)$, or in terms of the Lavrentiev dilatation and p-outer dilatation,

$$Q(z) = K^p_\mu(z)/K^O_{p,f}(z).$$

2.4. Let (X, d_X) and (Y, d_Y) be two metric spaces with distances d_X and d_Y, respectively. A family $\mathcal{F}$ of continuous mappings $f : X \to Y$ is called *equicontinuous at a point* $x_0 \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathcal{F}$ and any $x \in X$ provided that $d_X(x, x_0) < \delta$. The family $\mathcal{F}$ is called *equicontinuous in* X if $\mathcal{F}$ is equicontinuous for every $x_0 \in X$.

The following notion is closely related to equicontinuity. A family $\mathcal{F}$ is called *normal* if for any sequence $\{f_n\}$ of continuous mappings $f_n : X \to Y$ there exists a subsequence $\{f_{n_m}\}$ that converges uniformly on each compact set $E \subset X$.

The following well known Ascoli's theorem provides a sufficient condition for an equicontinuous family $\mathcal{F}$ to be normal; see e.g. [20, 21].

Proposition. *If T is a separable topological space and Y is a compact metric space, then every equicontinuous family $\mathcal{F}$ of mappings $f : T \to Y$ is a normal family.*

3. Hölder continuity. One of the interesting problems in geometric function theory is to find the conditions insuring the Hölder continuity of mappings. For the Q-homeomorphisms with respect to conformal module, Q of BMO-class (bounded mean oscillation), FMO-class (finite mean oscillation) and L^α, see e.g. [22].

In this section we establish that integrability of Q in G with an exponent $\alpha < 2/(2-p)$ implies the Hölder continuity of Q-homeomorphisms with respect to p-module with degree $1 - 2/(2-p)\alpha$.

Theorem 3.1. *Let G and G^* be domains in $\mathbb{C}$, and let $f : G \to G^*$ be a Q-homeomorphism with respect to p-module, $1 < p < 2$, with $Q(z) \in \in L^{\alpha}(G)$, $\alpha > \frac{2}{2-p}$. Then for an arbitrary compact set $F \subset G$ and for any pair of points $z, \zeta \in F$, such that $|z - \zeta| < \delta$, $\delta = \frac{1}{4}\mathrm{dist}\,(F, \partial G)$, the following inequality holds*

$$|f(z) - f(\zeta)| \le \lambda_p \|Q\|_{\alpha}^{\frac{1}{2-p}} |z - \zeta|^{1-\frac{2}{\alpha(2-p)}}, \tag{5}$$

with a constant λ_p depending only on p.

Proof. Consider an annulus $A = A(z, \varepsilon_1, \varepsilon_2)$ centered at $z \in G$ and radii $\varepsilon_1, \varepsilon_2$, $0 < \varepsilon_1 < \varepsilon_2 < \delta$, such that $A(z, \varepsilon_1, \varepsilon_2) \subset G$. Then $\left(fB\left(z, \varepsilon_2\right), \overline{fB\left(z, \varepsilon_1\right)}\right)$ is a condenser located in G^*. By (2)

$$\mathrm{cap}_p\left(fB(z, \varepsilon_2), \overline{fB(z, \varepsilon_1)}\right) = \mathcal{M}_p(\triangle(\partial fB(z, \varepsilon_2), \partial fB(z, \varepsilon_1); fA)),$$

where $B(z, r)$ is a disk centered at z of radius r. Since f is homeomorphism,

$$\triangle\left(\partial fB\left(z, \varepsilon_2\right), \partial fB\left(z, \varepsilon_1\right); fA\right) = f\left(\triangle\left(\partial B(z, \varepsilon_2), \partial B(z, \varepsilon_1); A\right)\right).$$

Now consider the function

$$\varrho(z) = \begin{cases} \frac{1}{\varepsilon_2 - \varepsilon_1}, & z \in A \\ 0, & z \notin A, \end{cases}$$

which is admissible for the family $\triangle(\partial fB(x, \varepsilon_2), \partial fB(x, \varepsilon_1); fA)$. Then from (1)

$$\mathrm{cap}_p\ \left(fB(z, \varepsilon_2), \overline{fB(z, \varepsilon_1)}\right) \le \frac{1}{(\varepsilon_2 - \varepsilon_1)^p} \int\limits_{R(z, \varepsilon_1, \varepsilon_2)} Q(z)\ dm(z)\ ,$$

and by the Hölder inequality

$$\mathrm{cap}_p\ \left(fB(z, \varepsilon_2), \overline{fB(z, \varepsilon_1)}\right) \le \frac{\left(\pi \varepsilon_2^2\right)^{\frac{\alpha-1}{\alpha}}}{(\varepsilon_2 - \varepsilon_1)^p} \|Q\|_{\alpha}\,. \tag{6}$$

Letting $\varepsilon = |z - \zeta|$, $\varepsilon_1 = 2\varepsilon$ and $\varepsilon_2 = 4\varepsilon$, one gets the upper bound for p-capacity

$$\mathrm{cap}_p\left(fB(z, 4\varepsilon), \overline{fB(z, 2\varepsilon)}\right) \le\ \gamma_1 \|Q\|_{\alpha} \varepsilon^{\frac{2\alpha - \alpha p - 2}{\alpha}}\ . \tag{7}$$

On the other hand, one can derive from the inequality (3) the following lower bound

$$\mathrm{cap}_p\left(fB(z, 4\varepsilon), \overline{fB(z, 2\varepsilon)}\right) \ge \gamma_2 \left[m(fB(z, 2\varepsilon))\right]^{\frac{2-p}{2}}\ , \tag{8}$$

where γ_2 is a positive constant depending only on p.

Combining the estimates (7) and (8), one gets the upper bound for the image of the disk $B(z,2\varepsilon)$,

$$m(fB(z,2\varepsilon)) \leqslant \gamma_3 \|Q\|_\alpha^{\frac{2}{2-p}} \varepsilon^{\frac{2(2\alpha-\alpha p-2)}{\alpha(2-p)}},$$

where γ_3 is also a constant depending only on p.

Now, letting in (6) $\varepsilon_1=\varepsilon$ and $\varepsilon_2=2\varepsilon$, one obtains

$$\operatorname{cap}_p\left(fB(z,2\varepsilon),\overline{fB(z,\varepsilon)}\right) \leq \gamma_4 \|Q\|_\alpha \varepsilon^{\frac{2\alpha-\alpha p-2}{\alpha}}, \tag{9}$$

and after applying the lower bound from (4),

$$\operatorname{cap}_p\left(fB(z,2\varepsilon),\overline{fB(z,\varepsilon)}\right) \geq \gamma_5 \frac{d^p(\overline{fB(z,\varepsilon)})}{m^{p-1}(fB(z,2\varepsilon))}. \tag{10}$$

The inequalities (9) and (10) result in

$$d(\overline{fB(z,\varepsilon)}) \leq \gamma \|Q\|_\alpha^{\frac{1}{2-p}} \varepsilon^{1-\frac{2}{\alpha(2-p)}}$$

with a constant γ depending only on p.

Now the desired estimate (5) follows from the obvious inequality $d(\overline{fB(z,\varepsilon)}) \geq |f(z)-f(\zeta)|$ and this completes the proof.

4. Logarithmic Hölder continuity. When Q is locally integrable with the exponent $2/(2-p)$, one can derive a stronger inequality than (5). This kind of continuity can be regarded as a logarithmic Hölder continuity.

Theorem 4.1. *Let G and G^* be two domains in $\mathbb{C}$, $\zeta \in G$, and*

$$Q \in L^{\frac{2}{2-p}}(B(\zeta,r_0)), \qquad r_0 \leq \min\left(1, \operatorname{dist}^4(\zeta,\partial \mathrm{G})\right).$$

Then for every Q-homeomorphism $f: G \to G^*$ with respect to p-module, $1<p<2$,

$$|f(z)-f(\zeta)| \left(\log \frac{1}{|z-\zeta|}\right)^{\frac{p}{2(2-p)}} \leq C_p \|Q\|_{\frac{2}{2-p}}^{\frac{1}{2-p}}, \qquad |z-\zeta|<r_0, \tag{11}$$

where $\|Q\|_{\frac{2}{2-p}} = \Big(\int\limits_{B(\zeta,r_0)} Q^{\frac{2}{2-p}}(z)\, dm(z)\Big)^{\frac{2-p}{2}}$ *and C_p is a positive constant depending only on p.*

Proof. Consider an annulus $A(\zeta,\varepsilon_1,\varepsilon_2)=\{x: \varepsilon_1<|z-\zeta|<\varepsilon_2\}$ with radii $0<\varepsilon_1<\varepsilon_2$ such that $A(\zeta,\varepsilon_1,\varepsilon_2)\subset G$.

Since both $f\big(B\,(\zeta,\varepsilon_2)\,,\overline{B\,(\zeta,\varepsilon_1)}\,\big)$ and $\big(fB\,(\zeta,\varepsilon_2)\,,f\overline{B\,(\zeta,\varepsilon_1)}\,\big)$ are ring-like condensers in G^* and coincide (since f is a homeomorphism), one gets from (2)

$$\mathrm{cap}_p\,(fB(\zeta,\varepsilon_2),\overline{fB(\zeta,\varepsilon_1)}) = M_p(\triangle(\partial fB(\zeta,\varepsilon_2),\partial fB(\zeta,\varepsilon_1); fA(\zeta,\varepsilon_1,\varepsilon_2)))\,.$$

By the same reason $\triangle\,(\partial fB\,(\zeta,\varepsilon_2)\,,\partial fB\,(\zeta,\varepsilon_1)\,;fA(\zeta,\varepsilon_1,\varepsilon_2)) = = f\,(\triangle\,(\partial B(\zeta,\varepsilon_2),\partial B(\zeta,\varepsilon_1);A(\zeta,\varepsilon_1,\varepsilon_2)))$. These equalities and the definition of Q-homeomorphisms with respect to p-module yield

$$\mathrm{cap}_p\ (fB(\zeta,\varepsilon_2),\overline{fB(\zeta,\varepsilon_1)}) \le \int\limits_{A(\zeta,\varepsilon_1,\varepsilon_2)} Q(z)\ \rho^p(z)\,dm(z)$$

for any function ρ admissible for the family $\triangle(\partial B(\zeta,\varepsilon_2),\ \partial B(\zeta,\varepsilon_1); A(x_0,\varepsilon_1,\varepsilon_2))$.

Obviously, the function

$$\rho(z) \ = \begin{cases} \frac{1}{|z-\zeta|\log\frac{\varepsilon_2}{\varepsilon_1}}, & \text{if } z\in A(\zeta,\varepsilon_1,\varepsilon_2), \\ 0, & \text{otherwise} \end{cases}$$

is admissible for this family, and hence

$$\mathrm{cap}_p\ (fB(\zeta,\varepsilon_2),\overline{fB(\zeta,\varepsilon_1)}) \le \frac{1}{\log^p\frac{\varepsilon_2}{\varepsilon_1}} \int\limits_{A(\zeta,\varepsilon_1,\varepsilon_2)} \frac{Q(z)}{|z-\zeta|^p}\,dm(z)\,.$$

Applying to the integral in the right-hand side the Hölder inequality yields

$$\mathrm{cap}_p\ (fB(\zeta,\varepsilon_2),\overline{fB(\zeta,\varepsilon_1)}) \le$$
$$\le \log^{-p}\frac{\varepsilon_2}{\varepsilon_1}\Bigg(\ \int\limits_{B(\zeta,r_0)} Q^{\frac{2}{2-p}}(z)\,dm(z)\Bigg)^{\frac{2-p}{2}}\Bigg(\ \int\limits_{A(\zeta,\varepsilon_1,\varepsilon_2)} \frac{dm(z)}{|z-\zeta|^2}\Bigg)^{\frac{p}{2}}.$$

Since $\int\limits_{A(\zeta,\varepsilon_1,\varepsilon_2)} \frac{dm(z)}{|z-\zeta|^2} = 2\pi\log(\varepsilon_2/\varepsilon_1)$, one derives

$$\mathrm{cap}_p\,(fB(\zeta,\varepsilon_2),\overline{fB(\zeta,\varepsilon_1)}) \le$$
$$\le \sqrt{(2\pi)^p}\Bigg(\ \int\limits_{B(\zeta,r_0)} Q^{\frac{2}{2-p}}(z)\,dm(z)\Bigg)^{\frac{2-p}{2}} \log^{-\frac{p}{2}}\frac{\varepsilon_2}{\varepsilon_1}\,. \tag{12}$$

Choosing $\varepsilon_1=\varepsilon$ and $\varepsilon_2=\sqrt{\varepsilon}$, we have

$$\mathrm{cap}_p\ (fB(\zeta,\sqrt{\varepsilon}),\overline{fB(\zeta,\varepsilon)}) \le$$
$$\le C_1\Bigg(\ \int\limits_{B(\zeta,r_0)} Q^{\frac{2}{2-p}}(z)\,dm(z)\Bigg)^{\frac{2-p}{2}} \log^{-\frac{p}{2}}\frac{1}{\varepsilon} \tag{13}$$

with C_1 depending only on p.

On the other hand, from (4),

$$\operatorname{cap}_p\,(fB(\zeta,\sqrt{\varepsilon}),\overline{fB(\zeta,\varepsilon)})\geq C_2\frac{d^p(\overline{fB(\zeta,\varepsilon)})}{m^{p-1}(fB(\zeta,\sqrt{\varepsilon}))},\tag{14}$$

where C_2 is a positive constant depending only on p.

Now, combining (13) and (14), we have

$$\frac{d^p(fB(\zeta,\varepsilon))}{m^{p-1}(fB(\zeta,\sqrt{\varepsilon}))}\leq C_3\Bigg(\int\limits_{B(\zeta,r_0)}Q^{\frac{2}{2-p}}(z)\,dm(z)\Bigg)^{\frac{2-p}{2}}\log^{-\frac{p}{2}}\frac{1}{\varepsilon}\,.\tag{15}$$

To find an upper bound for $m(fB(\zeta,\sqrt{\varepsilon}))$ in (15), pick in (12) $\varepsilon_1=\sqrt{\varepsilon}$ and $\varepsilon_2=\sqrt[4]{\varepsilon}$. Then one derives

$$\operatorname{cap}_p(fB(\zeta,\sqrt[4]{\varepsilon}),\overline{fB(\zeta,\sqrt{\varepsilon})})\leq$$
$$\leq C_4\Bigg(\int\limits_{B(\zeta,r_0)}Q^{\frac{2}{2-p}}(z)\,dm(z)\Bigg)^{\frac{2-p}{2}}\log^{-\frac{p}{2}}\frac{1}{\varepsilon}\,.\tag{16}$$

The capacity in the left-hand side of (16) is estimated by (3)

$$\operatorname{cap}_p(fB(\zeta,\sqrt[4]{\varepsilon}),\overline{fB(\zeta,\sqrt{\varepsilon})})\geq C_5\left[m(fB(\zeta,\sqrt{\varepsilon}))\right]^{\frac{2-p}{2}};\tag{17}$$

here C_5 depends only on p. Combining (16) and (17), we derive the desired estimate

$$m(fB(\zeta,\sqrt{\varepsilon}))\leq C_6\log^{-\frac{p}{2-p}}\frac{1}{\varepsilon}\int\limits_{B(\zeta,r_0)}Q^{\frac{2}{2-p}}(z)\,dm(z)$$

with a constant C_6 depending only on p.

Substituting this into (15), one obtains the estimate

$$d(fB(\zeta,\varepsilon))\left(\log\frac{1}{\varepsilon}\right)^{\frac{p}{2(2-p)}}\leq C_p\Bigg(\int\limits_{B(\zeta,r_0)}Q^{\frac{2}{2-p}}(z)\,dm(z)\Bigg)^{\frac{1}{2}}$$

which implies

$$|f(z)-f(\zeta)|\left(\log\frac{1}{|z-\zeta|}\right)^{\frac{p}{2(2-p)}}\leq d(fB(\zeta,\varepsilon))\left(\log\frac{1}{\varepsilon}\right)^{\frac{p}{2(2-p)}}\leq C_p\,\|Q\|_{\frac{2}{2-p}}^{\frac{1}{2-p}}\,,$$

where $\|Q\|_{\frac{2}{2-p}}=\Big(\int\limits_{B(\zeta,r_0)}Q^{\frac{2}{2-p}}(z)\,dm(z)\Big)^{\frac{2-p}{2}}$ and C_p is a positive constant depending only on p. The proof is completed.

5. Example. It seems likely that the assumption $Q \in L^{\frac{2}{2-p}}$ in Theorem 4.1 is not only sufficient for the logarithmic Hölder continuity, but also can be the necessary one.

Fix p satisfying $1 < p < 2$, and consider the automorphism $w = f : \Delta \to \Delta$ of the unit disk Δ given by

$$w = e^{i\theta}\left(1 + \frac{2-p}{p-1}\log\frac{1}{|z|}\right)^{-\frac{p-1}{2-p}}, \ z \neq 0, \text{ and } w(0) = 0, \ z = |z|e^{i\theta}. \quad (18)$$

This mapping is differentiable, has nonvanishing Jacobian at all points of Δ, except for the origin. Using the polar coordinates (ρ, ψ) and (r, θ) in the image and its inverse, respectively, one can rewrite the mapping at $z \neq 0$ in the form

$$w = \left\{\rho = \left(1 + \frac{2-p}{p-1}\log\frac{1}{r}\right)^{-\frac{p-1}{2-p}}, \psi = \theta, 0 < r < 1, 0 \leq \theta < 2\pi\right\}.$$

The semiaxes of the characteristic ellipse are $d\rho/dr$ and ρ/r, and a direct calculation implies

$$\frac{d\rho}{dr} = \left(1 + \frac{2-p}{p-1}\log\frac{1}{r}\right)^{-\frac{1}{2-p}}\frac{1}{r}, \qquad \frac{\rho}{r} = \left(1 + \frac{2-p}{p-1}\log\frac{1}{r}\right)^{-\frac{p-1}{2-p}}\frac{1}{r}.$$

Hence, $|w_z| - |w_{\bar{z}}| = d\rho/dr$ and $|w_z| + |w_{\bar{z}}| = \rho/r$, which yields that the p-inner dilatation equals $K^I_{p,w}(z) = r^{p-2}$.

For the mapping (18) $Q(z) \notin L^{\frac{2}{2-p}}$, because a direct calculation implies $\|Q\|_{\frac{2}{2-p}} = \infty$.

In this case the left-hand-side of (11) also increases to ∞ as $z \to 0$, which shows that Theorem 4.1 is somewhat sharp.

6. Equicontinuity and normality. The questions concerning equicontinuity and normality of various classes of mappings are of a special interest. For the classical quasiconformal mappings we refer to [20], for their generalization named mappings quasiconformal in the mean see [21] (cf. [22]).

Theorem 6.1. *Let G and G^* be two domains in $\mathbb{C}$, and let $\mathcal{F}_Q$ be a family of Q-homeomorphisms $f : G \to G^*$ with respect to p-module, $1 < p < 2$, with $Q(z) \in L^{\alpha}(G)$, $\alpha \geq \frac{2}{2-p}$. Then the family $\mathcal{F}_Q$ is equicontinuous.*

This theorem follows from Theorems 3.1 and 4.1. For a bounded image G^*, we have more

Corollary. *Let $\mathcal{F}_Q$ be a family of Q-homeomorphisms $f : G \to G^*$ with respect to p-module, $1 < p < 2$, with $Q(z) \in L^\alpha(G)$, $\alpha \geq \frac{2}{2-p}$. Then the family $\mathcal{F}_Q$ is normal.*

Observe, that the assumption $\alpha \geq \frac{2}{2-p}$ is essential, and the corresponding problem on Q-homeomorphisms with $1 < \alpha < \frac{2}{2-p}$ is still open.

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