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Investigation of nonequilibrium processes in vicinity of hydrodynamic states

The Chapman–Enskog method is generalized for the investigation of processes in the vicinity of hydrodynamic states of a gas. The generalization is made on the basis of the Bogolyubov idea of the functional hypothesis. A theory that describes a nonequilibrium state of a gas by the usual hydrodynamic variables and arbitrary additional local variables is constructed. The gradients of all these parameters and the deviations of the latter variables from their hydrodynamic values are assumed to be small and are estimated by two independent small parameters. The proposed theory is nonlinear in the additional variables too. It leads to linear integral equations with an operator, given by the linearized collision integral. Some of them are eigenvalue problems for this operator and describe kinetic modes of the system.

The proposed theory is applied to the solution of a modified Grad problem in which nonequilibrium states of a gas are described by the usual hydrodynamic variables and small deviations of the energy and momentum fluxes from their hydrodynamic values. In the simplest approximation this leads to a theory of the Maxwell relaxation. It is shown that the distribution function of the 13-moment Grad approximation corresponds to the approximation of zero order in gradients and to small fluxes. Moreover, in that theory the investigation of the relaxation phenomena in the system is reduced to a very approximate solution of the abovementioned eigenvalue problems. The Bogolyubov idea of the functional hypothesis gives an adequate solution of the problem.

Метод Чепмена–Енскога узагальнюється для дослідження процесів поблизу від гідродинамічних станів газу. Узагальнення робиться на основі ідеї функціональної гіпотези Боголюбова. Побудовано теорію, яка описує нерівноважні стани газу звичайними гідродинамічними змінними і довільними додатковими локальними змінними. Градієнти всіх цих параметрів і відхилення останніх змінних від їх гідродинамічних значень вважаються малими. Розроблена теорія є нелінійна також і по додаткових змінних. Вона веде до лінійних інтегральних рівнянь з оператором, що дається лінеаризованим інтегралом зіткнень. Деякі з них є спектральною задачею і описують кінетичні моди системи.

Розвинута теорія застосована до розв'язування модифікованої проблеми Греда, в якій нерівноважний стан газу описується звичайними гідродинамічними змінними і малими відхиленнями потоків енергії та імпульсу від їх гідродинамічних значень. У найпростішому наближенні це веде до теорії максвеллівської релаксації. Показується, що функція розподілу 13-моментного наближення Греда відповідає нульовому наближенню за градієнтами і малим потокам. Більше того, в цій теорії дослідження релаксаційних явищ в системі зводиться до дуже наближеного розв'язку вказаної спектральної задачі. Ідея функціональної гіпотези Боголюбова дає адекватний розв'язок проблеми.

1 Introduction

The problem of the solution of the Boltzmann equation in order to build hydrodynamic equations was posed by Boltzmann as soon as he derived his equation

$$\frac{\partial \mathrm{f}_p(x,t)}{\partial t} = -\upsilon_{lp}\frac{\partial \mathrm{f}_p(x,t)}{\partial x_l} + I_p(\mathrm{f}(x,t)), \tag{1}$$

which describes nonequilibrium states of a rarefied gas in terms of the distribution function (DF) $\mathrm{f}_p(x,t)$ ($I_p(\mathrm{f})$ is the collision integral, $\upsilon_{lp} \equiv p_l/m$; we restrict the discussion to a one-component system).

In any approach the starting point for the construction of hydrodynamic equations on the basis of the kinetic equation (1) is the energy, momentum, and particle number conservation laws in differential form, which follow from the relations

$$\int d^3p\, \zeta_{\mu p} I_p(\mathrm{f}) = 0, \qquad \left(\zeta_{\mu p}: \quad \varepsilon_p,\, p_l,\, m; \quad \varepsilon_p \equiv p^2/2m\right). \tag{2}$$

The above-mentioned conversation laws have the form

$$\frac{\partial \varepsilon}{\partial t} = -\frac{\partial q_n}{\partial x_n}, \qquad \frac{\partial \pi_l}{\partial t} = -\frac{\partial t_{ln}}{\partial x_n}, \qquad \frac{\partial \rho}{\partial t} = -\frac{\partial \pi_n}{\partial x_n}, \tag{3}$$

where the energy density ε, the momentum density π_l, the mass density ρ (variables $\zeta_\mu(x,t)$), and the densities of the corresponding fluxes q_n, t_{ln}, π_n are given by the formulas

$$\varepsilon = \int d^3p\, \varepsilon_p \mathrm{f}_p, \quad \pi_n = \int d^3p\, p_n \mathrm{f}_p, \quad \rho = \int d^3p\, m \mathrm{f}_p;$$

$$q_n = \int d^3p\, \varepsilon_p \upsilon_{np} \mathrm{f}_p, \quad t_{ln} = \int d^3p\, p_l \upsilon_{np} \mathrm{f}_p. \tag{4}$$

The mass velocity υ_n and temperature T are defined by the relations

$$\pi_n = \rho \upsilon_n, \quad \varepsilon \equiv \varepsilon^o + \rho \upsilon^2/2, \quad \varepsilon^o = 3nT/2 \qquad (\rho \equiv mn)\,. \tag{5}$$

An important role in hydrodynamics is played by the Galilean transformation from the laboratory reference frame (LRF) to the accompanying reference frame (ARF)

$$q_n = q_n^o + t_{nl}^o \upsilon_l + (\varepsilon^o + \rho \upsilon^2/2)\upsilon_n, \quad t_{ln} = t_{ln}^o + \rho \upsilon_l \upsilon_n \tag{6}$$

(A^o is a quantity A in the ARF). Finally, the time equations for usual hydrodynamic variables $T(x,t), \upsilon_n(x,t), n(x,t)$ (denoted below by $\xi_\mu(x,t)$) become

$$\frac{\partial T}{\partial t} = -\upsilon_n \frac{\partial T}{\partial x_n} - \frac{2}{3n}\left(\frac{\partial q_l^o}{\partial x_l} + t_{ln}^o \frac{\partial \upsilon_l}{\partial x_n}\right), \quad \frac{\partial \upsilon_l}{\partial t} = -\upsilon_n \frac{\partial \upsilon_l}{\partial x_n} - \frac{1}{mn}\frac{\partial t_{ln}^o}{\partial x_n},$$

$$\frac{\partial n}{\partial t} = -\frac{\partial n \upsilon_l}{\partial x_l}. \tag{7}$$

The next step is to express the fluxes q_l^o, t_{nl}^o in terms of the hydrodynamic variables $\xi_\mu(x,t)$ by functionals $q_l^o(x,\xi(t))$, $t_{nl}^o(x,\xi(t))$ that leads to closed equations of the form

$$\frac{\partial \xi_\mu(x,t)}{\partial t} = M_\mu(x,\xi(t)) \tag{8}$$

($M_\mu(x,\xi)$ are some functionals of $\xi_\mu(x)$). The hydrodynamic variables $\xi_\mu(x,t)$ are expressed only in terms of the simplest moments of the distribution function $\mathrm{f}_p(x,t)$. Therefore the number of parameters that describe the system state is reduced as we go from kinetics to hydrodynamics. Therefore the parameters $\xi_\mu(x,t)$ (or equivalent variables $\zeta_\mu(x,t)$) may be called reduced description parameters (RDP) of the system.

A considerable contribution into the solution of this problem was made by Hilbert [1], who formulated the concept of the normal solution $\mathrm{f}_p(x,\xi(t))$ of the kinetic equation (1). This solution is a functional of the hydrodynamic variables $\xi_\mu(x,t)$ as functions of the coordinates, and it depends on the time only through their mediation. Hilbert developed a perturbation theory in a small parameter g for calculation DF $\mathrm{f}_p(x,\xi)$ on the basis of the estimates $I_p(\mathrm{f}) \sim g^{-1}$, $\mathrm{f}_p(x,t) \sim g^0$. The parameter g (the Knudsen number) is given by the formula $g = l/L$ where l is the gas mean free path, and L is a characteristic dimension of the system inhomogeneities. In the main order of the perturbation theory the DF $\mathrm{f}_p(x,\xi)$ coincides with the Maxwell distribution w_p

$$w_p = w^o_{p-mv}, \qquad w^o_p \equiv \frac{n}{(2\pi mT)^{3/2}} e^{-\frac{\varepsilon_p}{T}} \qquad (I_p(w)=0). \tag{9}$$

In practical terms the Hilbert perturbation theory was improved by Enskog [2], who managed to derive hydrodynamic equations with account for dissipative processes. His method is called the Chapman–Enskog method [3], because the same results were obtained by Chapman on the basis of Maxwell's ideas. The Chapman–Enskog method reduces to the solution of linear integral equations of the form

$$\hat{K} g_p = h_p, \tag{10}$$

where g_p is the sought-for function, and h_p is a known function (the Fredholm integral equation of the first kind). The kernel $K_{pp'}$ of the operator $\hat{K}$ is defined by the linearized collision integral

$$\hat{K} g_p = \int d^3p' K_{pp'} g_{p'},$$

$$w_p K_{pp'} = -M_{pp'} w_{p'}, \quad M_{pp'} \equiv \left. \frac{\delta I_p(\mathrm{f})}{\delta \mathrm{f}_{p'}} \right|_{\mathrm{f}\to w}. \tag{11}$$

The integral equations are solved with additional conditions which follow from the definition of the hydrodynamic variables (4) and (5). In fact, in the Chapman–Enskog method the perturbation theory for the DF $\mathrm{f}_p(x,\xi)$ is built in small gradients of the hydrodynamic variables according to the estimate

$$\frac{\partial^s \xi_\mu(x)}{\partial x_{l_1}...\partial x_{l_s}} \sim g^s. \tag{12}$$

Burnett proposed the method of Sonine orthogonal polynomial $S_n^\alpha(x)$ expansion [4, 5] for the approximate solution of integral equations of the form (10) related to the Boltzmann equation. The use of these polynomials is due to the fact that the DF $\mathrm{f}_p(x,\xi)$ proves to be proportional to the Maxwell distribution, and therefore the orthogonality relation

$$\int d^3p w_p^o \varepsilon_p^{\alpha-1/2} S_s^\alpha(\beta\varepsilon_p) S_{s'}^\alpha(\beta\varepsilon_p) = nT^{\alpha-1/2}\frac{2\Gamma(s+\alpha+1)}{s!\pi^{1/2}}\delta_{ss'} \tag{13}$$

(α is some parameter; $\beta \equiv T^{-1}$) is obviously helpful. In fact, the polynomial series is truncated artificially, and one-, two-, etc. polynomial approximations are built. Kohler proposed a variational principle [6] for the solution of integral equations of the form (4) which is based on the Hilbert result that the bilinear form

$$\{g_p, h_p\} = \int d^3p\, d^3p' w_p K_{pp'} g_p h_{p'} \tag{14}$$

(g_p, h_p are arbitrary functions) has the properties

$$\{g_p, h_p\} = \{h_p, g_p\}, \quad \{g_p, g_p\} \geq 0;$$
$$\{g_p, g_p\} = 0 \quad \Rightarrow \quad g_p = \zeta_{\mu p}, \tag{15}$$

($\zeta_{\mu p}$ are defined in (2)). This variational principle justifies the convergence of the Burnett procedure of approximate solution of the integral equation (10) with increasing number of polynomials.

It is also worth noting at this point that on the basis of (15) it is easy to show, following Hilbert, the symmetry of the operator $\hat{K}$ (11), i.e. the relation $(g_p, \hat{K}h_p) = (\hat{K}g_p, h_p)$. Here, the scalar product is defined as

$$(g_p, h_p) = \langle g_p h_p \rangle, \quad \langle g_p \rangle \equiv \int d^3p\, w_p g_p. \tag{16}$$

The positiveness of the eigenvalues of the operator $\hat{K}$ also follows from (15). Its eigenfunctions can be assumed to be orthogonal in the introduced scalar product.

In this paper we construct theory which describes nonequilibrium states close to hydrodynamic ones. These states are described by usual hydrodynamic variables $\xi_\mu(x,t)$ (or $\zeta_\mu(x,t)$) and some additional variables $\varphi_i(x,t)$ that vanish at usual hydrodynamic evolution. Therefore, variables $\varphi_i(x,t)$ describe relaxation phenomena and one receives an opportunity to study forming the hydrodynamic evolution. The proposed theory is given by our generalization of the Chapman–Enskog method based on the Bogolyubov idea of the functional hypothesis [7] (see discussion this idea, for example, in [8]). Relaxation processes are considered at their end that allows to build a perturbation theory in magnitude of variables $\varphi_i(x,t)$ which is additional one to usual perturbation theory in gradients.

The idea of investigation of relaxation processes at their end was used in a series of papers: in theory of relaxation of polaron gas velocity and temperature in polar crystals [9], in hydrodynamics of phonon subsystem of dielectrics taking into account drift velocity relaxation [10], in hydrodynamics of two-component plasma taking into account temperature and velocity relaxation of the components [11].

Plan of the paper is as follows. In the Section 2 the Grad approach to solution of kinetic equations is discussed in connection with the Bogolyubov reduced description method. Particular attention is paid to the analysis of his 13-moment approximation (we call this theory the Grad problem). In the Section 3 the general theory is constructed which describes nonequilibrium processes in the vicinity of hydrodynamic states. Section 4 gives application of the developed theory to a modified Grad problem.

2 The Grad problem and the Bogolyubov reduced description method

Grad proposed a method [12] in which solutions of the Boltzmann equation are sought from the very outset as a truncated series in the orthogonal tensor Hermite polynomials $H_{l_1\ldots l_s}(\vec{x})$. The use of these polynomials is justified by the same reason as the use of the Sonine polynomials, namely, by the form of their normalization condition

$$\int d^3p\, w_p^o H_{l_1\ldots l_s}((\beta/m)^{1/2}\vec{p}) H_{l'_1\ldots l'_{s'}}((\beta/m)^{1/2}\vec{p}) = \\ = n\delta_{ss'} \sum_\sigma \delta_{l'_1 l_{\sigma(1)}} \ldots \delta_{l'_s l_{\sigma(s)}} \tag{17}$$

(σ is an arbitrary permutation of the numbers $1, ..., s$). In the Chapman–Enskog method, a hydrodynamic gas state is described by moments of the DF $\mathrm{f}_p(x,t)$ with the functions $\zeta_{\mu p}$ defined in (2). In the 13-moment Grad approximation the system is described by the moments of the DF with the functions

$$\zeta_{\mu p}, \quad \varepsilon_p v_{lp}, \quad h_{ln}(p)/m \qquad \big(h_{ln}(p) \equiv p_n p_l - \delta_{nl} p^2/3\big)$$

which are taken in the ARF. According to (4) this state is defined by the variables $\zeta_\mu(x,t)$, $q^o_l(x,t)$,

$$\pi^o_{ln}(x,t) \equiv t^o_{ln}(x,t) - \delta_{ln} t^o_{mm}(x,t)/3$$

where $q^o_l(x,t)$, $t^o_{ln}(x,t)$ are the densities of the gas energy and momentum fluxes in the ARF ($\xi_\mu(x,t)$ can be used instead of $\zeta_\mu(x,t)$). The Grad equations for fluxes are obtained from the kinetic equation by direct substitution of the DF expansion in the Hermite polynomials, which leads to their quadratic nonlinearity because the collision integral $I_p(\mathrm{f})$ is a quadratic function of the DF $\mathrm{f}_{p'}$. In the G-13 approximation, equations (7) are final equations, and they are supplemented by the time equations for the fluxes $q^o_l(x,t)$, $\pi^o_{ln}(x,t)$.

The equations of the G-13 approximation were considered by Grad as a means to investigate nonequilibrium states that precede standard hydrodynamic ones. On this basis he discussed [13] the hydrodynamic evolution, studying normal according to Hilbert solutions, with the fluxes $q^o_l(x,t)$, $\pi^o_{ln}(x,t)$ that are functionals of the usual hydrodynamic variables $q^o_l(x,\zeta(t))$, $\pi^o_{ln}(x,\zeta(t))$.

The situation may be clarified further if we will base the consideration on the Bogolyubov *idea of the functional hypothesis* and his *idea of hierarchy of nonequilibrium states of a system during its evolution*. These ideas are basis of *the Bogolyubov reduced description method* (RDM) of nonequilibrium processes [7] and can be applied to investigation of evolution of a system described by the Liouville equation or kinetic equations (see review in [8]). On this basis the Chapman–Enskog method is generalized in the present paper.

Usual hydrodynamic states are realized in the system at times $t \gg \tau_0$, where τ_0 is the mean free time. Nonequilibrium Grad states are realized at $t \gg \tau_1$, where τ_1 is a characteristic time such that $\tau_1 \ll \tau_0$. According to the idea of *the functional hypothesis* we have

$$\mathrm{f}_p(x,t) \xrightarrow[t \gg \tau_1]{} \mathrm{f}_p(x, \zeta(t), q^o(t), \pi^o(t)), \tag{18}$$

i.e. at times $t \gg \tau_1$ the DF $\mathrm{f}_p(x,t)$ becomes a functional of the RDP $\zeta_\mu(x,t)$, $q^o_n(x,t)$, $\pi^o_{ln}(x,t)$ as functions of the coordinates and depends on the time only through their mediation. This *functional is universal* in the sense that only the RDP $\zeta_\mu(x,t)$, $q^o_l(x,t)$, $\pi^o_{ln}(x,t)$ on the right-hand side of (18) depend on the initial system state described by the DF $\mathrm{f}_p(x,t=0)$. To the functional hypothesis (18) *definitions of the RDP*

$$\int d^3p\,\varepsilon_p v_{lp}\,\mathrm{f}_{p+mv(x)}(x,\zeta,q^o,\pi^o) = q^o_l(x),$$
$$\int d^3p\,h_{nl}(p)\mathrm{f}_{p+mv(x)}(x,\zeta,q^o,\pi^o)/m = \pi^o_{ln}(x),$$
$$\int d^3p\,\zeta_{\mu p}\mathrm{f}_p(x,\zeta,q^o,\pi^o) = \zeta_\mu(x) \tag{19}$$

must be added.

The idea of the functional hypothesis is obviously a generalization of the Hilbert idea of normal solutions. However, in Bogolyubov's research it became a result of his investigations into non-linear dynamic systems, in which the synchronization of the solutions of their dynamic equations with the evolution of some parameters is observed. The term "the functional hypothesis" was introduced by Uhlenbeck. By now, thanks to Peletminsky's investigations, this idea has largely lost the status of a hypothesis. In some important cases it can be proven [14]-[16] (see also [8]). The right hand side of the functional hypothesis (18) contains asymptotic value of the distribution function $\mathrm{f}_p(x,t)$. Transition to asymptotics implies some coarsening procedure. This procedure corresponds to possibilities of experimental observations and make possible the reduced description of the system.

In fact in the G-13 approximation, the DF $\mathrm{f}_p(x,\zeta,q^o,\pi^o)$ is given by the formula

$$\mathrm{f}_p(x,\zeta,q^o,\pi^o) = w^o_p(n,T)\Big\{1 + \frac{1}{2mnT^2}p_n p_l \pi^o_{nl}(x) + $$
$$+\frac{1}{nT^2}p_l\Big(\frac{2\varepsilon_p}{5T}-1\Big)q^o_l(x)\Big\}_{\substack{p\to p-mv(x)\\ n\to n(x),\ T\to T(x)}}, \tag{20}$$

to which there are no corrections in the framework of Grad's theory ($w^o_p(n,T) \equiv w^o_p$ in (9)). A comparison of (20) with the functional from (18) shows that (20) corresponds to an approximation of small fluxes $q^o_l(x,t)$,

$\pi^o_{ln}(x,t)$ and the zero-order approximation in gradients. Moreover, we show further on the basis of the developed in the Section 3 theory that expression (20) corresponds to one-polynomial approximation.

Below we call problem of reduced description of the nonequilibriun system by variables $\zeta_\mu(x,t)$, $q^o_l(x,t)$, $\pi^o_{ln}(x,t)$ *the Grad problem.*

3 Description of nonequilibrium processes in the vicinity of the hydrodynamic states

3.1 The basic equations of the theory

Consider a generalization of the Grad problem to the case of description of the system by arbitrary parameters that are additional to usual hydrodynamic variables. The corresponding stage of evolution precedes in time hydrodynamic stage. The use of the two reference frames (the laboratory and the accompanying ones) brings a certain complication to the theory. Therefore in this section we choose the densities of the integrals of motion $\zeta_\mu(x,t)$ as the basic hydrodynamic variables.

According to Bogolyubov, at the hydrodynamic stage of evolution the reduced description can be built on the basis of *the functional hypothesis*

$$\mathrm{f}_p(x,t) \xrightarrow[t\gg\tau_0]{} \tilde{\mathrm{f}}_p(x,\tilde{\zeta}(t)); \qquad \int d^3p\,\zeta_{\mu p}\,\tilde{\mathrm{f}}_p(x,\tilde{\zeta}) = \tilde{\zeta}_\mu(x). \tag{21}$$

Here, the second formula is the definition of the parameters $\tilde{\zeta}_\mu(x,t)$, for which the gas energy, momentum, and mass densities are taken (see (2) and (4)). Let at $t \gg \tau_1$ ($\tau_0 \gg \tau_1$) the gas be described by the hydrodynamic parameters $\zeta_\mu(x,t)$ and the deviations $\varphi_i(x,t)$ of the parameters with the microscopic values θ_{ip} from their hydrodynamic values $\theta_i(x,t)$ (notation $\varphi_i(x,t)$ is less descriptive than $\delta\theta_i(x,t)$ but leads to compact formulas). Then *the functional hypothesis* at these times has the form

$$\mathrm{f}_p(x,t) \xrightarrow[t\gg\tau_1]{} \mathrm{f}_p(x,\zeta(t),\varphi(t));$$

$$\int d^3p\,\theta_{ip}\mathrm{f}_p(x,\zeta,\varphi) = \varphi_i(x) + \theta_i(x,\zeta), \qquad \int d^3p\,\theta_{ip}\,\tilde{\mathrm{f}}_p(x,\tilde{\zeta}) \equiv \theta_i(x,\tilde{\zeta});$$

$$\int d^3p\,\zeta_{\mu p}\mathrm{f}_p(x,\zeta,\varphi) = \zeta_\mu(x), \tag{22}$$

where the last three formulas define the RDPs $\zeta_\mu(x,t)$ and $\varphi_i(x,t)$.

According to kinetic equation (1), the introduced parameters satisfy the following time equations

$$\frac{\partial\tilde{\zeta}_\mu(x,t)}{\partial t} = \tilde{L}_\mu(x,\tilde{\zeta}(t)), \quad \tilde{L}_\mu(x,\tilde{\zeta}) \equiv -\frac{\partial}{\partial x_n}\int d^3p\,\zeta_{\mu p}\upsilon_{np}\tilde{\mathrm{f}}_p(x,\tilde{\zeta}); \tag{23}$$

$$\frac{\partial\zeta_\mu(x,t)}{\partial t} = L_\mu(x,\zeta(t),\varphi(t)),$$
$$L_\mu(x,\zeta,\varphi) \equiv -\frac{\partial}{\partial x_n}\int d^3p\,\zeta_{\mu p}\upsilon_{np}\mathrm{f}_p(x,\zeta,\varphi); \tag{24}$$

$$\frac{\partial\varphi_i(x,t)}{\partial t} = L_i(x,\zeta(t),\varphi(t)),$$
$$L_i(x,\zeta,\varphi) \equiv -\sum_\mu \frac{\delta\theta_i(x,\zeta)}{\delta\zeta_\mu(x')}L_\mu(x',\zeta,\varphi) - \frac{\partial}{\partial x_n}\int d^3p\,\theta_{ip}\upsilon_{np}\mathrm{f}_p(x,\zeta,\varphi) +$$
$$+\int d^3p\,\theta_{ip}I_p(\mathrm{f}(x,\zeta,\varphi)). \tag{25}$$

The considered problem implies that the relations

$$\zeta_\mu(x,t)\xrightarrow[t\gg\tau_0]{}\tilde{\zeta}_\mu(x,t), \qquad \varphi_i(x,t)\xrightarrow[t\gg\tau_0]{}0;$$
$$\mathrm{f}_p(x,\zeta(t),\varphi(t))\xrightarrow[t\gg\tau_0]{}\tilde{\mathrm{f}}_p(x,\tilde{\zeta}(t)) \tag{26}$$

are true, whence we have the identities

$$\mathrm{f}_p(x,\zeta,\varphi=0) = \tilde{\mathrm{f}}_p(x,\zeta), \quad L_\mu(x,\zeta,\varphi=0) = \tilde{L}_\mu(x,\zeta),$$
$$L_i(x,\zeta,\varphi=0) = 0. \tag{27}$$

According to Bogolyubov's MRD, *the asymptotic DF are the exact solutions of the kinetic equation*

$$\frac{\partial\tilde{\mathrm{f}}_p(x,\tilde{\zeta}(t))}{\partial t} = -\upsilon_{lp}\frac{\partial\tilde{\mathrm{f}}_p(x,\tilde{\zeta}(t))}{\partial x_l} + I_p(\tilde{\mathrm{f}}_p(x,\tilde{\zeta}(t))),$$

$$\frac{\partial\mathrm{f}_p(x,\zeta(t),\varphi(t))}{\partial t} = -\upsilon_{lp}\frac{\partial\mathrm{f}_p(x,\zeta(t),\varphi(t))}{\partial x_l} + I_p(\mathrm{f}_p(x,\zeta(t),\varphi(t))). \tag{28}$$

By their meaning, they describe the system state at $t\gg\tau_0$ and $t\gg\tau_1$, respectively. However, the solution of equations (26) can be continued to

$t = 0$, that introduces the effective initial conditions to time equations (23)-(25).

Equations (28) together with relations (23)-(25) yield the following integro-differential equations for the DF $\tilde{\mathrm{f}}_p(x,\tilde{\zeta})$ and $\mathrm{f}_p(x,\zeta,\varphi)$

$$\sum_\mu \int d^3x' \frac{\delta\tilde{\mathrm{f}}_p(x,\zeta)}{\delta\zeta_\mu(x')}\tilde{L}_\mu(x',\zeta) = -\upsilon_{lp}\frac{\partial\tilde{\mathrm{f}}_p(x,\zeta)}{\partial x_l} + I_p(\tilde{\mathrm{f}}_p(x,\zeta)), \tag{29}$$

$$\sum_\mu \int d^3x' \frac{\delta\mathrm{f}_p(x,\zeta,\varphi)}{\delta\zeta_\mu(x')}L_\mu(x',\zeta,\varphi) + \sum_i \int d^3x' \frac{\delta\mathrm{f}_p(x,\zeta,\varphi)}{\delta\varphi_i(x')}L_i(x',\zeta,\varphi) =$$

$$= -\upsilon_{lp}\frac{\partial\mathrm{f}_p(x,\zeta,\varphi)}{\partial x_l} + I_p(\mathrm{f}_p(x,\zeta,\varphi)). \tag{30}$$

They should be solved in a perturbation theory in the gradients g of the RDP $\zeta_\mu(x)$, $\tilde{\zeta}_\mu(x)$, $\varphi_i(x)$ and in the parameter λ that estimates the order of the variables $\varphi_i(x)$ according to

$$\frac{\partial^s\zeta_\mu(x)}{\partial x_{l_1}...\partial x_{l_s}} \sim g^s, \quad \frac{\partial^s\tilde{\zeta}_\mu(x)}{\partial x_{l_1}...\partial x_{l_s}} \sim g^s, \quad \frac{\partial^s\varphi_i(x)}{\partial x_{l_1}...\partial x_{l_s}} \sim g^s\lambda^1. \tag{31}$$

In doing so,the RDP definitions (21) and (22) should be used as additional conditions.

3.2 Construction of the perturbation theory

According to (27), we can restrict ourselves to the solution of equation (30) only. The structure of its solution in the perturbation theory is given by the formulas

$$\mathrm{f}_p = \mathrm{f}_p^{(0)} + \mathrm{f}_p^{(1)} + O(g^2), \quad \mathrm{f}_p^{(0)} = \mathrm{f}_p^{(0,0)} + \mathrm{f}_p^{(0,1)} + \mathrm{f}_p^{(0,2)} + O(g^0\lambda^3),$$

$$\mathrm{f}_p^{(1)} = \mathrm{f}_p^{(1,0)} + \mathrm{f}_p^{(1,1)} + O(g^1\lambda^2);$$

$$\mathrm{f}_p^{(0,0)} = w_p, \quad \mathrm{f}_p^{(0,1)} = w_p\sum_i a_{ip}\varphi_i, \quad \mathrm{f}_p^{(0,2)} = w_p\sum_{ii'} b_{ii'p}\varphi_i\varphi_{i'};$$

$$\mathrm{f}_p^{(1,0)} = w_p^o\{1 + \frac{\partial T}{\partial x_n}A_p p_n + \frac{\partial\upsilon_n}{\partial x_l}B_p h_{nl}(p)\}_{p\to p-m\upsilon},$$

$$f_p^{(1,1)} = w_p\{\sum_i c_{ilp}\frac{\partial\varphi_i}{\partial x_l} + \sum_{i\mu} d_{i\mu lp}\varphi_i\frac{\partial\xi_\mu}{\partial x_l}\}. \tag{32}$$

Here and in what follows, $A^{(m)}$ is the contribution of the order g^m, and $A^{(m,n)}$ is the contribution of the order $g^m\lambda^n$ to the quantity A, and the results are given in terms of the gradients of the variables $\xi_\mu(x,t)$, i.e. $T(x,t)$, $\upsilon_n(x,t)$, $n(x,t)$. In (32) a_{ip}, $b_{ii'p}$, c_{ilp}, $d_{i\mu lp}$ are some momentum functions to be computed. The contribution $\mathrm{f}_p^{(1,0)}$ coincides with the first-order approximation in gradients of the usual Chapman–Enskog method. The scalar functions A_p, B_p satisfy the known integral equations

$$\hat{K}^o A_p p_l = \frac{1}{mT}\left(\frac{5}{2}-\frac{\varepsilon_p}{T}\right)p_l, \qquad \langle A_p\varepsilon_p\rangle^o = 0;$$

$$\hat{K}^o B_p h_{nl}(p) = -\frac{1}{mT}h_{nl}(p), \tag{33}$$

where

$$\hat{K}^o h_p = \int d^3p' w_p^o K_{pp'}^o h_{p'}, \qquad K_{p,p'}^o = K_{p+m\upsilon,p'+m\upsilon};$$

$$\langle h_p\rangle^o = \int d^3p\, w_p^o h_p \tag{34}$$

(the kernel $K_{pp'}$ is defined in (11); h_p is an arbitrary function). The second formula in (33) is the additional condition that follows from the RDP definition in (22). The functions $\mathrm{f}_p^{(0,0)}$ and $\mathrm{f}_p^{(1,0)}$ define the main contributions $M_\mu^{(1,0)}$ and $M_\mu^{(2,0)}$ to the usual hydrodynamic equations (8)

$$M_0^{(1,0)} = -\upsilon_n\frac{\partial T}{\partial x_n}-\frac{2}{3}T\frac{\partial\upsilon_n}{\partial x_n}, \quad M_l^{(1,0)} = -\upsilon_n\frac{\partial\upsilon_l}{\partial x_n}-\frac{T}{mn}\frac{\partial n}{\partial x_l}-\frac{1}{m}\frac{\partial T}{\partial x_l},$$

$$M_4^{(1,0)} = -\frac{\partial n\upsilon_l}{\partial x_l}; \tag{35}$$

$$M_0^{(2,0)} = -\frac{2}{3n}\left(\frac{\partial q_l^{o(1,0)}}{\partial x_l}+t_{ln}^{o(1,0)}\frac{\partial\upsilon_l}{\partial x_n}\right), \quad M_l^{(2,0)} = -\frac{1}{mn}\frac{\partial t_{ln}^{o(1,0)}}{\partial x_n},$$

$$M_4^{(2,0)} = 0. \tag{36}$$

In (35) the expressions for the reversible fluxes

$$q_n^{o(0,0)} = 0, \qquad t_{nl}^{o(0,0)} \equiv p\delta_{nl}, \qquad p = nT$$

are taken into account (p is the pressure). Equations (36) include the dissipative contributions to the fluxes

$$q_n^{o(1,0)} = -\kappa \frac{\partial T}{\partial x_n}, \qquad t_{ln}^{o(1,0)} = -\eta \left(\frac{\partial \upsilon_l}{\partial x_n} + \frac{\partial \upsilon_n}{\partial x_l} - \frac{2}{3}\delta_{ln}\frac{\partial \upsilon_m}{\partial x_m} \right),$$

$$\kappa \equiv -\frac{2}{3}\langle \varepsilon_p^2 A_p \rangle^o, \qquad \eta \equiv -\frac{4m}{15}\langle \varepsilon_p^2 B_p \rangle^o \tag{37}$$

where κ and η are the gas heat conductivity and shear viscosity, respectively.

According to (25) and (32), in the zero-order approximation in gradients the equation for the parameters φ_i has the form

$$\frac{\partial \varphi_i}{\partial t} = L_i^{(0,1)} + L_i^{(0,2)} + O(g^0\lambda^3, g^1),$$

$$L_i^{(0,1)} = -\sum_{i'} \mu_{ii'}\varphi_{i'}, \qquad L_i^{(0,2)} = -\sum_{i'i''} \nu_{ii'i''}\varphi_{i'}\varphi_{i''}, \tag{38}$$

where the coefficients

$$\mu_{ii'} = \{\theta_{ip}, a_{i'p}\},$$

$$\nu_{ii'i''} = \{\theta_{ip}, b_{i'i''p}\} + \frac{1}{2}\int d^3p\, d^3p' w_p \theta_{ip} K_{pp'p''} a_{i'p'} a_{i''p''} \tag{39}$$

and the function $K_{pp'p''}$

$$w_p K_{pp'p''} = -M_{pp'p''} w_{p'} w_{p''}, \quad M_{pp'p''} \equiv \left. \frac{\delta^2 I_p(\mathrm{f})}{\delta \mathrm{f}_{p'} \delta \mathrm{f}_{p''}} \right|_{\mathrm{f}\to w}. \tag{40}$$

are introduced. According to (22) and (30), these coefficients $\mu_{ii'}$, $\nu_{ii'i''}$ and functions a_{ip}, $b_{ii'p}$ from (32) are the solutions of the integral equations with the additional conditions

$$\hat{K} a_{ip} = \sum_{i'} a_{i'p}\mu_{i'i}, \quad \langle a_{ip}\zeta_{\mu p}\rangle = 0, \quad \langle a_{ip}\theta_{i'p}\rangle = \delta_{ii'}; \tag{41}$$

$$\hat{K} b_{i'i''p} = \sum_i a_{ip}\nu_{ii'i''} + \sum_i (b_{ii'p}\mu_{ii''} + b_{ii''p}\mu_{ii'}) -$$

$$-\frac{1}{2}\int d^3p' d^3p'' K_{pp'p''} a_{i'p'} a_{i''p''}, \quad \langle b_{i'i''p}\zeta_{\mu p}\rangle = 0, \quad \langle b_{i'i''p}\theta_{ip}\rangle = 0. \tag{42}$$

Expressions (39) for the coefficients $\mu_{ii'}$, $\nu_{ii'i''}$ follow also from the integral equations (41) and (42) when the last relations in (41) and (42) are taken into account. However, as will be shown below, these expressions for $\mu_{ii'}$, $\nu_{ii'i''}$ are not needed for the solution of equations (41) and (42).

Further analysis of the integral equations (41) and (42) without specifying the parameters φ_i (and the functions θ_{ip}) is difficult. However, it is easy to show that the quantities φ_i are linear combinations of the gas kinetic modes φ_α. To demonstrate this, consider the right and left eigenfunctions of the matrix $\mu_{ii'}$

$$\sum_{i'} \mu_{ii'} u_{i'\alpha} = \lambda_\alpha u_{i\alpha}, \quad \sum_i \upsilon_{i\alpha}\mu_{ii'} = \lambda_\alpha \upsilon_{i'\alpha};$$
$$\sum_i u_{i\alpha}\upsilon_{i\alpha'} = \delta_{\alpha\alpha'}, \qquad \sum_\alpha u_{i\alpha}\upsilon_{i'\alpha} = \delta_{ii'}, \tag{43}$$

with additional normalization and completeness conditions. Then

$$\varphi_i = \sum_\alpha u_{i\alpha}\varphi_\alpha, \qquad \varphi_\alpha \equiv \sum_i \varphi_i \upsilon_{i\alpha} \tag{44}$$

and, according to (38), the quantities φ_α satisfy the equation

$$\frac{\partial \varphi_\alpha}{\partial t} = -\lambda_\alpha \varphi_\alpha - \sum_{\alpha'\alpha''} \nu_{\alpha\alpha'\alpha''}\varphi_{\alpha'}\varphi_{\alpha''} + O(g^0\lambda^3, g^1),$$
$$\nu_{\alpha\alpha'\alpha''} \equiv \sum_{ii'i''} \upsilon_{i\alpha}\nu_{ii'i''}u_{i'\alpha'}u_{i''\alpha''} \tag{45}$$

In this case, (41) gives the integral equation

$$\hat{K} a_{\alpha p} = \lambda_\alpha a_{\alpha p}, \quad \langle a_{\alpha p}\zeta_{\mu p}\rangle = 0, \quad \langle a_{\alpha p}\theta_{\alpha' p}\rangle = \delta_{\alpha\alpha'} \tag{46}$$

for the functions $a_{\alpha p} \equiv \sum_i a_{ip}u_{i\alpha}$ ($\theta_{\alpha p} \equiv \sum_i \theta_{ip}\upsilon_{i\alpha}$). Thus, we have arrived at a spectral problem for the operator $\hat{K}$, i.e. for the linearized collision operator. The positiveness of its eigenvalues λ_α and the possibility of considering its eigenfunctions to be orthogonal in the scalar product (16) are mentioned above. The spectral problem (46) describes the kinetic modes of the system because the second condition in (46) means that the eigenfunctions $a_{\alpha p}$ are orthogonal to the hydrodynamic ones $\zeta_{\mu p}$ ($\hat{K}\zeta_{\mu p} = 0$). Thus the variables φ_i, as the problem under consideration implies, really attenuate with the time. The functions φ_α are the gas kinetic modes. Relations (46) define the type of these modes.

The integral equation (42) should be solved for the quantities $b_{ii'p}$ and $\nu_{ii'i''}$. This equation is simplified if, using the eigenfunctions (43), we introduce the variable

$$b_{\alpha\alpha' p} = \sum_{ii'} b_{ii'p} u_{i\alpha} u_{i'\alpha'} \tag{47}$$

that yields the equations

$$\hat{K} b_{\alpha'\alpha'' p} = \sum_{\alpha} a_{\alpha p} \nu_{\alpha\alpha'\alpha''} + b_{\alpha'\alpha'' p}(\lambda_{\alpha'} + \lambda_{\alpha''}) + h_{\alpha'\alpha'' p};$$

$$\langle b_{\alpha\alpha' p}\, \zeta_{\mu p}\rangle = 0, \quad \langle b_{\alpha\alpha' p}\, \theta_{\alpha'' p}\rangle = 0, \quad \langle h_{\alpha\alpha' p}\, \zeta_{\mu p}\rangle = 0 \tag{48}$$

for the quantities $b_{\alpha\alpha' p}$ and $\nu_{\alpha\alpha'\alpha''}$ (equations (48) may be called equations (42) in the α-representation). Here, $a_{\alpha p}$ is the eigenfunction that is found from equation (46), and $h_{\alpha\alpha' p}$ is a known function that depends on $a_{\alpha p}$. Besides the eigenfunctions $a_{\alpha p}$ (their number equals to the number of the parameters φ_i), the operator $\hat{K}$ has eigenfunctions $a_{\mu p}$ and additional ones a_{sp}. All these functions are orthogonal each to other and $a_{\mu p}$ are obtained by the orthogonalization of the functions $\zeta_{\mu p}$ ($\hat{K}\zeta_{\mu p} = 0$). The solution of the integral equation (48) can be sought in the form of expansion in the operator $\hat{K}$ eigenfunctions

$$b_{\alpha'\alpha'' p} = \sum_{\alpha} b^{\alpha}_{\alpha'\alpha''} a_{\alpha p} + \sum_{s} b^{s}_{\alpha'\alpha''} a_{sp} + \sum_{\mu} b^{\mu}_{\alpha'\alpha''} a_{\mu p},$$

$$h_{\alpha'\alpha'' p} = \sum_{\alpha} h^{\alpha}_{\alpha'\alpha''} a_{\alpha p} + \sum_{s} h^{s}_{\alpha'\alpha''} a_{sp} + \sum_{\mu} h^{\mu}_{\alpha'\alpha''} a_{\mu p}. \tag{49}$$

The second and fourth formulas in (48) show that the coefficients $b^{\mu}_{\alpha'\alpha''}$, $h^{\mu}_{\alpha'\alpha''}$ are equal to zero. Then the integral equation (48) yields

$$b^{\alpha}_{\alpha'\alpha''} = \frac{\nu_{\alpha\alpha'\alpha''} + h^{\alpha}_{\alpha'\alpha''}}{\lambda_\alpha - (\lambda_{\alpha'} + \lambda_{\alpha''})}, \qquad b^{s}_{\alpha'\alpha''} = \frac{h^{s}_{\alpha'\alpha''}}{\lambda_s - (\lambda_{\alpha'} + \lambda_{\alpha''})}. \tag{50}$$

The coefficients $\nu_{\alpha\alpha'\alpha''}$ are now found from the third formula in (48) with account for the last relation in (46), which yields

$$b^{\alpha}_{\alpha'\alpha''} + \sum_{s} b^{s}_{\alpha'\alpha''} \langle a_{sp}\theta_{\alpha p}\rangle = 0. \tag{51}$$

So, the integral equation with the additional conditions (42) has an unique solution for the quantities $b_{ii'p}$ and $\nu_{ii'i''}$.

Let us now discuss the calculation of the first-order contribution in gradients $L_i^{(1)}$ to equation (25) for the parameters φ_i. According to (25), with account for (22) and (27) the main contribution can be written as

$$L_i^{(1,1)} = -\frac{\partial}{\partial x_n}\int d^3p\,\theta_{ip}\upsilon_{np}\mathrm{f}_p^{(0,1)} - \sum_\mu \frac{\partial\langle\theta_{ip}\rangle}{\partial\zeta_\mu}L_\mu^{(1,1)} +$$

$$+\int d^3p\,d^3p'\theta_{ip}M_{pp'}\mathrm{f}_{p'}^{(1,1)} + \int d^3p\,d^3p'\,d^3p''\theta_{ip}M_{pp'p''}\mathrm{f}_{\mathrm{p}'}^{(1,0)}\mathrm{f}_{\mathrm{p}''}^{(0,1)} \qquad (52)$$

($L_i^{(1,0)} = 0$). Here, $L_\mu^{(1,1)}$ is the right-hand sides of the hydrodynamic equations for the variables ζ_μ, and thus for any function h of the hydrodynamic variables the following formula can be used

$$\sum_\mu \frac{\partial h}{\partial\zeta_\mu}L_\mu^{(1,1)} = \sum_\mu \frac{\partial h}{\partial\xi_\mu}M_\mu^{(1,1)}. \qquad (53)$$

Expressions for functions $M_\mu^{(1,1)}$ follows from equations (7) and (8)

$$M_0^{(1,1)} = -\frac{2}{3n}\left(\frac{\partial q_l^{o(0,1)}}{\partial x_l} + t_{ln}^{o(0,1)}\frac{\partial\upsilon_l}{\partial x_n}\right), \quad M_l^{(1,1)} = -\frac{1}{mn}\frac{\partial t_{ln}^{o(0,1)}}{\partial x_n},$$

$$M_4^{(1,1)} = 0 \qquad (54)$$

with fluxes

$$q_n^{o(0,1)} = \sum_i \langle\varepsilon_p\upsilon_{np}a_{i,p-m\upsilon}\rangle^o\varphi_i, \quad t_{ln}^{o(0,1)} = \sum_i \langle p_l\upsilon_{np}a_{i,p-m\upsilon}\rangle^o\varphi_i. \quad (55)$$

Further simplification of formula (52) for $L_i^{(1,1)}$ requires the specification of the parameters φ_i and the corresponding microscopic quantities θ_{ip}.

According to (30) and (53), the contribution $\mathrm{f}_p^{(1,1)}$ to the DF satisfies the equation

$$\sum_\mu \frac{\partial\mathrm{f}_p^{(0,1)}}{\partial\xi_\mu}M_\mu^{(1,0)} + \sum_\mu \frac{\partial w_p}{\partial\xi_\mu}M_\mu^{(1,1)} + \sum_i \frac{\partial\mathrm{f}_p^{(0,1)}}{\partial\varphi_i}L_i^{(1,1)} + \upsilon_{np}\frac{\partial\mathrm{f}_p^{(0,1)}}{\partial x_n} +$$

$$+\sum_i\left(\frac{\partial \mathrm{f}_p^{(1,1)}}{\partial\varphi_i}+\frac{\partial \mathrm{f}_p^{(1,1)}}{\partial\partial\varphi_i/\partial x_n}\frac{\partial}{\partial x_n}\right)L_i^{(0,1)}=$$

$$=\int d^3p'd^3p''M_{pp'p''}\mathrm{f}_{p'}^{(0,1)}\mathrm{f}_{p''}^{(1,0)}+\int d^3p'M_{pp'}\mathrm{f}_{p'}^{(1,1)}. \tag{56}$$

Taking into account expressions (32) for the DF and the expressions for the right-hand sides $M_\mu^{(1,0)}$, $L_i^{(0,1)}$, $L_i^{(1,1)}$, $M_\mu^{(1,1)}$ of the equations for RDP from (35), (38), (52), (54), we obtain the integral equations for the functions c_{inp}, $d_{i\mu np}$

$$\hat{K}c_{inp}=\sum_{i'}c_{i'np}\mu_{i'i}+\sum_{i'}a_{i'p}\alpha_{i'in}+h_{inp}, \tag{57}$$

$$\hat{K}d_{i\mu np}=\sum_{i'}d_{i'\mu np}\mu_{i'i}+\sum_{i'}a_{i'p}\beta_{i'i\mu n}+h_{i\mu np}, \tag{58}$$

which, according to (32), define the DF $\mathrm{f}_p^{(1,1)}$. Here h_{inp} and $h_{i\mu np}$ are known functions ($h_{i\mu np}$ depends on c_{inp}) and the coefficients $\alpha_{ii'n}$, $\beta_{ii'\mu n}$ are given by formulas

$$\alpha_{ii'n}=\{\theta_{ip},c_{i'np}\},\qquad \beta_{ii'\mu n}=\{\theta_{ip},d_{i'\mu np}\}. \tag{59}$$

The additional conditions for equations (57), (58) are given by the formulas respectively

$$\langle\zeta_{\mu p}c_{inp}\rangle=0,\quad \langle\theta_{ip}\,c_{i'np}\rangle=0,\quad \langle\zeta_{\mu p}h_{inp}\rangle=0; \tag{60}$$

$$\langle\zeta_{\mu p}d_{i\mu' np}\rangle=0,\quad \langle\theta_{ip}\,d_{i'\mu np}\rangle=0,\quad \langle\zeta_{\mu p}h_{i\mu' np}\rangle=0. \tag{61}$$

Expressions (59) for the coefficients $\alpha_{ii'n}$, $\beta_{ii'\mu n}$ follow from equations (57), (58) with account for the relation $\langle a_{ip}\theta_{i'p}\rangle=\delta_{ii'}$ from (41). However, these relations are not needed for the solution of equations (57), (58), and the integral equations are linear (this is similar to the situation with the solution of equations (41) and (42) without regard for (39)). Equations (57), (58), and (60), (61) are simplified in the α-representation

$$\begin{aligned}&\hat{K}c_{\alpha np}=c_{\alpha np}\lambda_\alpha+\sum_{\alpha'}a_{\alpha' p}\alpha_{\alpha'\alpha n}+h_{\alpha np},\\&\langle\zeta_{\mu p}c_{\alpha np}\rangle=0,\quad \langle\theta_{\alpha p}c_{\alpha' np}\rangle=0,\quad \langle\zeta_{\mu p}h_{\alpha np}\rangle=0;\end{aligned} \tag{62}$$

$$\begin{aligned}&\hat{K}d_{\alpha\mu np}=d_{\alpha\mu np}\lambda_\alpha+\sum_{\alpha'}a_{\alpha' p}\beta_{\alpha'\alpha\mu n}+h_{\alpha\mu np},\\&\langle\zeta_{\mu p}d_{\alpha\mu' np}\rangle=0,\quad \langle\theta_{\alpha p}d_{\alpha'\mu np}\rangle=0,\quad \langle\zeta_{\mu p}h_{\alpha\mu' np}\rangle=0.\end{aligned} \tag{63}$$

The solution of these equations may be discussed in a similar way to the solution of equation (48) and needs more information about parameters φ_i.

The calculation of the DF $f_p^{(1,1)}$ allows us to find the contribution $M_\mu^{(2,1)}$ to the right-hand side of the hydrodynamic equations (8). According to (8), the following formulas hold

$$M_0^{(2,1)} = -\frac{2}{3n}\left(\frac{\partial q_l^{o(1,1)}}{\partial x_l} + t_{ln}^{o(1,1)}\frac{\partial \upsilon_l}{\partial x_n}\right), \quad M_l^{(2,1)} = -\frac{1}{mn}\frac{\partial t_{ln}^{o(1,1)}}{\partial x_n},$$

$$M_4^{(2,1)} = 0 \tag{64}$$

with fluxes

$$q_n^{o(1,1)} = \int d^3p\varepsilon_p\upsilon_{np}f_p^{(1,1)}, \quad t_{ln}^{o(1,1)} = \int d^3pp_l\upsilon_{np}f_p^{(1,1)}. \tag{65}$$

In summary, we have investigated the equations for the RDP φ_i and ξ_μ in the following orders of the perturbation theory

$$\frac{\partial\varphi_i}{\partial t} = L_i^{(0,1)} + L_i^{(0,2)} + L_i^{(1,1)} + O(g^0\lambda^3, g^1\lambda^2, g^2\lambda^1, g^3),$$

$$\frac{\partial\xi_\mu}{\partial t} = M_\mu^{(1,0)} + M_\mu^{(1,1)} + M_\mu^{(2,0)} + M_\mu^{(2,1)} + O(g^1\lambda^2, g^2\lambda^2, g^3), \tag{66}$$

where the quantities $L_i^{(0,1)}$, $L_i^{(0,2)}$, $L_i^{(1,1)}$, $M_\mu^{(1,0)}$, $M_\mu^{(1,1)}$, $M_\mu^{(2,0)}$, $M_\mu^{(2,1)}$ are given in formulas (35), (36), (38), (52), (54), (64). Clearly the above-described procedure of sequential calculation of the DF and the right-hand sides of the RDP time equations can be continued. A detailed analysis of the obtained integral equations for the contributions to the DF is only possible when the parameters φ_i and the corresponding microscopic quantities θ_{ip} are specified. This will allow us to use rotational invariance considerations, which can greatly simplify the calculations, and to perform the required Galilean transformation.

4 Modified Grad problem

Consider application of the developed theory to the Grad problem in which nonequilibrium states of a gas are described by hydrodynamic variables

$\zeta_\mu(x,t)$ as well as by energy flux $q^o_n(x,t)$ and traceless momentum flux $\pi^o_{ln}(x,t)$ taken in ARF. Specification of RDP simplifies the consideration because allows us to make Galilean transformation for transition from LRF to ARF and to use rotational invariance in calculations.

It was stressed above that solution of the Grad problem in framework of the RDM can be based on the functional hypothesis (18) supplemented by definition of the RDP (19). It is obvious that the Grad DF (20) corresponds to the zero order approximation in gradients and to an approximation of small fluxes. In this section according to the general theory developed in Section 3 a modified Grad problem is investigated. In this problem the deviations of the fluxes $\delta q^o_n(x,t)$, $\delta\pi^o_{ln}(x,t)$ from their hydrodynamic values $q^o_n(x,\zeta(t))$, $\pi^o_{ln}(x,\zeta(t))$, which are functionals of the hydrodynamic variables $\zeta_\mu(x,t)$, are taken as the RDP. These deviations are assumed to be small values of the same order λ. Specification of the result of Section 3 for the problem considered here is quite simple. For example, the functional hypothesis considers the DF as a functional of the form $\mathrm{f}_p(x,\zeta(t),\delta q^o(t),\delta\pi^o(t))$.

The time equations for the fluxes q^o_n, π^o_{ln} follow from their definitions (19) and kinetic equation (1) and can be written as

$$\frac{\partial\pi^o_{ln}}{\partial t}=-\upsilon_m\frac{\partial\pi^o_{ln}}{\partial x_m}-\pi^o_{ln}\frac{\partial\upsilon_m}{\partial x_m}-\left(\pi^o_{lm}\frac{\partial\upsilon_n}{\partial x_m}+\pi^o_{nm}\frac{\partial\upsilon_l}{\partial x_m}-\frac{2}{3}\delta_{ln}\pi^o_{sm}\frac{\partial\upsilon_s}{\partial x_m}\right)+$$

$$-nT\left(\frac{\partial\upsilon_l}{\partial x_n}+\frac{\partial\upsilon_n}{\partial x_l}-\frac{2}{3}\delta_{ln}\frac{\partial\upsilon_m}{\partial x_m}\right)-\frac{\partial\pi_{ln,m}(\mathrm{f})}{\partial x_m}+R_{ln}(\mathrm{f}),\tag{67}$$

$$\frac{\partial q^o_l}{\partial t}=-\upsilon_n\frac{\partial q^o_l}{\partial x_n}-q^o_l\frac{\partial\upsilon_n}{\partial x_n}-\frac{5}{3}q^o_n\frac{\partial\upsilon_l}{\partial x_n}+\frac{1}{mn}\pi^o_{ln}\frac{\partial\pi^o_{nm}}{\partial x_m}+\frac{5}{2}\frac{T}{m}\frac{\partial\pi^o_{ln}}{\partial x_n}-$$

$$+\frac{1}{mn}(\pi^o_{ln}+\frac{5}{2}nT\delta_{ln})\frac{\partial nT}{\partial x_n}-\frac{\partial q_{ln}(\mathrm{f})}{\partial x_n}-\pi_{ln,m}(\mathrm{f})\frac{\partial\upsilon_n}{\partial x_m}+R_l(\mathrm{f})\tag{68}$$

where the notation

$$R_{ln}(\mathrm{f})=\frac{1}{m}\int d^3p\,h_{ln}(p)I_{p+m\upsilon}(\mathrm{f}),\quad R_l(\mathrm{f})=\frac{1}{m}\int d^3p\,\varepsilon_p p_l I_{p+m\upsilon}(\mathrm{f});$$

$$\pi_{ln,m}(\mathrm{f})=\frac{1}{m^2}\int d^3p\,h_{ln}(p)p_m\mathrm{f}_{p+m\upsilon},$$

$$q_{ln}(\mathrm{f})=\frac{1}{m^2}\int d^3p\,\varepsilon_p p_l p_n\mathrm{f}_{p+m\upsilon}\tag{69}$$

is introduced. Equations (67), (68) are satisfied by the functionals $q^o_n(x,\zeta(t)) \equiv \tilde{q}^o_n$, $\pi^o_{ln}(x,\zeta(t)) \equiv \tilde{\pi}^o_{ln}$ and the usual hydrodynamic DF $\tilde{\mathrm{f}}_p(x,\zeta) = w_p + \mathrm{f}^{(1,0)}_p + O(g^2)$ (see (27), (32)). Therefore, according to (67), (68) , the exact time equations for the deviations have the form

$$\frac{\partial\delta\pi^o_{ln}}{\partial t} = -\upsilon_m\frac{\partial\delta\pi^o_{ln}}{\partial x_m} - \delta\pi^o_{ln}\frac{\partial\upsilon_m}{\partial x_m} - $$
$$-\left(\delta\pi^o_{lm}\frac{\partial\upsilon_n}{\partial x_m} + \delta\pi^o_{nm}\frac{\partial\upsilon_l}{\partial x_m} - \frac{2}{3}\delta_{ln}\delta\pi^o_{sm}\frac{\partial\upsilon_s}{\partial x_m}\right) +$$
$$-\frac{\partial\delta\pi_{ln,m}}{\partial x_m} + \delta R_{ln} \equiv L_{ln}, \tag{70}$$

$$\frac{\partial\delta q^o_l}{\partial t} = -\upsilon_n\frac{\partial\delta q^o_l}{\partial x_n} - \delta q^o_l\frac{\partial\upsilon_n}{\partial x_n} - \frac{5}{3}\delta q^o_n\frac{\partial\upsilon_l}{\partial x_n} + \frac{5}{2}\frac{T}{m}\frac{\partial\delta\pi^o_{ln}}{\partial x_n} +$$
$$+\frac{1}{mn}\delta\pi^o_{ln}\frac{\partial\delta\pi^o_{nm}}{\partial x_m} + \frac{1}{mn}\tilde{\pi}^o_{ln}\frac{\partial\delta\pi^o_{nm}}{\partial x_m} + \frac{1}{mn}\delta\pi^o_{ln}\frac{\partial\tilde{\pi}^o_{nm}}{\partial x_m} +$$
$$+\frac{1}{mn}\delta\pi^o_{ln}\frac{\partial nT}{\partial x_n} - \frac{\partial\delta q_{ln}}{\partial x_n} - \delta\pi_{ln,m}\frac{\partial\upsilon_n}{\partial x_m} + \delta R_l \equiv L_l \tag{71}$$

where the notations

$$\delta R_{ln} \equiv R_{ln}(\mathrm{f}) - R_{ln}(\tilde{\mathrm{f}}), \quad \delta R_l \equiv R_l(\mathrm{f}) - R_l(\tilde{\mathrm{f}});$$
$$\delta\pi_{ln,m} \equiv \pi_{ln,m}(\mathrm{f}) - \pi_{ln,m}(\tilde{\mathrm{f}}), \quad \delta q_{ln} \equiv q_{ln}(\mathrm{f}) - q_{ln}(\tilde{\mathrm{f}}) \tag{72}$$

are introduced. To continue the derivation of the time equations, one needs to calculate DF $\mathrm{f}_p(x,\zeta,\delta q^o,\delta\pi^o)$ using the general theory developed in Section 3 and substitute it into (72).

Here we restrict ourselves to the calculation of the contribution $\mathrm{f}^{(0)}_p$ of zero order in gradients to this DF. According to (32) in this approximation the DF $\mathrm{f}_p(x,\zeta,\delta q^o,\delta\pi^o)$ has the structure

$$\mathrm{f}^{(0)}_p = w^o_p\{1 + a_{np}\,\delta q^o_n + a_{lnp}\,\delta\pi^o_{ln} + O(g^0\lambda^2)\}_{p\to p-m\upsilon},$$
$$a_{np} \equiv a_p p_n, \quad a_{nlp} \equiv b_p\,h_{nl}(p) \tag{73}$$

where a_p, b_p are some scalar functions. In view of (70)-(72), the time equations for the deviations of the fluxes q^o_n, π^o_{ln} in the zero order in gradients have the form

$$\frac{\partial\delta q^o_l}{\partial t} = -\lambda_q\delta q^o_l + O(g^0\lambda^2, g^1),$$

$$\frac{\partial \delta\pi^o_{ln}}{\partial t} = -\lambda_\pi \delta\pi^o_{ln} + O(g^0\lambda^2, g^1) \tag{74}$$

where

$$\lambda_q = \frac{1}{3m}\{p_l\varepsilon_p, p_l a_p\}^o, \quad \lambda_\pi = \frac{1}{5m}\{p_l p_n, h_{ln}(p)b_p\}^o. \tag{75}$$

These quantities are written using the following bilinear form

$$\{g_p, h_p\}^o = \int d^3p d^3p' w^o_p K^o_{pp'} g_p h_{p'} \tag{76}$$

which is a specification of the form (14) $\{g_p, h_p\}$ for the ARF (see also (9), (11), (34)). According to formulas (32), (41) of the general theory, the functions a_p, b_p from (73) satisfy the integral equations with the additional conditions

$$\hat{K}^o a_p p_l = \lambda_q a_p p_l, \quad \langle\varepsilon_p a_p\rangle^o = 0, \quad \langle\varepsilon^2_p a_p\rangle^o = 3/2; \tag{77}$$

$$\hat{K}^o b_p h_{ln}(p) = \lambda_\pi b_p h_{ln}(p), \quad \langle\varepsilon^2_p b_p\rangle^o = \frac{15}{8m}. \tag{78}$$

As would be expected, these equations are eigenvalue problems for the operator $\hat{K}^o$ defined in (34). According to the remark given after formulas (16), its eigenvalues are positive and equations (71) describe attenuation of the flux deviations $\delta q^o_n(x,t)$, $\delta\pi^o_{ln}(x,t)$ i.e. the processes

$$q^o_n(x,t) \xrightarrow[t\gg\tau_0]{} q^o_n(x,\zeta(t)), \qquad \pi^o_{ln}(x,t) \xrightarrow[t\gg\tau_0]{} \pi^o_{ln}(x,\zeta(t)). \tag{79}$$

This phenomenon is called the Maxwell relaxation. In the Grad theory [13] relaxation equations of the type (74) for fluxes $q^o_n(x,t)$, $\pi^o_{ln}(x,t)$ are obtained too, but describe simple attenuation of these fluxes.

Equations (74) give contributions $L^{(0,1)}_l$, $L^{(0,1)}_{ln}$ to the right-hand sides L_l, L_{ln} of the time equations for RDP (70), (71). According to the general theory, contributions to L_l, L_{ln} that do not depend on the parameters δq^o_n, $\delta\pi^o_{ln}$ are absent, and therefore

$$L^{(1,0)}_l = 0, \quad L^{(1,0)}_{ln} = 0; \quad L^{(2,0)}_l = 0, \quad L^{(2,0)}_{ln} = 0 \tag{80}$$

(see, for example, equations (66)). In the present paper other contributions will not be discussed. Consider only approximate solution of the integral equations (77), (78) using the Burnett method of a truncated Sonine polynomial expansion.

Solution of the equation (77) with account for its tensor dimensionality is found in the form of the series

$$a_p = \sum_{s=0}^{\infty} a_s S_s^{3/2}(\beta\varepsilon_p) \tag{81}$$

($\beta = T^{-1}$). Additional conditions (77) thus give

$$a_0 = 0, \qquad a_1 = -2\beta^2/5n. \tag{82}$$

Integral equation (77) leads to the following infinite set of linear equations for the coefficients a_s

$$\sum_{s'=1}^{\infty} A_{ss'}\tilde{a}_{s'} = \tilde{\lambda}_q \tilde{a}_s \tag{83}$$

where the notations

$$A_{ss'} = \{p_l S_s^{3/2}(\beta\varepsilon_p), p_l S_{s'}^{3/2}(\beta\varepsilon_p)\}^o (x_s x_{s'})^{-1/2},$$

$$\tilde{a}_s = a_s x_s^{1/2}, \qquad \tilde{\lambda}_q = \frac{2mn}{\beta}\lambda_q, \qquad x_s \equiv \frac{2\Gamma(s+5/2)}{s!\pi^{1/2}} \tag{84}$$

are introduced. According to the properties of the bilinear form (76) the matrix $A_{ss'}$ is symmetric and positively defined. Solution of equations (83) in one- and two-polynomial approximations gives

$$a_1^{[1]} = a_1, \qquad \tilde{\lambda}_q^{[1]} = A_{11};$$

$$a_1^{[2]} = a_1, \qquad a_2^{[2]} = \frac{2}{7^{1/2}} \frac{\tilde{\lambda}_q^{[2]} - A_{11}}{A_{12}} a_1,$$

$$\tilde{\lambda}_q^{[2]} = \{(A_{11} + A_{22}) - [(A_{11} - A_{22})^2 + 4A_{12}^2]^{1/2}\}/2 \tag{85}$$

(here $A^{[n]}$ is a quantity A taken in n-polynomial approximation). Note that in the two-polynomial approximation eigenvalue $\tilde{\lambda}_q^{[2]}$ of the smallest value was chosen.

With account for the tensor dimensionality of equation (78) its solution is found as the following series expansion

$$b_p = \sum_{s=0}^{\infty} b_s S_s^{5/2}(\beta\varepsilon_p). \tag{86}$$

Additional condition (78) define the first coefficient of the expansion

$$b_0 = \beta^2/2mn. \tag{87}$$

Integral equation (78) leads to the infinite set of linear equations for the coefficients b_s

$$\sum_{s'=0}^{\infty} B_{ss'}\tilde{b}_{s'} = \tilde{\lambda}_\pi \tilde{b}_s \tag{88}$$

where the notations

$$B_{ss'} = \{h_{ln}(p)S_s^{5/2}(\beta\varepsilon_p), h_{ln}(p)S_{s'}^{5/2}(\beta\varepsilon_p)\}^o(y_s y_{s'})^{-1/2},$$

$$\tilde{b}_s = b_s y_s^{1/2}, \qquad \tilde{\lambda}_\pi = \frac{8m^2n}{3\beta^2}\lambda_\pi, \qquad y_s \equiv \frac{2\Gamma(s+7/2)}{s!\pi^{1/2}} \tag{89}$$

are introduced. According to the properties of the bilinear form (76) the matrix $B_{ss'}$ is symmetric and positively defined one. Solution of equations (88) in one- and two-polynomial approximations gives

$$b_0^{[1]} = b_0, \qquad \tilde{\lambda}_\pi^{[1]} = B_{00};$$

$$b_0^{[2]} = b_0, \qquad \tilde{b}_1^{[2]} = \frac{2^{1/2}}{7^{1/2}} \frac{\tilde{\lambda}_\pi^{[2]} - B_{00}}{B_{01}} b_0,$$

$$\tilde{\lambda}_\pi^{[2]} = \{(B_{00} + B_{11}) - [(B_{00} - B_{11})^2 + 4B_{01}^2]^{1/2}\}/2. \tag{90}$$

Note that in the two-polynomial approximation the eigenvalue $\tilde{\lambda}_\pi^{[2]}$ of the smallest value was chosen.

So, in the one-polynomial approximation the following expression for DF (73) of the zero order in gradients is obtained

$$\mathrm{f}_p^{(0)} = w_p^o \left\{ 1 + \frac{1}{2mnT^2} p_n p_l \delta\pi_{ln}^o + \right.$$

$$\left. + \frac{1}{nT^2} p_n (\frac{2}{5}\frac{\varepsilon_p}{T} - 1)\delta q_n^o + O(g^0\lambda^2) \right\}_{p \to p - mv} \tag{91}$$

For the selected independent variables this expression coincides with the Grad DF (20). Therefore, the statement given at the end of section 2 is confirmed and his DF contains only contributions of the orders $g^0\lambda^0$, $g^0\lambda^1$ and takes them in the one-polynomial approximation.

In the one-polynomial approximation our attenuation constants are given by the formulas

$$\lambda_q^{[1]} = \frac{2}{15mnT^3}\{p_l\varepsilon_p, p_l\varepsilon_p\}^o = \frac{5}{2}\frac{nT}{m\kappa^{[1]}},$$

$$\lambda_\pi^{[1]} = \frac{1}{10m^2nT^2}\{h_{ln}(p), h_{ln}(p)\}^o = \frac{nT}{\eta^{[1]}} \tag{92}$$

where the expressions for the heat conductivity $\kappa^{[1]}$ and the shear viscosity $\eta^{[1]}$ in the same approximation are used to compare. Famous result of the theory [3] is given by the formula

$$\kappa^{[1]} = 15\eta^{[1]}/4m \tag{93}$$

which leads to the relations

$$\lambda_q^{[1]} = 2\lambda_\pi^{[1]}/3. \tag{94}$$

Note that the Grad theory [12] gives also expressions (92) for attenuation constants. However, in his theory this constants describe unphysical attenuation of the fluxes $q_l^o(x,t)$, $\pi_{ln}^o(x,t)$ to zero and cannot be corrected.

As the final remark note that it is not possible to rigorously prove the method of a truncated polynomial expansion for solution of eigenvalue problem for operator $\hat{K}$. However, the proposed calculations additionally show limitation of the Grad method as an alternative to the Bogolyubov reduced description method.

5 Conclusion

The Chapman–Enskog method has been generalized for the investigation of processes in the vicinity of hydrodynamic states of a gas. The generalization is made on the basis of the Bogolyubov idea of the functional hypothesis. A theory that describes a nonequilibrium state of a gas by the usual hydrodynamic variables $\zeta_\mu(x,t)$ and arbitrary additional local variables $\theta_i(x,t)$ has been constructed. The gradients of all these parameters and the deviations $\varphi_i(x,t)$ of the variables $\theta_i(x,t)$ from their hydrodynamic values $\theta_i(x,\zeta(t))$ are assumed to be small and are estimated by two independent small parameters g, λ. The proposed theory is nonlinear in the variables $\varphi_i(x,t)$ too.

The usual Chapman–Enskog method leads to the solution of Fredholm integral equations of the first kind with an operator $\hat{K}$ given by the linearized collision integral. The proposed theory leads to the solution of linear integral equations of a more complicated nature with the same operator $\hat{K}$. Some of them are eigenvalue problems for the operator $\hat{K}$ and describe the kinetic modes of the system.

The proposed theory is applied to the solution of a modified Grad problem. Grad formulated his problem in his 13-moment approximation for the solution of kinetic equations. In his theory nonequilibrium states of a gas are described, in addition to the usual hydrodynamic variables, by the fluxes of energy $q^o_n(x,t)$ and traceless momentum $\pi^o_{ln}(x,t) \equiv t^o_{ln}(x,t) - t^o_{mm}(x,t)\delta_{ln}/3$ in the accompanying reference frame. In fact these fluxes are considered as small quantities of the same order λ and the Grad distribution function includes only terms of the orders $g^0\lambda^0$, $g^0\lambda^1$. Moreover, it corresponds to the one-polynomial approximation. In our modification of the Grad problem a nonequilibrium state of a gas is described by the usual hydrodynamic variables and small deviations $\delta q^o_n(x,t)$, $\delta\pi^o_{ln}(x,t)$ of the above-mentioned fluxes from their hydrodynamic values $q^o_n(x,\zeta(t))$, $\pi^o_{ln}(x,\zeta(t))$. In the simplest approximation this leads to a theory of the Maxwell relaxation.

The consideration shows that in the 13-moment Grad approximation the investigation of the relaxation phenomena in the system is reduced to a very approximate solution of the eigenvalue problem for the operator $\hat{K}$. The Bogolyubov reduced description method, based on his idea of the functional hypothesis, gives an adequate solution of the problem.

The proposed theory can be applied to evolution described by arbitrary kinetic equations, and to the evolution of dense systems described by the Liouville equation.

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