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The sea-based gravity monotower: coupling external hydrodynamics, sloshing, soil, and structural vibrations. The multimodal modelling *

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За допомогою лінійного мультимодального метода виводено наближену аналітичну модель тривимірної лінійної динаміки морських гравітаційних башт. Модель набуває формі системи лінійних [інтегро-] диференціальних рівнянь. Вона враховує реакцію ґрунту, вібрації структури, коливання рідини в шахті башти та морські хвилі. Зовнішні навантаження можуть пов'язуватися з набігаючими морськими хвилями, операціями на платформі, а також землетрусом.

Используя линейный мультимодальный метод, выводится приближенная аналитическая модель трехмерной динамики морских гравитационных башень. Модель принимает вид системы линейных [интегро-] дифференциальных уравнений. Она учитывает реакцию грунта, вибрации структуры, колебания жидкости в шахте башни и морские волны. Внешние нагрузки могут связываться с набегающими морскими волнами, операциями на платформе, а также землетрясением.

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1. Introduction

The sea-based gravity platforms are common for the oil industry and wind turbine towers. As a necessary supporting element, they include one or several concrete steel-reinforced towers with operational platforms on the top. The complex structure remains stable in its upright static state facilitated by a heavy fundament set upon the hard bedrock. The towers have, normally, shafts partly filled by a liquid. The contained liquid is not necessarily the sea water. The entire structure can oscillate, horizontally and vertically (lifting), due to external loads caused by incident waves, forces and moments applied to the top operational platform, and, generally speaking, the Earthquake. The maximum structural response is expected at resonant frequencies which are not the same as the eigenfrequencies of the "dry" structure.

Sloshing in the tower shaft can play the same role as in the so-called Tuned Liquid Dampers (TLD) installed on the roof of tall buildings for mitigating their vibrations [7, 8]. The lowest natural sloshing frequency should then be close to the lowest structural eigenfrequency. Such a closeness is found out, e.g., for the Draugen platform [1].

Direct numerical simulations of the structure–fluids–soil motions based on the space-and-time discretisation could be a rather difficult task, especially, when the research goal is a parameter study. The latter prefers a relatively simple approximate mathematical model of the complex multi-component mechanical system. An attempt to construct such a model is presented in Sect. 5.4.5 of [4], where the main assumptions are (i) the structural motions are strongly two-dimensional occurring in the meridional tower plane, (ii) the tower shell is modelled as the Euler–Bernoulli beam, (iii) the fundament is rigid and clamped with an absolutely rigid soil, (iv) effect of internal pipes (submerged into the tower shaft) is neglected, (v) the top operational platform is a mass point. The key analytical tool for describing the liquid sloshing in the tower shaft was the linear *multimodal method*. The method replaces, in a rigorous mathematical way, the original free-surface problem with a system of ordinary differential equation coupling the generalised coordinates responsible for amplification of the natural sloshing modes. The method is efficient for solving the coupled liquid–tank problems (see, e.g., a recent study on the elevated water tank [6]) when the tank is *rigid*. In Sect. 5.4.5 of [4], the method was applied to the monotower problem assuming that the upper wetted shaft part (whose height is approximately equal to the three shaft radii) moves horizontally as a rigid body. This assumption is quite disputable but the authors had no other choice – the multimodal method was well elaborated only for the rigid tank.

The goal of the present paper is fourfold. *First*, we consider a more general mechanical model of the sea-based gravity monotower by (a) suggesting the three-dimensional fluids-structure-soil motions, (b) modelling the tower shell as the Euler–Bernoulli beam with varying stiffness admitting both horizontal beam vibrations and, it is also a novelty, the beam lifting (as a rigid body), (c) accounting for the soil feedback by using the Winkel approach (the soil effect is accounted for in many publications on the elevated water tanks, see, e.g., [2, 3] and references therein, but not for the gravity platforms – an exception is in [10] where a semi-empirical analysis is given in the two-dimensional statement), (d) accounting for the submerged pipes effect on the natural sloshing frequencies (even though we demonstrated in Sect. 4.11.5 of [4] how to account for the pipes effect, the simplified analysis in Sect. 5.4.5 and other works [an exception is [1]] neglects it), (e) modelling the top-installed operational platform as a rigid plate of the non-small thickness (not a mass point!). *Second*, we introduce and analyse eigenoscillations of the monotower without the surface wave effect of both internal and external fluids (pseudo-dry conditions). *Third*, we generalise the linear multimodal method for sloshing in a beam-type tank. The method introduces and deals with an infinite series of the Stokes–Joukowski potentials. The corresponding linear modal system is derived with respect to the generalised coordinates responsible for amplification of the natural sloshing modes. The system is coupled with the structural dynamics. *Fourth*, we consider the linear modal equations for the entire mechanical system in which all the generalised coordinates (due to structural vibrations and sloshing) are linearly coupled. When the external wave effect is neglected, the modal equations give rise as a system of ordinary differential equations. Additional convolution integral terms should appear when accounting for the external water waves.

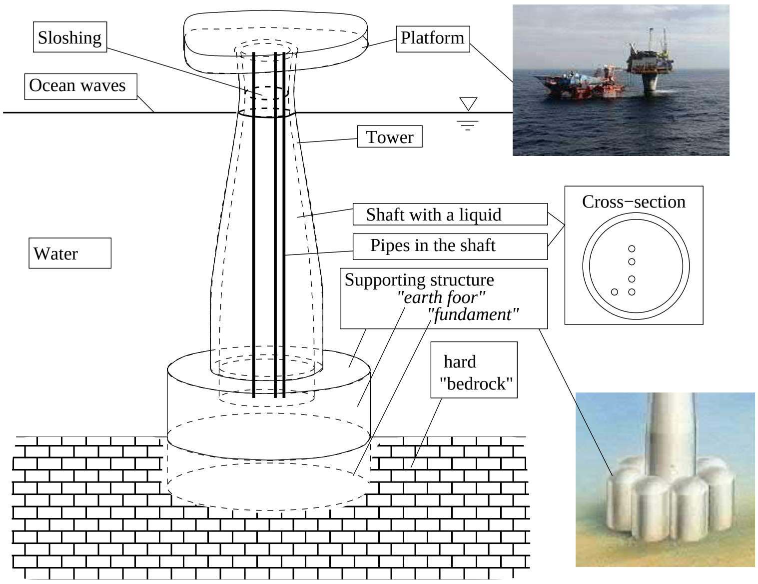

Fig 1. Sketch of a gravity monotower. The structural elements are the rigid platform installed on the tower top and modelled as a rigid plate of finite height, the hollow axisymmetric tower whose shaft can contain vertical pipes, the heavy supporting structure which is formally divided into the fundament (underground part) and the earth floor (wetted "soil-erecting" part). For the Draugen platform, the supporting structure consists of a set of cylindrical cells with spherical ends, the cells are coupled and reinforced (especially strongly for the underground part) alongside. The shaft embeds into the supporting structure. Two fluid domains are associated with the external water and the contained liquid inside the shaft. The vertical structure is assumed be stable in its static upright state under the gravitation force. The stability is facilitated by the massive (heavy) fundament laying on the hard soil (bedrock).

2. Mechanical model: governing equations and boundary conditions

2.1. General notations and notes

Fig. 1 schematically depicts the considered fluids-structure-soil mechanical system. The caption describes its components: (a) the top-placed platform is a rigid plate with a finite height (thickness), (b) the hollow steel-reinforced concrete axisymmetric tower is the Euler–Bernoulli beam deflecting horizontally and admitting vertical displacements as a rigid body, the tower shaft can contain rigid vertical pipes whose effect on the contained liquid sloshing frequencies may be important but their interaction with structural elements is neglected, (c) the supporting structure of a complex shape (e.g., a set of relatively long cylindrical concrete cells with spherical ends filled by a liquid) is reinforced by skirts at the soil–fundament contact area; the structure is formally divided into the [underground] fundament and the [ground-erecting] "earth floor". The "earth floor" is shorter for the Draugen platform but can be of comparable heigh, e.g., for the sea-based gravity wind turbines. The "earth floor" will therefore be modelled as the Euler–Bernoulli beam with appropriate Young modulus and mass density. With increasing rigidity, it can move as a rigid body. The structural elements (a), (b), and (c) are rigidly coupled with each other at the end-sides so that the two beams (b) and (c) are, for simplicity of derivations, considered as a single Euler–Bernoulli beam with the piece-wise geometrical and physical characteristics.

The coupled fluids–structure–soil motions are analysed in the linear approximation implying small-magnitude motions with respect to the static state. The three different excitations are related to (1) external loads applied to the rigid platform (due to, e.g., the workers and production activities); (2) the Earthquake appearing as a given translatory soil motion; (3) the sea wave loads introduced as appropriate conditions at infinity for the external water problem (the incident velocity potential).

The fluids are incompressible and inviscid with irrotational flows. The sea water and the contained liquid (in the shaft) can, generally, be of different depths so that their mean free surfaces have then different vertical levels.

The gravity platform is kept in the stable static upright state. Even though there are a scour protection around the fundament and a gathering of naturally occurring sediments that is broken down by processes of weathering and erosion, e.g., the sand, the seabed is assumed be totally flat.

2.1.1. Rigid platform

The *rigid platform* of the mass M_p is modelled as a rigid plate of the cross-section $\mathcal{S}_p$ and the height h_p (see, Fig. 2 a) whose weight (mass) centre S_p belongs to the symmetry axis of the beam. When introducing

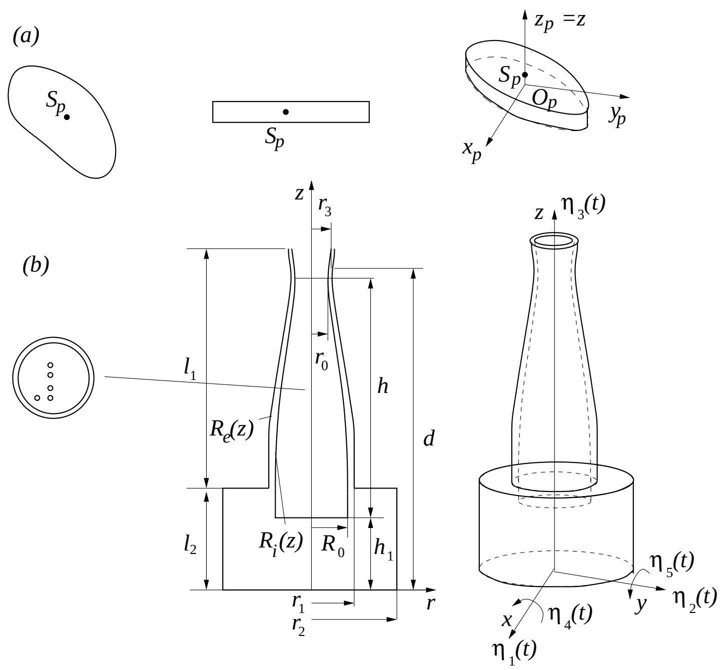

Fig 2. Structural components of the considered mechanical system the rigid platform, the tower and the "earth floor". The meridional cross-section and the three-dimensional view. The point S_p is the platform mass centre belonging to the symmetry axis. The coordinate system $O_p x_p y_p z_p$ is rigidly fixed with the platform bottom in its static state. The function $R_i(z)$ implies the shaft radius which equals to zero for $0 < z < h_1$. The function $R_e(z)$ governs the external radius of the axisymmetric structure; by definition, $r_1 = R_e(l_2+)$ and $r_2 = R_e(l_2-)$. The distance h is the liquid depth in the shaft and d is the sea depth. In the present paper, we assume that $h_1 < l_2$ (the shaft continues into the earth floor).

the local coordinate system $O_p x_p y_p z_p$ (rigidly fixed with the platform bottom in its static state), the mass centre is $(0, 0, z_p^C)$ so that $z_p^C = \frac{1}{2} h_p$ in the case of the homogeneous plate with the constant mass density ρ_p. We postulate, for simplicity of consideration, that the inertia tensor I^p computed with respect to $O_p x_p y_p z_p$ has the same structure as for the homogeneous plate, namely,

$$\begin{aligned}
&I_{11}^p = \int_0^{h_p}\int_{\mathcal{S}_p} \rho_p (y_p^2 + z_p^2) dS dz_p, \quad I_{22}^p = \int_0^{h_p}\int_{\mathcal{S}_p} \rho_p (x_p^2 + z_p^2) dS dz_p, \\
&I_{33}^p = \int_0^{h_p}\int_{\mathcal{S}_p} \rho_p (x_p^2 + y_p^2) dS dz_p, \\
&I_{12}^p = I_{21}^p = -\int_0^{h_p}\int_{\mathcal{S}_p} \rho_p x_p y_p \, dS dz_p, \quad I_{13}^p = I_{31}^p = I_{23}^p = I_{32}^p = 0,
\end{aligned} \tag{1}$$

where the platform density is $\rho_p = \rho_p(x_p, y_p, z_p)$ and, generally speaking, $I_{12}^p \neq 0$.

External loads applied to the rigid platform are associated with the given force $(\mathcal{P}_1(t), \mathcal{P}_2(t), \mathcal{P}_3(t))$ and moment $(\mathcal{P}_4(t), \mathcal{P}_5(t), 0)$ (with respect to O_p). The yaw motions are not considered and, therefore, we postulate that the torque $\mathcal{P}_6(t) \equiv 0$.

2.1.2. Beam

The complex *Euler–Bernoulli beam* (b+c) has the total length $L = l_1 + l_2$. Its motions are considered in the inertial Earth-fixed coordinate system $Oxyz$ whose vertical axis coincides with the beam symmetry axis in the static upright state (see, Fig. 2 b) and Oz is counter-directed to the gravity acceleration vector. The Oxy plane coincides with the unperturbed seabed plane.

The horizontal beam deviations along the Ox and Oy axes are governed by $v(z, t)$ and $w(z, t)$, $0 < z < L$, respectively. The vertical beam oscillations (as a rigid body, the so-called lifting) are described by the generalised coordinate $\eta_3(t)$. This means, for instance, that the velocity of a fixed point on the beam (internally, on the shaft surface, or, externally, on the wetted beam surface) is the vector $(\dot{v}(z, t), \dot{w}(z, t), \dot{\eta}_3(t))$ in the $Oxyz$-coordinate system.

Important geometric and physical characteristics of the beam are: the inner radius $R_i(z)$, the external radius $R_e(z)$, the Young modulus $E(z)$, the second moment of inertia $I(z) = \frac{1}{4}\pi(R_e^4(z) - R_i^4(z))$, the beam mass density $\rho_b(z)$, and the cross-sectional area of the beam $S_b(z) = \pi(R_e^2(z) - R_i^2(z))$. These characteristics are piece-wise functions of z so that jumps are possible at $z = h_1$ and l_2. We denote

$$r_1 = R_e(l_2+), \quad r_2 = R_e(l_2-) = R_e(0), \quad r_3 = R_e(d),$$
$$R_0 = R_i(h_1+), \quad r_0 = R_i(h + h_1)$$

with $r_1 < r_2$. In the present paper, we suggest, for clarity, that $h_1 < l_2$ (*the shaft continues into the earth floor part*).

Even though the theory operates with the deviations v and w defined for $0 < z < L$, we will, when necessarily, consider the structure as a beam-beam-beam (BBB) mechanical system (the beam-shell-beam and beam-beam-beam systems are in some detail studied in [9]) and, therefore, v and w will fall into the three sets of functions: $v_1(z,t), w_1(z,t)$ on $(0,h_1)$, $v_2(z,t), w_2(z,t)$ on (h_1,l_2), and $v_3(z,t), w_3(z,t)$ on (l_2,L). This divides $[0,L]$ into there subintervals, $k = 1,2,3$. The functions $v_k(z,t)$ and $w_k(z,t)$, $j = 1,2,3$, should then have up to fourth continuous derivatives by z on the introduced subintervals but only first derivative is continuous at h_1 and l_2 implying the so-called kinematic transmission conditions

$$\begin{aligned} &v_1(h_1,t) = v_2(h_1,t), \; w_1(h_1,t) = w_2(h_1,t), \\ &v_1'(h_1,t) = v_2'(h_1,t), \; w_1'(h_1,t) = w_2'(h_1,t), \\ &v_2(l_2,t) = v_3(l_2,t), \; w_2(l_2,t) = w_3(l_2,t), \\ &v_2'(l_2,t) = v_3'(l_2,t), \; w_2'(l_2,t) = w_3'(l_2,t). \end{aligned} \tag{2}$$

These condition should be *a priori* satisfied when working with the Lagrange type variational formulations. Additional transmission conditions at $z = h_1$ and l_2 are derivable from these formulations introducing relationships between the bending moments and the shear forces.

2.1.3. Soil

Adopting the Winkel approach is common for elevated water tanks [3,10]. This relates the soil feedback to artificial springs and dampers whose characteristics are taken from experiments. The feedback is transmitted to the earth floor (here, the cylinder of the height l_2 and the radius r_2). The small-magnitude sway $\eta_1(t)$, surge $\eta_2(t)$, heave $\eta_3(t)$ (lifting), roll $\eta_4(t)$, and pitch $\eta_5(t)$ motions relative to the $Oxyz$ system correspond to

$$\eta_1(t) = v(0,t), \; \eta_2(t) = w(0,t), \; \eta_4(t) = w'(0,t), \; \eta_5(t) = -v'(0,t) \tag{3}$$

rewritten in terms of the introduced horizontal beam deviations $v(z,t)$ and $w(z,t)$ at the lower end $x=0$.

The Earthquake is defined by the given functions $E_1(t), E_2(t)$, and $E_3(t)$ responsible for small-amplitude translatory motions of the origin O. Restricting to translatory excitations (no angular solid ones) is acceptable when the Earthquake wave length is much longer than the fundament diameter $2r_2$. The elastic fundament feedback is decoupled in terms of the generalised coordinates $\eta_i(t)$.

The force and moment applied to the lower beam end due to the elastic soil feedback are $(F_1(t), F_2(t), F_3(t))$ and $(F_4(t), F_5(t), 0)$, respectively; by definition, the torque $F_6(t) \equiv 0$. The horizontal forces are

$$F_1 = -k_1(\eta_1 - E_1) - d_1(\dot{\eta}_1 - \dot{E}_1) = \\ = -k_1(v(0,t) - E_1(t)) - d_1(\dot{v}(0,t) - \dot{E}_1(t)), \quad \text{(4a)}$$

$$F_2 = -k_1(\eta_2 - E_2) - d_1(\dot{\eta}_2 - \dot{E}_2) = \\ = -k_1(w(0,t) - E_2(t)) - d_1(\dot{w}(0,t) - \dot{E}_2(t)), \quad \text{(4b)}$$

where k_1 is the "Hooke coefficient" and d_1 is the damping coefficient.

To find the vertical force $F_3(t)$ and the moments $F_4(t), F_5(t)$, we introduce the normal elastic soil feedback applied to the beam end through the pressure at each point of the beam bottom ($z=0$)

$$\begin{aligned} p_s(x,y,t) &= -\frac{1}{\pi r_2^2}\left[k_2 v_n + d_2 \dot{v}_n\right], \\ v_s(x,y,t) &= (\eta_3(t) - E_3(t)) + y\eta_4(t) - x\eta_5(t) \end{aligned} \quad (5)$$

where $v_s(x,y,t)$ is a small relative normal deviation of the supporting structure in the Oxy-plane, k_2 and d_2 are the Hooke and damping coefficients, respectively.

The pressure p_s is a function of x and y. Integrating (5) over the cross-section of the supporting structure at $z=0$ deduces the vertical force component ((r,θ,z) is the corresponding cylindrical coordinate system):

$$F_3 = \int_0^{r_2}\int_0^{2\pi} r p_s d\theta dr = -k_2(\eta_3(t) - E_3(t)) - d_2(\dot{\eta}_3(t) - \dot{E}_3(t)). \quad (6)$$

The tangential forces F_1 and F_2 do not influence the moments as passing through O. However, the pressure p_s contributes the moment

$$\int_0^{r_2} r \int_0^{2\pi} \boldsymbol{r} \times (p_s\,\boldsymbol{k}) d\theta dr,$$

where $\boldsymbol{k}$ is the vertical unit vector and $\boldsymbol{r} = (x, y, z)$. Derivations give

$$F_4 = -\tfrac{1}{4} r_2^2 \left[k_2 \eta_4(t) + d_2 \dot{\eta}_4(t)\right] = -\tfrac{1}{4} r_2^2 \left[k_2 w'(0,t) + d_2 \dot{w}'(0,t)\right], \quad (7a)$$

$$F_5 = -\tfrac{1}{4} r_2^2 \left[k_2 \eta_5(t) + d_2 \dot{\eta}_5(t)\right] = \tfrac{1}{4} r_2^2 \left[k_2 v'(0,t) + d_2 \dot{v}'(0,t)\right]. \quad (7b)$$

The limit case k_1, $k_2 \to \infty$ (the rigidity increases) implies no sway, surge, heave, roll, and pitch motions of the beam end and, thereby, leads to the clamped end at $z = 0$.

2.1.4. Contained liquid

The absolute velocity potential of the contained liquid motions can be found from the linear sloshing problem formulated in the cylindrical coordinates (r, θ, z) as

$$\nabla^2 \varphi_i = 0 \quad \text{in} \quad Q_0, \quad (8a)$$

$$\frac{\partial \varphi_i}{\partial n} = (\cos\theta\, \dot{v} + \sin\theta\, \dot{w} - R_i' \dot{\eta}_3) \,/ \sqrt{1 + R_i'^2} \ \text{ on } \ W_0, \quad (8b)$$

$$\frac{\partial \varphi_i}{\partial n} = -\frac{\partial \varphi_i}{\partial z} = r\cos\theta\, \dot{v}' + r\sin\theta\, \dot{w}' - \dot{\eta}_3 \quad \text{on} \quad B_0, \quad (8c)$$

$$\frac{\partial \varphi_i}{\partial z} = \dot{\zeta}, \quad \dot{\varphi}_i + g(\zeta - \eta_3 - h - h_1) = 0 \ \text{ on } \ \Sigma_0, \quad (8d)$$

where $z = \zeta(r, \theta, t)$ determines the absolute small-magnitude free-surface elevations, $\Sigma_0 = \{(r, \theta, h_1 + h) : 0 \le r < R_i(h + h_1) = r_0,\ 0 < \theta \le 2\pi\}$ is the mean free surface, $W_0 = \{(R_i(z), \theta, z) :\ h_1 < z < h_1 + h,\ 0 < \theta \le 2\pi\}$ is the mean wall shape, $B_0 = \{(r, \theta, h_1) :\ 0 \le r < R_i(h_1+) = R_0, 0 < \theta \le 2\pi\}$ is the mean tank bottom, Q_0 is the mean liquid domain confined by Σ_0, B_0, and W_0, and n is the outer normal. The boundary conditions of (8) keep conserved the liquid volume

$$V_l = \pi \int_{h_1}^{h} R_i^2(z) dz = \text{const},$$

($M_l = \rho_l V_l$ is the liquid mass) provided by

$$\frac{1}{\pi r_0^2} \int_0^{r_0} \int_0^{2\pi} r\zeta(r, \theta, t)\, d\theta dr = h_1 + h + \eta_3(t). \quad (9)$$

This means that defining the free surface as $z = h_1 + h + \eta_3(t) + \zeta_0(r, \theta, t)$ extracts ζ_0 which is responsible for the surface waves (sloshing) on Σ_0.

When formulating (8), we have neglected the vertical pipes in the shaft, but, furthermore, we will follow Sect. 4.11.5 of [4] to estimate the natural sloshing frequencies correction due to the pipes.

We should note that the sloshing problem is normally formulated in the rigid-body fixed coordinate system [4] (implying the free-surface equation describes the relative motions) but the present problem prefers using the absolute inertial coordinate system $Oxyz$. This leads to the homogeneous kinematic condition (the first one in (8d)) while the analogous linear sloshing problem in Chap. 5 of the book [4] (in the tank-fixed coordinate system) contains extra inhomogeneous quantities. Both formulations are equivalent as discussed on page 198 of the book.

As long as we know φ_i from (8), one can compute the linear dynamic pressure $-\rho_l\dot{\varphi}_i$ acting on the shaft walls. The horizontal forces (per the vertical length) along the Ox and Oy axes read then as

$$\begin{aligned} f_{vi}(z,t) &= -\chi_{[h_1,h_1+h]}(z)\rho_l \int_0^{2\pi} R_i(z)\cos\theta\,\dot{\varphi}_i(R_i(z),\theta,z)\,d\theta, \\ f_{wi}(z,t) &= -\chi_{[h_1,h_1+h]}(z)\rho_l \int_0^{2\pi} R_i(z)\sin\theta\,\dot{\varphi}_i(R_i(z),\theta,z)\,d\theta, \end{aligned} \tag{10}$$

where $\chi_{[a,b]}(z)$ is the *Heaviside function* implying 1 on the interval $[a,b]$ and 0 otherwise. The dynamic pressure leads to the resulting vertical force always appearing as the inertia force of the contained liquid

$$F_{ui} = -M_l\,\ddot{\eta}_3, \tag{11}$$

where ρ_l=const is the liquid density. In the linear case, F_{ui} does not depend on sloshing as discussed in Ch. 5 of [4].

The dynamic pressure on the bottom at $z = h_1$ yields the hydrodynamic moment relative to the bottom centre. The scalar components of the moment around Ox and Oy axes are

$$\begin{aligned} P_x^{h_1} &= \rho_l \int_0^{R_0}\int_0^{2\pi} r^2\sin\theta\dot{\varphi}_i(r,\theta,h_1)d\theta dr, \\ P_y^{h_1} &= -\rho_l \int_0^{R_0}\int_0^{2\pi} r^2\cos\theta\dot{\varphi}_i(r,\theta,h_1)d\theta dr. \end{aligned} \tag{12}$$

In our forthcoming analysis, the velocity potential will be presented as a superposition of two components implying flows without the free-surface elevations ($\zeta_0 = 0$) and sloshing. The resulting vertical force (11) does not depend on the sloshing component, but expressions (10) and (12) fall into the corresponding sums

$$f_{vi} = f_{vi0} + f_{vis},\ f_{wi} = f_{wi0} + f_{wis};\ P_x^{h_1} = P_{x0}^{h_1} + P_{xs}^{h_1},\ P_y^{h_1} = P_{y0}^{h_1} + P_{ys}^{h_1}. \tag{13}$$

Finally, one should remember that the pressure has the following framed hydrostatic component

$$p_i = -\chi_{[h_1,h_1+h]}(z)\rho_l \left(\dot{\varphi}_i + \boxed{g(z-\eta_3-h-h_1)}\right). \tag{14}$$

which yields the quasi-static moment at each z associated with the liquid weight (per length). Following the paper [12], the corresponding quantity will be introduced in section 2.2.

2.1.5. External sea water problem

When φ_e is the absolute velocity potential and the function Z in the equation $z = d + Z(r,\theta,t)$ defines the water waves, the following linear boundary problem appears

$$\nabla^2 \varphi_e = 0 \quad \text{in} \quad Q, \tag{15a}$$

$$\frac{\partial \varphi_e}{\partial n} = (\cos\theta\,\dot{v} + \sin\theta\,\dot{w} - R_e'\dot{\eta}_3)\,/\sqrt{1+R_e'^2} \quad \text{on}$$
$$W = \{(R_e(z),\theta,z) :\ 0 < \theta \le 2\pi,\ 0 < z < l_2,\ l_2 < z < d\}, \tag{15b}$$

$$\frac{\partial \varphi_e}{\partial n} = \frac{\partial \varphi_e}{\partial z} = \dot{\eta}_3 - r\cos\theta\,\dot{v}' - r\sin\theta\,\dot{w}' \quad \text{on}$$
$$P = \{(r,\theta,l_2) :\ r_1 < r < r_2,\ 0 < \theta \le 2\pi\}, \tag{15c}$$

$$\frac{\partial \varphi_e}{\partial z} = \dot{E}_3(t)\, w_E(r,\theta) \quad \text{at} \quad z = d_3, \tag{15d}$$

$$\frac{\partial \varphi_e}{\partial z} = \dot{Z}, \quad \dot{\varphi}_e + gZ = 0 \quad \text{on} \quad \Sigma, \tag{15e}$$

$$\nabla(\varphi_e - \varphi_I) \to 0, \quad (\dot{\varphi}_e - \dot{\varphi}_I) \to 0, \quad (\varphi_e - \varphi_I) \to 0, \quad \text{as} \quad r \to \infty, \tag{15f}$$

where (15f) involves a given incident wave potential φ_I and $(\varphi_e - \varphi_I)$ is, by definition, the scattering velocity potential, n is the inner normal with respect to the water domain, $E_3(t)$ governs the vertical soil motions due to the Earthquake in a neightborhood of the tower, and $w_E(r,\theta)$ is the weight coefficient reflecting the fact that the vertical Earthquake motions vanish away from the monotower, i.e. $w_E(r,\theta)=1$ somewhere about the tower but $w_E(r,\theta)=0$ far from the tower.

An important note is that the incident wave (associated with φ_I) is, in the linear approximation, a superposition of progressive incoming waves

$$Z = Z_a \sin(\omega t - kr\sin(\theta-\theta_0)), \quad \varphi_I = \frac{gZ_a}{\omega}\frac{\cosh(kz)}{\cosh(kd)}\cos(\omega t - kr\sin(\theta-\theta_0)), \tag{16}$$

where $\omega^2 = gk\tanh(kd)$ and θ_0=const is the angle between the Oy axis and the wave direction.

As long as we know φ_e from (15), one can find the horizontal force (per the length) along the external beam walls

$$\begin{aligned} f_{ve}(z,t) &= \chi_{[0,d]}(z)\rho_w \int_0^{2\pi} [R_e(z)\cos\theta]\dot\varphi_e(R_e(z),\theta,z)\,d\theta, \\ f_{we}(z,t) &= \chi_{[0,d]}(z)\rho_w \int_0^{2\pi} [R_e(z)\sin\theta]\dot\varphi_e(R_e(z),\theta,z)\,d\theta, \end{aligned} \tag{17}$$

(one should note that the sign in (17) is opposite to that in (10) for the internal hydrodynamic loads) where ρ_w is the sea water density.

The vertical hydrodynamic force is as follows

$$F_{ue_1}(t) = -\rho_w \int_{l_2}^{d} R_e(z)R_e'(z)\int_0^{2\pi}\dot\varphi_e(R_e(z),\theta,z)\,d\theta dz + \\ + \rho_w \int_{r_1}^{r_2} r \int_0^{2\pi} \dot\varphi_e(r,\theta,l_2)\,d\theta dr. \tag{18}$$

In addition, as explained in the previous section for the sloshing problem, the hydrodynamic pressure acting on the annular earth floor roof P yields the moments at the level $z=l_2$

$$\begin{aligned} P_x^{l_2} &= \rho_w \int_{r_1}^{r_2}\int_0^{2\pi} r^2\sin\theta\dot\varphi_e(r,\theta,h_1)d\theta dr, \\ P_y^{l_2} &= -\rho_w \int_{r_1}^{r_2}\int_0^{2\pi} r^2\cos\theta\dot\varphi_e(r,\theta,h_1)d\theta dr. \end{aligned} \tag{19}$$

The boundary problem (15) is linear and contains three types of inhomogeneous terms caused by the Earthquake (the Neumann condition (15d) on the seabed), incoming waves (the non-zero incident wave potential φ_I), and the beam vibrations (the Neumann conditions (15b) and (15c)). This means that the hydrodynamic loads (17), (18), and (19) can formally be divided into the corresponding three components

$$\begin{aligned} & f_{ve} = f_{veE} + f_{veI} + f_{veB}, \quad f_{we} = f_{weE} + f_{weI} + f_{weB}, \\ & P_x^{l_2} = P_{xE}^{l_2} + P_{xI}^{l_2} + P_{xB}^{l_2}, \quad P_y^{l_2} = P_{yE}^{l_2} + P_{yI}^{l_2} + P_{yB}^{l_2}, \\ & F_{ue_1} = F_{ue_1E} + F_{ue_1I} + F_{ue_1B}. \end{aligned} \tag{20}$$

In our forthcoming analysis, we will also distinguish the two subcomponents for $f_{veB}, f_{weB}, P_{xB}^{l_2}, P_{xB}^{l_2}$, and F_{ue_1B} (loads caused by flows due to the beam vibrations):

$$\begin{aligned} & f_{veB} = f_{veB0} + f_{veBS}, \quad f_{weB} = f_{weB0} + f_{weBS}, \\ & P_{xB}^{l_2} = P_{xB0}^{l_2} + P_{xBS}^{l_2}, \quad P_{yB}^{l_2} = P_{yB0}^{l_2} + P_{yBS}^{l_2}, \\ & F_{ue_1B} = F_{ue_1B0} + F_{ue_1BS} \end{aligned} \tag{21}$$

responsible for flows without free-surface motions and those when the surface waves are not negligible.

Finally, the external water pressure contains the hydrostatic component, i.e.,

$$p_e = -\chi_{[0,d]}(z)\rho_w\left(\dot{\varphi}_e + g(z-d)\right). \tag{22}$$

The hydrostatic pressure effect causes an extra moment due to the beam deviations. This weight forces effect will be accounted for in section 2.2. The vertical Archimedes force contributes the restoring force

$$F_{ue_2} = -\pi\rho_w g(r_2^2 - r_3^2)\eta_3. \tag{23}$$

2.2. Horizontal beam vibrations

2.2.1. Variational formulation

Variational formulation of the beam problem can be based on the *virtual work principle*. According to this principle, the full variation of the beam potential energy

$$\delta W = \int_0^L EI\left(v''(\delta v)'' + w''(\delta w)''\right) dz, \tag{24}$$

where δv and δw are admissible variations of the beam deviations, should be equal to the sum of the virtual works caused by different-type loads applied to the beam, i.e.

$$\delta W = \sum_{i=1}^{N_f} \delta A_i. \tag{25}$$

The following loads ($N_f = 7$) should be accounted for:

(i) the inertia force and moment due to the beam vibrations;

(ii) the body-and-fluids weight moments due to the local beam inclination;

(iii) the force and moment associated with the inertia of the rigid platform;

(iv) the external force and moment applied to the rigid top platform;

(v) the force and moment due the bedrock (soil) feedback;

(vi) the horizontal hydrodynamic loads;

(vii) the hydrodynamic moments at $z = h_1$ (shaft bottom) and l_2 (the ring-shaped earth floor roof).

The virtual work due to inertial properties of the beam reads as

$$\delta A_1 = -\int_0^L \rho_b S_b (\ddot{v}\delta v + \ddot{w}\delta w) dz, \tag{26}$$

where $\rho_b(z)$ is the beam mass density and $S_b(z)$ is the cross-sectional area of the beam.

Following Volmir [12], one can define the virtual work due to the weight-caused moment along the beam axis

$$\begin{aligned} &\delta A_2 = \int_0^L N(z)(v'\delta v' + w'\delta w')dz = \int_0^L N(z)(v'\delta v' + w'\delta w')dz, \\ &N(z) = N_b(z) + N_i(z) + N_e(z), \end{aligned} \tag{27}$$

where $N(z)$ in the continuous function on $(0, L)$ except at $z = h_1$ and l_2,

$$N_b(z) = M_p g + \rho_b S_b(z) g(L - z) \tag{28}$$

expresses the top platform and, properly, the beam weight effect (the weight per the beam length), this term was included into analysis of the elevated water tank [6];

$$N_i(z) = \chi_{[h_1,h_1+h]}(z)\rho_l S_i(z) g(h_1 + h - z) \tag{29}$$

($S_i(z) = \pi R_i^2(z)$ is the area of the shaft cross-section) corresponds to the weight per length due to the contained liquid; and

$$N_e(z) = -\chi_{[0,d]}(z)\rho_w S_e(z) g(d - z) \tag{30}$$

($S_e(z) = \pi R_e^2(z)$ is the area of the external beam cross-section) implies the weight-related moment caused by the sea water.
N.B. Adding the virtual work δA_2 makes it possible to account for the hydrostatic pressure components in (14) and (22).

To get the virtual work δA_3 due to the inertia force and moment (with respect to the upper beam end, O_p) of the rigid platform, we introduce the six local degree of freedom of the rigid platform $\eta_1^p(t)$, $\eta_2^p(t), \eta_3^p(t), \eta_4^p(t), \eta_5^p(t)$, and $\eta_6^p(t) = 0$ so that

$$\eta_1^p = v(L,t),\ \eta_2^p = w(L,t),\ \eta_3^p = \eta_3,\ \eta_4^p = -w'(L,t),\ \eta_5^p = v'(L,t). \tag{31}$$

The inertia forces are

$$\begin{aligned} P_1 &= M_p\left(-\ddot{\eta}_1^p - z_p^C\ddot{\eta}_5^p\right) = -M_p\left(\ddot{v}(L,t) + z_p^C\ddot{v}'(L,t)\right)\\ P_2 &= M_p\left(-\ddot{\eta}_2^p + z_p^C\ddot{\eta}_4^p\right) = -M_p\left(\ddot{w}(L,t) + z_p^C\ddot{w}'(L,t)\right),\\ P_3 &= -M_p\ddot{\eta}_3, \end{aligned} \tag{32}$$

and the moments are

$$\begin{aligned} P_4 &= z_p^C M_p\left(\ddot{\eta}_2^p + g\eta_4^p\right) - I_{11}^p\ddot{\eta}_4^p - I_{12}^p\ddot{\eta}_5^p =\\ &= z_p^C M_p\left(\ddot{w}(L,t) - g w'(L,t)\right) + I_{11}^p\ddot{w}'(L,t) - I_{12}^p\ddot{v}'(L,t),\\ P_5 &= z_p^C M_p\left(-\ddot{\eta}_1^p + g\eta_5^p\right) - I_{12}^p\ddot{\eta}_4^p - I_{22}^p\ddot{\eta}_5^p =\\ &= z_p^C M_p\left(-\ddot{v}(L,t) + g v'(L,t)\right) + I_{12}^p\ddot{w}'(L,t) - I_{22}^p\ddot{v}'(L,t),\\ P_6 &= 0. \end{aligned} \tag{33}$$

Based on (32) and (33),

$$\delta A_3 = -M_p\left(\ddot{v}(L,t) + z_p^C\ddot{v}'(L,t)\right)\delta v(L,t) +$$

$$+\left(z_p^C M_p\left(-\ddot{v}(L,t)+gv'(L,t)\right)+I_{12}^p\ddot{w}'(L,t)-I_{22}^p\ddot{v}'(L,t)\right)\delta v'(L,t)-$$
$$-M_p\left(\ddot{w}(L,t)+z_p^C\ddot{w}'(L,t)\right)\delta w(L,t)+\left(z_p^C M_p\left(-\ddot{w}(L,t)+gw'(L,t)\right)-\right.$$
$$\left.-I_{11}^p\ddot{w}'(L,t)+I_{12}^p\ddot{v}'(L,t)\right)\delta w'(L,t). \quad (34)$$

The virtual work due to external loads to the platform is

$$\delta A_4=\mathcal{P}_1\delta v(L,t)+\mathcal{P}_2\delta w(L,t)-\mathcal{P}_4\delta w'(L,t)+\mathcal{P}_5\delta v'(L,t), \quad (35)$$

where $(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$ and $(\mathcal{P}_4,\mathcal{P}_5,\mathcal{P}_6)$ are the forces and moments (with respect O_p) be given *a priori*.

The virtual work due to the soil feedback, δA_5, is associated with the forces (4) and moments (7) applied at the lower beam end

$$\delta A_5=F_1\delta v(0,t)+F_2\delta w(0,t)+F_4\delta w'(0,t)-F_5\delta v'(0,t)=$$
$$=-(k_1(v(0,t)-E_1)+d_1(\dot{v}(0,t)-\dot{E}_1))\,\delta v(0,t)-$$
$$-(k_1(w(0,t)-E_2)+d_1(\dot{w}(0,t)-\dot{E}_2))\,\delta w(0,t)-$$
$$-\tfrac{1}{2}r_2^2(k_2w'(0,t)+d_2\dot{w}'(0,t))\,\delta w'(0,t)-$$
$$-\tfrac{1}{2}r_2^2(k_2v'(0,t)+d_2\dot{v}'(0,t))\,\delta v'(0,t). \quad (36)$$

The hydrodynamic pressure causes the horizontal hydrodynamic forces (per length) and the moments at $z=h_1$ (the shaft bottom) and l_2 (the annular earth floor roof). The horizontal hydrodynamic forces are assumed be explicitly known from (10) and (17). The virtual work due to these forces is

$$\delta A_6=\int_0^L((f_{vi}+f_{ve})\delta v+(f_{wi}+f_{we})\delta w)dz. \quad (37)$$

The virtual work due to the moments at $z=h_1$ and l_2 is defined by (12) and (19)

$$\delta A_7=P_x^{h_1}(-\delta w'(h_1))+P_y^{h_1}\delta v'(h_1)+P_x^{l_2}(-\delta w'(l_2))+P_y^{l_2}\delta v'(l_2)=$$
$$=-\rho_l\int_0^{R_0}\int_0^{2\pi}r^2\dot{\varphi}_i(r,\theta,h_1)[\sin\theta\delta w'(h_1)+\cos\theta\delta v'(h_1)]d\theta dr-$$
$$-\rho_w\int_{r_1}^{r_2}\int_0^{2\pi}r^2\dot{\varphi}_e(r,\theta,l_2)[\sin\theta\delta w'(l_2)+\cos\theta\delta v'(l_2)]d\theta dr. \quad (38)$$

When introducing the virtual works δA_6 and δA_7, we should recall that the horizontal hydrodynamic forces (per the vertical length) and moments at $z=h_1$ and l_2 can formally be divided into several components as discussed about (13), (20), and (21).

2.2.2. Differential formulation

Expressions for δW and δA_i suggest that functions v, w, v', and w' are continuous but the second derivatives may possess jumps at h_1 and l_2. This mean that all integrals in the variational formulation are defined in the usual sense. However, using the variational formulation for derivation of the corresponding differential problem leads, formally, to the fourth-order derivatives by z. This introduces the generalised derivatives. A way for avoiding them consists in considering the tower as the BBB–system with the kinematic transmission conditions (2).

Integrating by parts in (24) and (27) on intervals $(0, h_1)$, $k = 1$, (h_1, l_2), $k = 2$, and (l_2, L), $k = 3$, using calculus of variations in the virtual work principle (25) within $\delta A_i, i = 1, ..., N_f$ derives the governing equations

$$(EIv_k'')'' + \rho_b S_b \ddot{v}_k + (Nv_k')' = f_{vi} + f_{ve}, \tag{39a}$$

$$(EIw_k'')'' + \rho_b S_b \ddot{w}_k + (Nw_k')' = f_{wi} + f_{we}, \tag{39b}$$

for the *horizontal beams vibrations* and the following natural boundary conditions at the upper beam end

$$-(EIv_3'')' + M_p \left(\ddot{v}_3 + z_p^C \, \ddot{v}_3'\right) - Nv_3' = \mathcal{P}_1, \tag{40a}$$

$$-(EIw_3'')' + M_p \left(\ddot{w}_3 + z_p^C \, \ddot{w}_3'\right) - Nw_3' = \mathcal{P}_2. \tag{40b}$$

$$EIw_3'' + z_p^C \, M_p \left(\ddot{w}_3 - gw_3'\right) + I_{11}^p \ddot{w}_3' - I_{12}^p \ddot{v}_3' = -\mathcal{P}_4, \tag{40c}$$

$$EIv_3'' + z_p^C \, M_p \left(\ddot{v}_3 - gv_3'\right) - I_{12}^p \ddot{w}_3' + I_{22}^p \ddot{v}_3' = \mathcal{P}_5 \tag{40d}$$

at $z = L$, and

$$(EIv_1'')' + Nv_1' + k_1(v_1 - E_1) + d_1(\dot{v}_1 - \dot{E}_1) = 0, \tag{41a}$$

$$(EIw_1'')' + Nw_1' + k_1(w_1 - E_2) + d_1(\dot{w}_1 - \dot{E}_2) = 0, \tag{41b}$$

$$-EIw_1'' + \tfrac{1}{4} r_2^2 (k_2 w_1' + d_2 \dot{w}_1') = 0, \tag{41c}$$

$$-EIv_1'' + \tfrac{1}{4} r_2^2 (k_2 v_1' + d_2 \dot{v}_1') = 0 \tag{41d}$$

at $z = 0$ (the lower beam end).

When working with the trial functions at $z = h_1$ and l_2 and accounting for the kinematic transmission conditions (2) deduces the following natural (dynamic) transmission conditions

$$\begin{aligned}(EIv_2'')|_{h_1+} - (EIv_1'')|_{h_1-} &= \rho_l \int_0^{R_0}\int_0^{2\pi} r^2\cos\theta\dot{\varphi}_i(r,\theta,h_1)d\theta dr,\\ (EIw_2'')|_{h_1+} - (EIw_1'')|_{h_1-} &= \rho_l \int_0^{R_0}\int_0^{2\pi} r^2\sin\theta\dot{\varphi}_i(r,\theta,h_1)d\theta dr,\end{aligned} \tag{42a}$$

$$\begin{aligned}(EIv_3'')|_{l_2+} - (EIv_2'')|_{l_2-} &= \rho_w \int_{r_1}^{r_2}\int_0^{2\pi} r^2\cos\theta\dot{\varphi}_e(r,\theta,l_2)d\theta dr,\\ (EIw_3'')|_{l_2+} - (EIw_2'')|_{l_2-} &= \rho_w \int_{r_1}^{r_2}\int_0^{2\pi} r^2\sin\theta\dot{\varphi}_e(r,\theta,l_2)d\theta dr\end{aligned} \tag{42b}$$

implying a jump for the bending moment at $z = h_1$ and $z = l_2$, respectively, due to the hydrodynamic loads as well as the transmission conditions for the shear forces

$$(EIv_2'')'|_{h_1+} = (EIv_1'')'|_{h_1-}\,, \quad (EIw_2'')'|_{h_1+} = (EIw_1'')'|_{h_1-}\,, \tag{43a}$$

$$(EIv_3'')'|_{l_2+} = (EIv_2'')'|_{l_2-}\,, \quad (EIw_3'')'|_{l_2+} = (EIw_2'')'|_{l_2-}\,. \tag{43b}$$

Mathematically, (42) and (43) confirm that the second- and third-order derivatives by z are not continuous functions at $z = h_1$ and l_2.

2.3. Governing equation for the vertical motions

The boundary value problem (39), (40), (41) does not govern the vertical motions of the beam as a rigid body (lifting). These vertical motions are described by the generalised coordinate $\eta_3(t)$ which is a solution of the ordinary differential equation

$$\begin{aligned}(M_p + M_b + M_l)\,\ddot{\eta}_3 + d_2(\dot{\eta}_3 - \dot{E}_3) + k_2(\eta_3 - E_3) &=\\ = \underbrace{-\pi\rho_w g(r_1^2 - r_3^2)\eta_3}_{F_{ue_2}} + F_{ue_1}\,,\end{aligned} \tag{44}$$

where M_l is the contained liquid mass, M_b is the total beam mass, and F_{ue_1} is defined by (18).

3. Eigenoscillations without the surface-wave effect

Assuming the zero-damping (due to soil and wave radiation) and external excitations as well as suggesting that there is no the surface-wave effect, $\zeta_0 = Z \equiv 0$ in the free-surface equations, lead to the following four things: [i] the damping coefficients d_1 and d_2 are zeros, [ii] the force and moment loads to the platform are zeros ($\mathcal{P}_i(t) = 0$, $i = 1, \ldots, 6$), [iii] the incident wave potential is absent, $\varphi_I = 0$ in (15), and [iv] the internal (8d) and external (15e) kinematic surface-wave conditions replace to

$$\frac{\partial \varphi_i}{\partial z} = \dot{\eta}_3(t) \quad \text{on} \quad \Sigma_0 \qquad \text{and} \qquad \frac{\partial \varphi_e}{\partial z} = 0 \quad \text{on} \quad \Sigma, \tag{45}$$

respectively. The first condition on Σ_0 means that the frozen (flat) free surface moves vertically together with the structure. Furthermore, the hydrodynamic forces and moments at $z = h_1$ and l_2 contain only the components $f_{vi0}, f_{wi0}, f_{veB0}, f_{weB0}$ and $P_{x0}^{h_1}, P_{y0}^{h_1}, P_{xB0}^{l_2}, P_{yB0}^{l_2}$, and F_{ue_1B0} introduced in (13), (20), and (21).

Pursuing an eigenvalue problem introduces the harmonic solution

$$v_k(z,t) = \exp(\mathrm{i}\sigma t)V_k(z), \quad w_k(z,t) = \exp(\mathrm{i}\sigma t)W_k(z),$$
$$\eta_3(t) = \exp(\mathrm{i}\sigma t)\eta_0, \quad k = 1,2,3, \tag{46a}$$

$$\varphi_i(r,\theta,z,t) = \mathrm{i}\sigma \exp(\mathrm{i}\sigma t)\left[\eta_0 z + \Omega_V(r,z)\cos\theta + \Omega_W(r,z)\sin\theta\right], \tag{46b}$$

$$\varphi_e(r,\theta,z,t) = \mathrm{i}\sigma \exp(\mathrm{i}\sigma t)\left(\eta_0\omega_\eta(r,z) + \right.$$
$$\left. + \omega_V(r,z)\cos\theta + \omega_W(r,z)\sin\theta\right), \tag{46c}$$

where $\mathrm{i}^2 = -1$. Here, we used that the velocity potentials fall into the sum of axisymmetric (affecting the lifting), $\cos\theta$ (v-type), and $\sin\theta$ (w-type) components.

3.1. Spectral problem

3.1.1. Differential formulation

Substituting (46) into the governing equations and boundary conditions from section 2.2.2 and accounting for the replacement (45) derive the spectral boundary problem (σ^2 is the spectral parameter) consisting of the governing equations

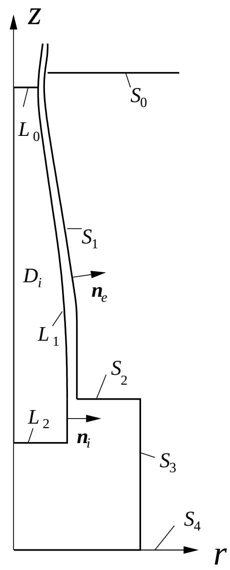

Fig 3. Meridional cross-section of the tower and adopted notations for the associated hydrodynamic problems.

$$(EIV_k'')'' + (NV_k')' = \sigma^2 \left[\chi_{[h_1,h_1+h]}(z)\pi\rho_l R_i(z)\Omega_v(R_i(z),z)- \right. \\ \left. -\chi_{[0,d]}(z)\pi\rho_w R_e(z)\omega_v(R_e(z),z) + \rho_b(z)S_b(z)V_k\right], \quad (47a)$$

$$(EIW_k'')'' + (NW_k')' = \sigma^2 \left[\chi_{[h_1,h_1+h]}(z)\pi\rho_l R_i(z)\Omega_w(R_i(z),z)- \right. \\ \left. -\chi_{[0,d]}(z)\pi\rho_w R_e(z)\omega_w(R_e(z),z) + \rho_b(z)S_b(z)W_k\right] \quad (47b)$$

on the introduced intervals $(0, h_1)$, (h_1, l_2), and (l_1, L) $(k = 1, 2, 3)$. The equations contain the functions $\Omega_V(r, x), \Omega_W(r, z), \omega_V(r, z)$, and $\omega_W(r, z)$ which are solutions of the Neumann boundary value problems

$$r^2\left(\frac{\partial^2\Omega_F}{\partial r^2} + \frac{\partial^2\Omega_F}{\partial z^2}\right) + r\frac{\partial\Omega_F}{\partial r} - \Omega_F = 0 \text{ in } D_i, \\ \frac{\partial\Omega_F}{\partial n_i} = \frac{\partial\Omega_F}{\partial z} = 0 \text{ on } L_0, \quad \frac{\partial\Omega_F}{\partial n_i} = -\frac{\partial\Omega_F}{\partial z} = rF'(h_1) \text{ on } B_0, \\ \frac{\partial\Omega_F}{\partial n_i} = \frac{F(z)}{\sqrt{1+R_i'^2(z)}} \text{ on } W_0 \quad (48)$$

and

$$
\begin{aligned}
& r^2\left(\frac{\partial^2\omega_F}{\partial r^2}+\frac{\partial^2\omega_F}{\partial z^2}\right)+r\frac{\partial\omega_F}{\partial r}-\omega_F=0 \ \text{ in } \ D_e,\\
& \frac{\partial\omega_F}{\partial z}=0 \text{ on } \ S_0\cup S_4; \ \ \frac{\partial\omega_F}{\partial r}=0 \text{ on } \ S_3,\\
& \frac{\partial\omega_F}{\partial n_e}=\frac{\partial\omega_F}{\partial z}=-rF'(l_2) \text{ on } \ S_2, \ \frac{\partial\omega_F}{\partial n_e}=\frac{F(z)}{\sqrt{1+R_e'^2(z)}} \ \text{ on } \ S_1,\\
& \omega_F, \ \ \frac{\partial\omega_F}{\partial r}\to 0 \ \text{ as } \ r\to\infty
\end{aligned}
\tag{49}
$$

with notations in Fig. 3. The problem is formulated in the meridional cross-section.

The equations (47) need the free-end boundary conditions

$$
(EIV_1'')'+NV_1'+k_1V_1=0, \quad -EIV_1''+\tfrac{1}{4}r_2^2k_2V_1'=0, \tag{50a}
$$

$$
(EIW_1'')'+NW_1'+k_1W_1=0, \quad -EIW_1''+\tfrac{1}{4}r_2^2k_2W_1'=0 \tag{50b}
$$

at $z=0$ and

$$
\begin{aligned}
& -(EIV_3'')'-NV_3'=\sigma^2M_p(V_3+z_p^CV_3'),\\
& EIV_3''-z_p^CgM_pV_3'=\sigma^2\left(z_p^CM_pV_3+I_{22}^pV_3'-\underline{I_{12}^pW_3'}\right),
\end{aligned}
\tag{51a}
$$

$$
\begin{aligned}
& -(EIW_3'')'-NW_3'=\sigma^2M_p(W_3+z_p^CW_3'),\\
& EIW_3''-z_p^CgM_pW_3'=\sigma^2\left(z_p^CM_pW_3+I_{11}^pW_3'-\underline{I_{12}^pV_3'}\right),
\end{aligned}
\tag{51b}
$$

at $z=L$.

The transmission conditions take the form

$$
\begin{aligned}
& V_1(h_1)=V_2(h_1), \ W_1(h_1)=W_2(h_1),\\
& V_1'(h_1)=V_2'(h_1), \ W_1'(h_1)=W_2'(h_1),\\
& V_2(l_2)=V_3(l_2), \ W_2(l_2)=W_3(l_2),\\
& V_2'(l_2)=V_3'(l_2), \ W_2'(l_2)=W_3'(l_2);
\end{aligned}
\tag{52a}
$$

$$
\begin{aligned}
&(EIV_1'')|_{h_1-} - (EIV_2'')|_{h_1+} = \sigma^2 \pi \rho_l \int_0^{R_0} r^2 \Omega_V(r, h_1) dr,\\
&(EIW_1'')|_{h_1-} - (EIW_2'')|_{h_1+} = \sigma^2 \pi \rho_l \int_0^{R_0} r^2 \Omega_W(r, h_1) dr,\\
&(EIV_2'')|_{l_2+} - (EIV_3'')|_{l_2-} = \sigma^2 \pi \rho_w \int_{r_1}^{r_2} r^2 \omega_V(r, l_2) dr,\\
&(EIW_2'')|_{l_2+} - (EIW_3'')|_{l_2-} = \sigma^2 \pi \rho_w \int_{r_1}^{r_2} r^2 \omega_W(r, l_2) dr;
\end{aligned}
\tag{52b}
$$

$$
\begin{aligned}
&(EIV_2'')'|_{h_1+} = (EIV_1'')'|_{h_1-}\,, \quad (EIV_2'')'|_{h_1+} = (EIW_1'')'|_{h_1-}\,,\\
&(EIV_3'')'|_{l_2+} = (EIV_2'')'|_{l_2-}\,, \quad (EIW_3'')'|_{l_2+} = (EIW_2'')'|_{l_2-}\,.
\end{aligned}
\tag{52c}
$$

Specifically, equations (47) and boundary/transmission conditions (51) and (52b) contain the spectral parameter σ^2. The spectral boundary problem (47)–(52) becomes decoupled with respect to V and W (as well as Ω_V, ω_V and Ω_W, ω_W) as $I_{12}^p = 0$ (the underlined terms vanish).

3.1.2. Variational formulation

The spectral boundary problem admits the variational formulation following from that in section 2.2.1. We switch from $v_k(z,t), w_k(z,t), k = 1, 2, 3$ in definition (46a) to the solution defined on the entire interval $v(z,t), w(z,t)$, $0 < z < L$ and introduce

$$
v(z,t) = \exp(\mathrm{i}\sigma t)V(z), \quad w(z,t) = \exp(\mathrm{i}\sigma t)W(z), \quad 0 < z < L, \tag{53}
$$

where the functions $V(z)$ and $W(z)$ are continuous together with the first derivative. Using the variational equation implies finding the eigensolution of the spectral type variational equation

$$
\begin{aligned}
&\int_0^L (EI[V''\delta V'' + W''\delta W''] + N[V'\delta V' + W'\delta W'])dz -\\
&\qquad - z_p^C M_p g[V'(L)\delta V'(L) + W'(L)\delta W'(L)] +\\
&+ k_1[V(0)\delta V(0) + W(0)\delta W(0)] + \tfrac{1}{2} k_2 r_2^2 [V'(0)\delta V'(0) + W'(0)\delta W'(0)] =\\
&= \sigma^2 \Bigg[\int_0^L \rho_b S_b (V\delta V + W\delta W) dz + M_p[V(L)\delta V(L) + W(L)\delta W(L)] +\\
&+ M_p z_p^C [V'(L)\delta V(L) + W'(L)\delta W(L) + V(L)\delta V'(L) + W(L)\delta W'(L)] +\\
&\quad + [I_{22}^p V'(L) - I_{12}^p W'(L)]\delta V'(L) + [I_{11}^p W'(L) - I_{12}^p V'(L)]\delta W'(L) +
\end{aligned}
$$

$$+\pi\rho_l\left(\int_{h_1}^{h_1+h} R_i(\Omega_V(R_i,z)\delta V+\Omega_W(R_i,z)\delta W)dz+\right.$$
$$\left.+\int_0^{R_0} r^2(\Omega_V(r,h_1)\delta V'(h_1)+\Omega_W(r,h_1)\delta W'(h_1))dr\right)+$$
$$+\pi\rho_w\left(-\int_0^d R_e(\omega_V(R_e,z)\delta V+\omega_W(R_e,z)\delta W)dz+\right.$$
$$\left.+\int_{r_1}^{r_2} r^2(\omega_V(r,l_2)\delta V'(l_2)+\omega_W(r,l_2)\delta W'(l_2))dr\right) \quad (54)$$

for arbitrary continuous (together with the first derivative) trial functions δV and δW and the constraints (48) and (49) are satisfied.

3.2. Eigenfrequency for lifting

Utilising the harmonic solution (46) in the equation (44) without damping and forcing terms computes the eigenfrequency σ found from the relation

$$\sigma^2\left(M_p+M_b+M_l+2\pi\left[\int_{r_1}^{r_2} r\omega_\eta(r,l_2)dr+\right.\right.$$
$$\left.\left.-\int_{l_2}^{d} R_e(z)R_e'(z)\omega_\eta(R_e(z),z)dz\right]\right)=k_2+\rho_w g\pi(r_1^2-r_2^2), \quad (55)$$

where ω_η is an axisymmetric solution of the Neumann boundary value problem in the meridional cross-section D_e (see, Fig. 3)

$$\begin{aligned}
&\frac{\partial^2\omega_\eta}{\partial r^2}+\frac{1}{r}\frac{\partial^2\omega_\eta}{\partial r}+\frac{\partial^2\omega_\eta}{\partial z^2}=0 \ \text{ in } \ D_e,\\
&\frac{\partial\omega_\eta}{\partial n_e}=0 \ \text{ on } \ S_0\cup S_4\cup S_3,\\
&\frac{\partial\omega_\eta}{\partial n_e}=-\frac{R_e'(z)}{\sqrt{1+R_e'^2(z)}} \ \text{ on } \ S_1, \quad \frac{\partial\omega_\eta}{\partial n_e}=1 \ \text{ on } \ S_2,\\
&\omega_\eta,\ \frac{\partial\omega_\eta}{\partial r}\to 0 \ \text{ as } \ r\to\infty.
\end{aligned} \quad (56)$$

3.3. Solution method

3.3.1. Global Galerkin variational scheme

Even though the transmission problem from section 3.1.1 looks quite difficult in view of constructing its approximate solution, it may seriously simplify when employing the variational statement from section 3.1.2. The Galerkin scheme can be adopted assuming that we have an appropriate functional basis $\xi_k(z)$, $k = 1, \ldots$ which is continuous together with the first-order derivative on $[0, L]$. The approximate solution takes then the form

$$V(z) = \sum_{k=1}^{Q_b} a_k \xi_k(z), \quad W(z) = \sum_{k=1}^{Q_b} a_{k+M} \xi_k(z), \quad 0 < z < L, \tag{57}$$

and the trial functions δV and δW are chosen from the same functional set $\xi_k(z)$, $k = 1, \ldots, Q_b$.

Substituting the Galerkin solution (57) into the variational equation (54) leads to the generalised spectral matrix problem $(A - \sigma^2 B)\boldsymbol{a} = 0$ where the symmetric matrices A and B have the structure

$$A = \left\| \begin{matrix} M_A & 0 \\ 0 & M_A \end{matrix} \right\|, \quad B = \left\| \begin{matrix} M_{B_2} & M_b \\ M_b & M_{B_1} \end{matrix} \right\|, \tag{58}$$

with elements $M_A = \{m^A_{mn}\}$, $M_{B_l} = \{m^{B_l}_{mn}\}$, and $M_b = \{m^b_{mn}\}$ computed by the formulas

$$m^A_{mn} = \int_0^L (E(z)I(z)\xi''_n(z)\xi''_m(z) + N(z)\xi'_n(z)\xi'_m(z))dz - \\ - z^C_p M_p g \xi'_n(L)\xi'_m(L) + k_1 \xi_n(0)\xi_m(0) + \tfrac{1}{2} k_2 r_2^2 \xi'_n(0)\xi'_m(0), \tag{59a}$$

$$m^{B_l}_{mn} = \int_0^L \rho_b(z) S_b(z) \xi_n(z)\xi_m(z) dz + M_p \xi_n(L)\xi_m(L) + \\ + \pi\rho_l \left(\int_{h_1}^{h+h_1} R_i(z)\Omega_{\xi_n}(R_i(z), z)\xi_m(z)dz + + \int_0^{R_0} r^2 \Omega_{\xi_n}(r, h_1)\xi'_m(h_1)dr \right) + \\ + \pi\rho_w \left(- \int_0^d R_e(z)\omega_{\xi_n}(R_e(z), z)\xi_m(z)dz + \int_{r_1}^{r_2} r^2 \omega_{\xi_n}(r, l_2)\xi'_m(l_2)dr \right) + \\ + M_p z^C_p (\xi'_n(L)\xi_m(L) + \xi_n(L)\xi'_m(L)) + I^p_{ll}\xi'_n(L)\xi'_m(L), \tag{59b}$$

$$m^b_{mn} = -I^p_{12}\xi'_n(L)\xi'_m(L), \quad n, m = 1, \dots, Q_b. \tag{59c}$$

Here, the functions Ω_{ξ_n} and ω_{ξ_n} are solutions of the Neumann boundary problems (48) and (49) (after substituting $\xi_n \to F$). Using the Green identity shows that the integral terms involving Ω_{ξ_n} and ω_{ξ_n} are symmetric by the indexes n and m.

3.3.2. The Trefftz solution of the hydrodynamic boundary problems (48), (49), and (56)

Solution of (48). For the upright circular cylindrical domain, the solution can be found by Joukowski's method (see, p. 236 in [4]). For arbitrary $R_i(z)$, the method yields the Trefftz approximation

$$\Omega_{\xi_j}(r,z) = \frac{\xi'_j(h_1)}{2h}\left[(z-h-h_1)^2 r - \tfrac{1}{4}r^3\right] + \\ + \sum_{k=0}^{q_1} b_k^{(\xi_j)} \underbrace{\frac{I_1(s_k r)}{I_1(s_k R_0)}\cos(s_k(z-h_1))}_{\phi_k(r,z)}, \quad s_k = \frac{\pi k}{h-h_1}, \tag{60}$$

where $\phi_0 = r/R_0$ and $I_1(\cdot)$ is the modified Bessel function of the first kind. The approximation satisfies the governing equation and the Neumann boundary conditions on B_0 and L_0, but the boundary condition on L_1 should be approximated by choosing appropriate $b_k^{(\xi_j)}$. The standard variational scheme leads to the matrix problem $M_\phi \boldsymbol{b}^{(\xi_j)} = \boldsymbol{d}^{(\xi_j)}$, where $\boldsymbol{b}^{(\xi_j)} = (b_0^{(\xi_j)}, ..., b_{q_1}^{(\xi_j)})$, $M_\phi = \{m^\phi_{nk}\}$, and $\boldsymbol{d}^{(\xi_j)} = \{d_n^{(\xi_j)}\}$ so that

$$m^\phi_{nk} = \int_{h_1}^{h} R_i(z)\left[\frac{\partial \phi_k}{\partial r} - R'{}_i(z)\frac{\partial \phi_k}{\partial z}\right]_{r=R_i(z)} \phi_n(R_i(z), z)\, dz, \tag{61a}$$

$$d_n^{(\xi_j)} = \Bigg\{\int_{h_1}^{h} R_i(z)\Big[\xi_j(z) - \frac{\xi'_j(h_1)}{2h}((z-h-h_1)^2 - \\ - \tfrac{3}{4}R_i^2(z) - R'_i(z)(z-h-h_1)R_i(z))\Big]\phi_n(R_i(z),z)\, dz\Bigg\}. \tag{61b}$$

Solution of (49). Analogously to (60), the Trefftz approximate solution satisfying the zero-Neumann boundary conditions on S_0 and S_4 as well as the condition at the infinity possesses the form

$$\omega_{\xi_j}(r,z) = \sum_{k=0}^{q_2} c_k^{(\xi_j)} \underbrace{\frac{K_1(t_k r)}{K_1(t_k r_3)} \cos(t_k z)}_{\phi_k^{(1)}(r,z)}, \quad t_k = \frac{\pi k}{d}, \tag{62}$$

where $K_1(\cdot)$ is the modified Bessel function of the second kind and $\phi_0^{(1)} = r_3/r$. To find the unknown coefficients $c_k^{(\xi_j)}$, $k = 0, ..., q_2$, we arrive at the matrix problem $M_{\phi_1} \boldsymbol{c}^{(\xi_j)} = \boldsymbol{e}^{(\xi_j)}$ with the elements

$$m_{nk}^{\phi_1} = \int_0^d R_e(z) \left[\frac{\partial \phi_k^{(1)}}{\partial r} - R_e' \frac{\partial \phi_k^{(1)}}{\partial z} \right]_{r=R_e(z)} \phi_n^{(1)}(R_e(z), z) dz + \\ + \int_{r_1}^{r_2} r \frac{\partial \phi_k^{(1)}}{\partial z}(r, l_2) \phi_n^{(1)}(r,z) dr, \tag{63a}$$

$$e_n^{(\xi_j)} = \int_0^d R_e(z) \xi_j(z) \phi_n^{(1)}(R_e(z), z) dz - \int_{r_1}^{r_2} r^2 \xi_j(l_2) \phi_n^{(1)}(r, l_2) dr. \tag{63b}$$

Solution of (56). The Trefftz approximation takes the form

$$\omega_\eta(r,z) = \sum_{k=1}^{q_3} f_k \underbrace{\frac{K_0(t_k r)}{K_0(t_k r_3)} \cos(t_k z)}_{\phi_k^{(2)}(r,z)}, \quad t_k = \frac{\pi k}{d}, \tag{64}$$

where the corresponding matrix problem $M_{\phi_2} \boldsymbol{f} = \boldsymbol{p}$ is characterised by the coefficients

$$m_{nk}^{\phi_2} = \int_0^d R_e(z) \left[\frac{\partial \phi_k^{(2)}}{\partial r} - R_e' \frac{\partial \phi_k^{(2)}}{\partial z} \right]_{r=R_e(z)} \phi_n^{(2)}(R_e(z), z) dz + \\ + \int_{r_1}^{r_2} r \frac{\partial \phi_k^{(2)}}{\partial z}(r, l_2) \phi_n^{(2)}(r,z) dr, \tag{65a}$$

$$p_n = -\int_0^d R_e(z) R_e'(z) \phi_n^{(2)}(R_e(z), z) dz + \int_{r_1}^{r_2} r \phi_n^{(2)}(r, l_2) dr. \tag{65b}$$

3.4. The eigensolution

The eigensolution consists of the degenerate (double multiplicity) positive eigenvalues $\sigma_m^2, m \geq 1$. The degenerate eigenfunctions (having the same eigenvalues) are associated with the pairs $(\xi_m^{(1)}(z), \xi_m^{(2)}(z))$ and $(\xi_m^{(2)}(z), \xi_m^{(1)}(z))$ for the beam-related eigenfunctions $(V(z), W(z))$, $0 < z < L$. The Stokes–Joukowski potentials $\Omega_{\xi_m^{(1)}}$ and $\Omega_{\xi_m^{(2)}}$ as well as $\omega_{\xi_m^{(1)}}$ and $\omega_{\xi_m^{(2)}}$ are found from the boundary value problems (48) and (49), respectively. If $I_{12}^p = 0$, the component with the upper index (2) disappears.

Employing the Galerkin method from section 3.3.1 gives the approximate eigenvalues and modes. As usually for variational methods, the best approximation is expected for the lower eigenvalues and modes. Furthermore, the tower mass is much higher of the top platform mass and $|I_{12}^p| \lesssim I_{11}^p \sim I_{22}^p$. This means that $||\xi_m^{(2)}|| \lesssim ||\xi_m^{(1)}||$ and we will take $\xi_m(z) = \xi_m^{(1)}(z)$, $m = 1, \ldots, q_b$ as a functional basis in the forthcoming analysis with $q_s < Q_s$.

4. Multimodal method for sloshing in the beam shaft

When constructing the multimodal solution of the sloshing problem (8) from section 2.1.4, we noted that the forcing is exclusively associated with the beam vibrations. These vibrations can be presented in terms of the generalised coordinates $a_{vk}(t), a_{wk}(t)$ as follows

$$v(z,t) = \sum_{k=1}^{q_b} a_{vk}(t)\, \xi_k^{(1)}(z), \quad w(z,t) = \sum_{k=1}^{q_b} a_{wk}(t)\, \xi_k^{(1)}(z), \qquad (66)$$

where, as we remarked in section 3.4, the functional basis is associated with the $\xi_k^{(1)}(z)$-component of the eigensolution. Developing the multimodal method for sloshing in the beam shaft suggests the generalised coordinates are the known input functions.

4.1. Natural sloshing modes and frequencies

4.1.1. The modes of the beam type

There are the two sets of the natural sloshing modes for an axisymmetric rigid tank which take the form $\cos(m\theta)F_{mk}(r, z)$ and

$\sin(m\theta)F_{mk}(r,z)$, $m \geq 0$, $k \geq 1$. However, only the beam-type modes, $\cos\theta\Phi_k$ and $\sin\theta\Phi_k$, corresponding to $m=1$ can be *linearly* excited [5] for the rigid tank. The same is true for (8) as it follows from analysing the inhomogeneous (input) terms.

The eigenfunctions Φ_i are solutions of the following spectral boundary problem

$$r^2\left(\frac{\partial^2\Phi_k}{\partial r^2}+\frac{\partial^2\Phi_k}{\partial z^2}\right)+r\frac{\partial\Phi_k}{\partial r}-\Phi_k=0 \quad \text{in} \quad D_0, \tag{67a}$$

$$\frac{\partial\Phi_k}{\partial n_i}=\left(\frac{\partial\Phi_k}{\partial r}-R_i'\frac{\partial\Phi_k}{\partial z}\right)/\sqrt{1+R_i'^2}=0 \quad \text{on} \quad L_1,$$

$$\frac{\partial\Phi_k}{\partial n_i}=-\frac{\partial\Phi_k}{\partial z}=0 \quad \text{on} \quad L_2, \tag{67b}$$

$$\frac{\partial\Phi_k}{\partial z}=\kappa_k\Phi_k \quad \text{on} \quad L_0 \tag{67c}$$

formulated in the meridional cross-section (see Fig. 3).

The spectral problem (67) has the positive eigenvalues $\kappa_k, k \geq 1$. The eigenfunctions (found within to a multiplier) are orthogonal, i.e.

$$f_k(r)=\Phi_k(r,h_1+h), \quad \int_0^{r_0} rf_kf_n\,dr=\delta_{kn}\int_0^{r_0} rf_k^2\,dr=\delta_{kn}||f_k||^2, \tag{68}$$

where δ_{kn} is the Kronecker delta. The natural sloshing frequencies are

$$\sigma_k=\sqrt{g\kappa_k}. \tag{69}$$

4.1.2. Approximate modes and frequencies

The considered tower shafts are thin and long (high) with approximately upright walls in the vicinity of the mean free surface. Their non-cylindrical part is geometrically close to an axisymmetric conical shape. The natural sloshing frequencies and modes in the truncated conical tanks are analysed in [5]. When the semi-apex angle is small, these frequencies and modes are well approximated by those taken for the corresponding upright circular cylindrical tank. This means that the approximate analytical eigenfunctions Φ_k and eigenvalues κ_k of (67) can be written down as

$$\Phi_n(r,z) = \frac{J_1(K_n r)}{J_1(K_n r_0)} \frac{\cosh(K_n(z-h_1))}{\cosh(K_n(h-h_1))}, \quad J_1'(\underbrace{K_n r_0}_{\alpha_n}) = 0,$$
$$\kappa_n = K_n \tanh(K_n(h-h_1)), \quad ||f_n||^2 = r_0^2 \frac{\alpha_n^2 - 1}{2\alpha_n^2}, \tag{70}$$

and, as a consequence

$$\sigma_n^2 \approx g\kappa_n = gK_n \tanh(K_n(h-h_1)) \approx gK_n \tag{71}$$

accounting for the deep water approximation ($3r_0 \lesssim (h-h_1)$).

4.1.3. Effect of vertical pipes

The natural sloshing frequencies and modes slightly modify due to vertical pipes of small radii $r_{0j} \ll r_0$, $j = 1, \ldots, Pi$ (Pi is the number of these vertical pipes) installed in the shaft. An asymptotic estimate of the modification is given in Sect. 4.11.5 by [4] assuming the pipes are situated far from each others (there is no the proximity effect). The maximum decrease of the natural sloshing frequencies is estimated as

$${\sigma'}_n^2 \approx gK_n \left[1 - 2\sum_{j=1}^{Pi} \frac{J_1^2(K_n r_{0j})}{(\alpha_n)^2 - 1} \right]. \tag{72}$$

This implies that the squares of the actual natural sloshing frequencies should be somewhere between (72) and (71), namely,

$$gK_n \left[1 - 2\sum_{j=1}^{Pi} \frac{J_1^2(K_n r_{0j})}{(\alpha_n)^2 - 1} \right] \le \sigma_n^2 \le gK_n. \tag{73}$$

The vertical pipes should also correct the natural sloshing modes. An rough estimate of this correction is given in Sect. 4.11.5 of [4]. The present paper assumes that the modes remain approximately the same as for the clean tank.

4.2. The modal solution and equations

When employing the natural sloshing modes as a Fourier basis and accounting for (9), the free-surface elevations in (8) can be approximated by the multimodal solution

$$\zeta = h + h_1 + \eta_3(t) + \sum_{j=1}^{q_s} f_j(r)(\beta_{vj}(t)\cos\theta + \beta_{wj}(t)\sin\theta), \tag{74}$$

where $\beta_{vj}(t)$ and $\beta_{wj}(t)$ play the role of the sloshing-related generalised coordinates. The multimodal solution for the velocity potential takes the form

$$\varphi_i(r,\theta,z,t) = \dot\eta_3(t)(z-h-h_1) + \sum_{k=1}^{q_b} \Omega_{\xi_k^{(1)}}(r,z)(\cos\theta\,\dot a_{vk}(t) + \sin\theta\,\dot a_{wk}(t)) + \\ + \sum_{j=1}^{q_s} \Phi_j(r,z)(R_{vj}(t)\cos\theta + R_{wj}(t)\sin\theta), \tag{75}$$

where $\Omega^{(1)}_{\xi_k}$ is defined by (48) and treated as the Stokes–Joukowski potentials for the considered sloshing problem. The time-dependent functions $R_{vj}(t)$ and $R_{wk}(t)$ are governed by

$$R_{vj} = \kappa_j^{-1}\dot\beta_{vj}, \quad R_{wj} = \kappa_j^{-1}\dot\beta_{wj}, \quad j = 1,...,q_s, \tag{76}$$

due to the kinematic boundary condition of (8d).

Substituting (74) and (75) into (8) and following the projective [variational] scheme from Ch. 5 of [4] derives the following modal (ordinary differential) equations with respect to the generalised coordinates β_{vj} and β_{wj}:

$$\begin{aligned} &\pi\rho_l||f_j||^2\left(\kappa_j^{-1}\ddot\beta_{vj}(t) + g\beta_{vj}(t)\right) + \sum_{k=1}^{q_b}\ddot a_{vk}(t)D_{j,k} = 0, \\ &\pi\rho_l||f_j||^2\left(\kappa_j^{-1}\ddot\beta_{wj}(t) + g\beta_{wj}(t)\right) + \sum_{k=1}^{q_b}\ddot a_{wk}(t)D_{j,k} = 0, \end{aligned} \tag{77}$$

($j = 1,...,q_s$), where the non–symmetric $[q_s \times q_b]$-matrix $D_{bs} = \{D_{j,k}\}$ consists of the elements

$$D_{j,k} = \pi\rho_l\int_0^{r_0} r\Omega_k(r,h+h_1)f_j(r)\,dr. \tag{78}$$

One can correct the natural sloshing frequencies due to the pipes effect by varying κ_j so that $\sigma_j^2 = g\kappa_j$ belongs to the interval (73).

The contained linear flows contribute the horizontal forces f_{vi}, f_{wi} and moments $P_x^{h_1}, P_y^{h_1}$ at $x = h_1$ introduced in (10) and (12). The two components of the forces and moments (defined in (13)) can explicitly be expressed in terms of the generalized coordinates as

$$f_{vi0} = -\chi_{[h_1,h_1+h]}\rho_l \pi R_i(z) \sum_{k=1}^{q_b} \Omega_{\xi_k^{(1)}}(R_i(z), z)\, \ddot{a}_{vk}(t), \tag{79a}$$

$$f_{wi0} = -\chi_{[h_1,h_1+h]}\rho_l \pi R_i(z) \sum_{k=1}^{q_b} \Omega_{\xi_k^{(1)}}(R_i(z), z)\, \ddot{a}_{wk}(t), \tag{79b}$$

$$f_{vis} = -\chi_{[h_1,h_1+h]}\rho_l \pi R_i(z) \sum_{j=1}^{q_s} \frac{\Phi_j(R_i(z), z)}{\kappa_j}\, \ddot{\beta}_{vj}(t), \tag{79c}$$

$$f_{wis} = -\chi_{[h_1,h_1+h]}\rho_l \pi R_i(z) \sum_{j=1}^{q_s} \frac{\Phi_j(R_i(z), z)}{\kappa_j}\, \ddot{\beta}_{wj}(t), \tag{79d}$$

and

$$P_{x0}^{h_1} = \pi \rho_l \sum_{k=1}^{q_b} \int_0^{R_0} r^2 \Omega_{\xi_k^{(1)}}(r, h_1) dr\, \ddot{a}_{wk}(t), \tag{80a}$$

$$P_{y0}^{h_1} = -\pi \rho_l \sum_{k=1}^{q_b} \int_0^{R_0} r^2 \Omega_{\xi_k^{(1)}}(r, h_1) dr\, \ddot{a}_{vk}(t), \tag{80b}$$

$$P_{xs}^{h_1} = \pi \rho_l \sum_{j=1}^{q_s} \int_0^{R_0} r^2 \frac{\Phi_j(r, h_1)}{\kappa_j} dr\, \ddot{\beta}_{wj}(t), \tag{80c}$$

$$P_{ys}^{h_1} = -\pi \rho_l \sum_{j=1}^{q_s} \int_0^{R_0} r^2 \frac{\Phi_j(r, h_1)}{\kappa_j} dr\, \ddot{\beta}_{vj}(t). \tag{80d}$$

5. Coupling the sloshing and structural vibrations

The modal equations (77) are the Euler–Lagrange equations which make it possible to describe sloshing (the sloshing-related generalised coordinates $\beta_{vk}(t), \beta_{wk}(t)$, $k = 1, ..., q_s$) as functions of the tower vibrations (the structure-related generalised coordinates $a_{vk}(t), a_{wk}(t)$, $k = 1, ..., q_b$). Analysing the coupled motions requires the Euler–Lagrange equations for the structure-related generalised coordinates. These equations may follow from the virtual work principle which is, in fact, the first variation the Lagrangian on the kinematic transmission conditions (2) are satisfied. In the present section, we derive these equations under the assumptions: (i) the sloshing feedback is fully accounted so that the hydrodynamic loads (13) are computed by (79) and (80) and substituted into (37) and (38), (ii) the external hydrodynamic loads are computed for the case when the surface wave effect is neglected implying the non-zero $f_{veB0}, f_{weB0}, P^{l_2}_{xB0}$ and $P^{l_2}_{yB0}$ while other components in (20) and (21) are zeros; $f_{ve} = f_{veB0}, f_{we} = f_{weB0}, P^{l_2}_x = P^{l_2}_{xB0}, P^{l_2}_y = P^{l_2}_{yB0}$ should be substituted into (37) and (38), and (iii) since the water waves are neglected, we must postulate the zero incident wave $\varphi_I = 0$ and there is no the Earthquake, $E_1 = E_2 = E_3 = 0$ but the external forces and moments to the top operational platform can be non-zeros, i.e. the virtual work (35) should be included.

5.1. Modal-type expressions for the only non-zero components $f_{veB0}, f_{weB0}, P^{l_2}_{xB0}, P^{l_2}_{yB0}$ and F_{ue_1B0}

Neglecting the water wave effect implies the velocity potential in the form

$$\varphi_e = \dot{\eta}_3(t)\omega_\eta(r,z) + \sum_{k=1}^{q_b} \omega_{\xi_k^{(1)}}(r,z)(\dot{a}_{vk}(t)\cos\theta + \dot{a}_{wk}(t)\sin\theta) \qquad (81)$$

with definitions (49) and (56). Choosing this velocity potential satisfies the Laplace equations, the decaying condition at infinity and all the boundary condition including on the mean free surface replaced by the zero–Neumann condition as explained for the case when the water waves are neglected.

Substituting (81) into expressions (17) and (19) derives

$$f_{veB0} = \chi_{[0,d]}\rho_w \pi R_e(z) \sum_{k=1}^{q_b} \omega_{\xi_k^{(1)}}(R_e(z),z)\,\ddot{a}_{vk}(t), \qquad (82a)$$

$$f_{weB0} = \chi_{[0,d]}\rho_w \pi R_e(z) \sum_{k=1}^{q_b} \omega_{\xi_k^{(1)}}(R_e(z),z)\,\ddot{a}_{wk}(t) \qquad (82b)$$

and

$$P^{l_2}_{xB0} = \pi\rho_w \sum_{k=1}^{q_b} \int_{r_1}^{r_2} r^2 \omega_{\xi_k^{(1)}}(r,l_2)dr\,\ddot{a}_{wk}(t), \qquad (83a)$$

$$P^{l_2}_{yB0} = -\pi\rho_w \sum_{k=1}^{q_b} \int_{r_1}^{r_2} r^2 \omega_{\xi_k^{(1)}}(r, l_2) dr\, \ddot{a}_{vk}(t). \tag{83b}$$

Finally, the axisymmetric component of the velocity potential (81) contributes F_{ue_1B0} that is the only non-zero summand for F_{ue_1} by (18). The result is

$$F_{ue_1B0} = 2\pi\ddot{\eta}_3(t)\rho_w \Big[-\int_{l_2}^{d} R_e(z)R_e'(z)\omega_\eta(R_e(z), z)dz + \\ + \int_{r_1}^{r_2} r\omega_\eta(r, l_2)dr \Big] = -\ddot{\eta}_3(t)\ M_{eB0}. \tag{84}$$

5.2. Modal equations

Under assumptions of the present section, one can employ the virtual work principle from section 2.2.1 for derivation of the modal equations coupling the generalised coordinates $a_{vk}(t)$ and $a_{wk}(t)$, $k = 1, \dots, q_b$. The procedure suggests that the trial functions δv and δw consequently change with $\xi_k^{(1)}$, $k = 1, \dots, q_b$. This leads to the corresponding [modal] system of the $2q_b$-ordinary differential equations. Introducing the $2(q_b + q_s)$-set of generalised coordinates

$$\boldsymbol{p} = (a_{v1}, ..., a_{vq_b}, a_{w1}, ..., a_{wq_b}, \beta_{v1}, ..., \beta_{vq_s}, \beta_{w1}, ..., \beta_{wq_s})$$

and joining the obtained system of ordinary differential equations with the modal equations (77) leads to the system

$$B\ddot{\boldsymbol{p}} + D\dot{\boldsymbol{p}} + A\boldsymbol{p} = \boldsymbol{f}, \tag{85}$$

where the symmetric positive matrices A and B take the form

$$A = \left\| \begin{matrix} M_A & 0 & 0 & 0 \\ 0 & M_A & 0 & 0 \\ 0 & 0 & S_A & 0 \\ 0 & 0 & 0 & S_A \end{matrix} \right\|, \tag{86a}$$

$$B = \left\| \begin{matrix} M_{B_2} & M_b & (D_{bs})^T & 0 \\ M_b & M_{B_1} & 0 & (D_{bs})^T \\ D_{bs} & 0 & S_B & 0 \\ 0 & D_{bs} & 0 & S_B \end{matrix} \right\| \tag{86b}$$

Here, we suggest the replacement of $\xi_k(z)$ by $\xi_k^{(1)}$, the $q_b \times q_b$ sub-matrices M_A, M_{B_2}, M_{B_1}, and M_b are defined by (59), the $q_s \times q_s$ sub-matrices S_A, S_B are diagonal with elements $\pi g \rho_l ||f_j||^2$ and $\pi \rho_l ||f_j||^2 \kappa_j^{-1}$, $j = 1, \ldots, q_s$, respectively, but the $q_s \times q_b$-dimensional matrix D_{bs} is introduced in (78).

In addition, we see in (85) the damping matrix

$$D = \begin{Vmatrix} d_b & 0 & 0 & 0 \\ 0 & d_b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}, \tag{87}$$

in which the $q_b \times q_b$-dimensional matrix d_d consists of the elements $k_1 \xi_n^{(1)}(0)\xi_m^{(1)}(0) + \frac{1}{4} r_2^2 k_2 (\xi_n^{(1)})'(0)(\xi_m^{(1)})'(0)$, $n, m = 1, ..., q_b$ (damping due to the soil) and the right-hand side implying the forces and moments applied to the operational top platform $\boldsymbol{f}(t) = (o_{v1}, ..., o_{vq_b}, o_{w1}, ..., o_{wq_b}, 0, ..., 0)$ where $o_{vk}(t) = \mathcal{P}_1(t)\xi_k^{1)}(L) + \mathcal{P}_5(t)(\xi_k^{(1)})'(L)$ and $o_{wk}(t) = \mathcal{P}_2(t)\xi_k^{1)}(L) - \mathcal{P}_4(t)(\xi_k^{(1)})'(L)$.

To deduce the structure of B regarding D_{bs} we have used the Green identity in the following derivation line

$$\int_0^{r_0} r\Omega_{\xi_k^{(1)}}(r, h_1 + h) f_j(r) dr = \frac{1}{\kappa_j} \int_0^{r_0} r\ \Omega_{\xi_k^{(1)}} \frac{\partial \Phi_j}{\partial z}\bigg|_{z=h+h_1} dr =$$
$$= \frac{1}{\kappa_j} \int_{L_0+L_1+L_2} \Omega_{\xi_k^{(1)}} \frac{\partial \Phi_j}{\partial n_i} dS = \int_{L_0+L_1+L_2} \frac{\Phi_j}{\kappa_j} \frac{\Omega_{\xi_k^{(1)}}}{\partial n_i} dS =$$
$$= \int_{h_1}^{h+h_1} R_i(z) \frac{\Phi_j(R_i(z), z)}{\kappa_j} \xi_k^{(1)}(z) dz + \int_0^{R_0} r^2 \frac{\Phi_j(r, h_1)}{\kappa_j} (\xi_k^{(1)})'(h_1) dr.$$

The ordinary differential equation (44) governing the lifting takes, accounting for (84), the form

$$(M_p + M_b + M_l + M_{eB0})\, \ddot{\eta}_3 + d_2 \dot{\eta}_3 + \eta_3 (k_2 + \pi \rho_w g (r_1^2 - r_3^2)) = \mathcal{P}_3. \tag{88}$$

5.3. Eigenoscillations

Assuming the zero-damping and external forcing, we can consider the eigenoscillatons associated with the $2\pi/\sigma$-periodic generalised coordinates. This leads to the generalised spectral matrix problem $(A -$

$\sigma^2 B)\boldsymbol{p}_0 = 0$ which of the higher dimension than that for the problem where liquid sloshing is neglected (section 3.3.1). Here, $\boldsymbol{p}_0(t) = \exp(\mathrm{i}\sigma t)\boldsymbol{p}_0$.

The eigenfrequency associated with the lifting follows from (88) and is found from

$$(M_p + M_b + M_l + M_{eB0})\,\sigma^2 = (k_2 + \pi\rho_w g(r_1^2 - r_3^2)). \tag{89}$$

6. Accounting for the external water waves

The external water wave problem (15) can be solved by using the Fourier transform. This gives the velocity potential as the sum of (81) and a convolution integral. The derivation should give the right-hand side vector of (85) with

$$o_{vk}(t) = \mathcal{P}_1(t)\xi_k^{1)}(L) + \mathcal{P}_5(t)(\xi_k^{(1)})'(L) + o'_{vk}(t) + \int_{-\infty}^{+\infty}\sum_{l=1}^{2q_b} K_{vl}(t-\tau)p_l(\tau)d\tau, \tag{90a}$$

$$o_{wk}(t) = \mathcal{P}_2(t)\xi_k^{1)}(L) - \mathcal{P}_4(t)(\xi_k^{(1)})'(L) + o'_{wk}(t) + \int_{-\infty}^{+\infty}\sum_{l=1}^{2q_b} K_{wl}(t-\tau)p_l(\tau)d\tau, \tag{90b}$$

where $o'_{vl}(t), o'_{wl}(t)$ are the explicitly given functions of the incident wave and the integral kernels should also be explicitly derived. Derivation of the integral quantities due to water wave, studying the corresponding integro-differential equations are tedious and independent tasks and deserve an independent publication.

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