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Transversality and Lipschitz-Fredholm maps

We study transversality for Lipschitz-Fredholm maps in the context of bounded Fréchet manifolds. We show that the set of all Lipschitz-Fredholm maps of a fixed index between Fréchet spaces has the transverse stability property. We give a straightforward extension of the Smale transversality theorem by using the generalized Sard's theorem for this category of manifolds. We also provide an answer to the well known problem concerning the existence of a submanifold structure on the preimage of a transversal submanifold.

Вивчається поняття трансверсальності відображень Ліпшица-Фредгольма у контексті обмежених многовидів Фреше. Доведено, що множина всіх відображень Ліпшица-Фредгольма фіксованого індексу між просторами Фреше має властивість стійкості трансверсальних перетинів. Дано пряме узагальнення теореми Смейла про трансверсальність, для доведення якого використовується узагальнення теореми Сарда на цю категорію многовидів. Також отримано відповідь на відоме питання про існування структури підмноговиду на прообразі трансверсального підмноговиду.

1. INTRODUCTION

In [1] we proved a version of the classical Sard-Smale theorem for a category of generalized Fréchet manifolds, bounded (or MC^k) Fréchet manifolds, introduced in [2]. Our approach to the theorem's generalization is based on the assumption that Fredholm operators need to be globally Lipschitz. A reason for this interest is that there exists an appropriate topology on $\mathcal{L}(E, F)$, the space of all linear globally Lipschitz maps between Fréchet spaces E and

F, that leads to the openness of the set of linear isomorphisms in $\mathcal{L}(E, F)$, [1, Proposition 2.2]. This result in turn yields the openness of the collection of Fredholm operators in $\mathcal{L}(E, F)$, [1, Theorem 3.2]. The other reason is that Lipschitzness is consistent with the notion of differentiability, bounded (or MC^k-) differentiability, that we apply. If E, F are Fréchet spaces and if U is an open subset of E, a map $f \colon U \to F$ is called bounded (or MC^1-) differentiable if it is Keller-differentiable, the directional derivative $\mathrm{d}\, f(p)$ belongs to $\mathcal{L}(E, F)$ for all $p \in U$, and the induced map $\mathrm{d}\, f \colon U \to \mathcal{L}(E, F)$ is continuous. Thus, we can naturally define the index of a Fredholm map between manifolds.

We should point out that the mentioned results stems from the essential fact that under a certain condition we can endow the space $\mathcal{L}(E, F)$ with a topological group structure. Also, the group of automorphisms of a Fréchet space E, $\mathrm{Aut}(E)$, is open in $\mathcal{L}(E, E)$ [3, Proposition 2.1]. But, in general, the group of automorphisms of a Fréchet space does not admit a non-trivial topological group structure. Thus, without some restrictions it would be impossible to establish openness of sets of linear isomorphisms and Fredholm operators. This is a major obstruction in developing the Fredholm theory for Fréchet spaces.

A crucial step in the proof of an infinite dimensional version of Sard's theorem is that, roughly speaking, for a Fredholm map $f \colon M \to N$ of manifolds, at each point $p \in M$, we may find local charts $(p \in U \subseteq M, \phi)$ and $(f(p) \in V \subseteq N, \psi)$ such that in the charts f has a representation of the form $f(u, v) = (u, \eta(u, v))$, where $\eta \colon \phi(U) \to \mathbb{R}^n$ is a smooth map. This is a consequence of an inverse function theorem. One of the main significance of the category of bounded Fréchet manifolds is the availability of an inverse function theorem in the sense of Nash and Moser [2, Theorem 4.7]. However, the bounded differentiability is strong and in some cases the class of bounded maps can be quite small, e.g. when the identity component of $\mathcal{L}(E, F)$ contains only the zero map [3, Remark 2.16].

We have argued that why we have utilized this particular category of Fréchet manifolds. A salient example of these manifolds is the space of all smooth sections of a fiber bundle over closed or non-compact manifolds ([2, Theorem 3.34]). On the other hand, it turns out that these generalized manifolds can surpass the geometry of Fréchet manifolds. On these manifold we are able to give a precise analytic meaning to some essential geometric objects (such as connection maps, vector fields and integral curves), [4]. Therefore, we would expect their applications to problems in global analysis.

The present work studies the differential topology of Lipschitz-Fredholm maps in the bounded Fréchet setting. We show that the set of Lipschitz-Fredholm operators of index l between Fréchet spaces E and F is open in the space of linear globally Lipschitz maps endowed with the fine topology (Proposition 3.5). We say that a set of maps has the transverse stability property for the fine topology if maps in a fine neighborhood of a given map have the same transversality property i.e. if $f\colon E \to F$ is a map transversal to a closed subspace $\mathbb{F}$ of F, then any map in a fine neighborhood of f is transversal to $\mathbb{F}$. We then prove that the set of all Lipschitz-Fredholm maps of a fixed index between Fréchet spaces has the transverse stability property (Theorem 3.6). We also study transversality for Lipschitz-Fredholm maps between manifolds. We give a straightforward generalization of the Smale transversality theorem ([5, Theorem 3.1]) by using our generalized Sard's theorem (Theorem 3.9). Finally, we prove that if $f\colon M \to N$ is an MC^k Lipschitz-Fredholm map of manifolds which is transversal to a finite dimensional submanifold $\mathcal{A}$ of N, then $f^{-1}(\mathcal{A})$ is a submanifold (Theorem 4.2).

We stress that these results can not be proved without strong restrictions. However, the basic concepts of infinite dimensional differential topology such as submanifold and transversality can be simply come over from the Banach setting.

Our motivation for the present work, in the light of [4], lay in the desire to develop transversality tools for the degree theory, including the Leray-Schauder degree, for Lipschitz-Fredholm maps, to derive applications to the study of solutions to systems of nonlinear partial differential or integral equations on spaces of smooth sections which are not linear.

2. Preliminaries

We shall recall the required definitions from the category of MC^k manifolds briefly but in a self-contained way for the convenience of the reader, which also give us the opportunity to establish our notations for the rest of the paper. For more studies we refer to [1, 2, 4].

Let (F, d) be a Fréchet space whose topology is defined by a complete translational-invariant metric d. A metric with absolutely convex balls will be called a standard metric. Every Fréchet space admits a standard metric which defines its topology. We shall always define the topology of Fréchet spaces with this type of metrics.

Let (E, g) and (F, d) be Fréchet spaces and let $\mathcal{L}_{g,d}(E, F)$ be the set of all linear maps $L : E \to F$ such that

$$\mathrm{Lip}(L)_{g,d} := \sup_{x \in E \setminus \{0\}} \frac{d(L(x), 0)}{g(x, 0)} < \infty.$$

The transversal-invariant metric

$$D_{g,d}(L, H) = \mathrm{Lip}(L - H)_{g,d} \qquad (2.1)$$

on $\mathcal{L}_{g,d}(E, F)$ turns it into an Abelian topological group ([1, Remark 2.1]). A map $\varphi \in \mathcal{L}_{g,d}(E, F)$ is called Lipschitz-Fredholm operator if its kernel has finite dimension and its image is closed and has finite co-dimension. The index of φ, $\mathrm{Ind}\,\varphi$, is defined by $\mathrm{Ind}\,\varphi = \dim \ker \varphi - \mathrm{codim}\,\mathrm{Img}\,\varphi$. We denote by $\mathcal{LF}(E, F)$ the set of all Lipschitz-Fredholm operators, and by $\mathcal{LF}_l(E, F)$ the subset of $\mathcal{LF}(E, F)$ consisting of those operators of index l.

Proposition 2.1. [1, Proposition 2.2] *The set of linear iso-morphisms from E into F, $\mathrm{Iso}\,(E,F)$, is open in $\mathcal{L}_{g,d}(E,F)$ with respect to the topology induced by the Metric (2.1).*

Theorem 2.2. [1, Theorem 3.2] $\mathcal{LF}(E,F)$ *is open in $\mathcal{L}_{g,d}(E,F)$ with respect to the topology defined by the metric (2.1). Furthermore, the function $T \to \mathrm{Ind}\,T$ is continuous on $\mathcal{LF}(E,F)$, hence constant on connected components of $\mathcal{LF}(E,F)$.*

A subset G of a Fréchet space F is called topologically complemented (or it splits in F), if F is homeomorphic to the topological direct sum $G \oplus H$, where H is a subspace of F. We call H a topological complement of G in F.

Theorem 2.3. [2, Theorem 3.14] *Let E be a Fréchet space. Then*
(1) *Every finite-dimensional subspace of E is closed.*
(2) *Every closed subspace $G \subset E$ with*

$$\mathrm{codim}(G) = \dim(E/G) < \infty$$

   is topologically complemented in E.
(3) *Every finite-dimensional subspace of E is topologically complemented.*
(4) *A linear subspace G of E has a topological complement H if and only if there exists a continuous projection Pr of E onto H, see [6].*

Let E, F be Fréchet spaces, U an open subset of E, and

$$P \colon U \to F$$

a continuous map. Let $CL(E,F)$ be the space of all continuous linear maps from E to F topologized by the compact-open topology. We say P is differentiable at the point $p \in U$ if the directional derivative $\mathrm{d}P(p)$ exists in all directions $h \in E$. If P is differentiable at all points $p \in U$, if $\mathrm{d}P(p) \colon U \to CL(E,F)$ is continuous for all $p \in U$ and if the induced map

$$P' \colon U \times E \to F, \qquad (u,h) \mapsto \mathrm{d}P(u)h$$

is continuous in the product topology, then we say that P is Keller-differentiable. We define $P^{(k+1)} \colon U \times E^{k+1} \to F$ in the obvious inductive fashion.

If P is Keller-differentiable, $\mathrm{d}P(p) \in \mathcal{L}_{g,d}(E, F)$ for all $p \in U$, and the induced map $\mathrm{d}P(p) \colon U \to \mathcal{L}_{g,d}(E, F)$ is continuous, then P is called bounded differentiable. We say P is MC^0 and write $P^0 = P$ if it is continuous. We say P is an MC^1 and write $P^{(1)} = P'$ if it is bounded differentiable. Let $\mathcal{L}_{g,d}(E, F)_0$ be the connected component of $\mathcal{L}_{g,d}(E, F)$ containing the zero map. If P is bounded differentiable and if $V \subseteq U$ is a connected open neighborhood of $x_0 \in U$, then $P'(V)$ is connected and hence contained in the connected component $P'(x_0) + \mathcal{L}_{g,d}(E, F)_0$ of $P'(x_0)$ in $\mathcal{L}_{g,d}(E, F)$. Thus, the map $Q_{x_0} \colon V \to \mathcal{L}_{g,d}(E, F)_0$ defined by

$$Q_{x_0}(y) = P'(y) - P'(x_0)$$

is again a map between subsets of Fréchet spaces. This enables a recursive definition: if P is MC^1 and V can be chosen for each $x_0 \in U$ such that $Q_{x_0} \colon V \to \mathcal{L}_{g,d}(E, F)_0$ is MC^{k-1}, then P is called an MC^k-map. We make a piecewise definition of $P^{(k)}$ by $P^{(k)}|_V := (Q_{x_0})^{(k-1)}$ for x_0 and V as before. The map P is MC^∞ (or smooth) if it is MC^k for all $k \in \mathbb{N}_0$. We shall denote the derivative of P at p by $\mathrm{D}\,P(p)$. Note that MC^k-differentiability implies the usual C^k-differentiability for maps of finite dimensional manifolds.

Within this framework we can define MC^k Fréchet manifolds, MC^k-maps of manifolds and tangent bundle over MC^k manifolds in obvious fashion way. We assume that manifolds are connected and second countable.

Let $f \colon M \to N$ $(k \geqq 1)$ be an MC^k-map of manifolds. We denote by $T_x f \colon T_x M \to T_{f(x)} N$ the tangent map of f at $x \in M$ from the tangent space $T_x M$ to the tangent space $T_{f(x)} N$. We say that f is an immersion (resp. submersion) provided $T_x f$ is injective (resp. surjective) and the range $\mathrm{Img}(T_x f)$ (resp. the kernel $\ker(T_x f)$) splits in $T_{f(x)} N$ (resp. $T_x M$) for any $x \in M$. An injective immersion $f \colon M \to N$ which gives an isomorphism

onto a submanifold of N is called an embedding. A point $x \in M$ is called a regular point if $\mathrm{D}\, f(x) \colon T_x M \longrightarrow T_{f(x)} N$ is surjective. The corresponding value $f(x)$ is a regular value. Points and values other than regular are called critical points and values, respectively.

Let M and N be MC^k manifolds, $k \geqq 1$. A Lipschitz-Fredholm map is an MC^1-map $f \colon M \to N$ such that for each $x \in M$ the derivative $\mathrm{D}\, f(x) \colon T_x M \longrightarrow T_{f(x)} N$ is a Lipschitz-Fredholm operator. The index of f, denoted by $\mathrm{Ind}\, f$, is defined to be the index of $\mathrm{D}\, f(x)$ for some x. Since f is MC^k and M is connected in the light of Theorem 2.2 the definition does not depend on the choice of x.

3. Transversality and openness

Let F_1 be a linear closed subspace of a Fréchet space F that splits in F. Given MC^k manifold M modelled on F, a subset M_1 of M is a submanifold of M modelled on F_1 provided there is MC^k-atlas $\{(U_i, \phi_i)\}_{i \in I}$ on M that induces an atlas on M_1, i.e. for any $i \in I$ there are open subsets V_i, W_i of F, F_1 such that $\phi_i(U_i) = V_i \oplus W_i$ and $\phi_i(U_i \cap M_1) = V_i \oplus \{0\}$ is open in F_1. We say that M_1 is a submanifold of Banach type if F_1 is a Banach space, and a submanifold of finite type if $F_1 = \mathbb{R}^n$ for some $n \in \mathbb{N}$.

Let $C(E, \mathbb{R}^+)$ be the set of all continuous functions from E into $\mathbb{R}^+$, $h \in \mathcal{L}_{g,d}(E, F)$ and $\varepsilon \in C(E, \mathbb{R}^+)$. A map $f \in \mathcal{L}_{g,d}(E, F)$ is called a ε-approximation to h if $d(f(x), h(x)) < \varepsilon(x)$ for all $x \in E$, we write $d(f, h) < \varepsilon$ for short. If we take the ε-approximation to h to be a neighborhood of h in the set $\mathcal{L}_{g,d}(E, F)$, then we obtain a topology. This topology is called the fine topology and we denote the resulting space by $L^0_{fine}(E, F)$.

Let M and N be MC^k manifolds modelled on Fréchet spaces E and F, respectively. Let $MC^k(M, N), 1 \leqq k \leqq \infty$, be the set of MC^k-maps from M into N. Two maps $f, h \in MC^k(M, N)$ are said to be k-equivalent at $x \in M$ if $T_x^k f = T_x^k h$, where T^k is the k-th tangent map. We define the k-jet of f at x, $j_x^k f$, to be the equivalence class of f. Let d_k be a fiber metric on the tangent

space $T_x^k M$ that induces a Fréchet topology which is isomorphic to E. We describe the fine topology of order k on $MC^k(M, N)$ as follows. Let $\varphi \in MC^k(M, N)$ and $\Omega \coloneqq \{V_i\}_{i \in I}$ be a locally finite cover of M. Let $\epsilon_i \colon V_i \to \mathbb{R}^+$ be continuous for all $i \in I$. Then, the sets

$$\Theta(\varphi, V_i, \epsilon_i) \coloneqq \{\phi \in MC^k(M, N) \mid d_k(j_x^k \phi, j_x^k \varphi) < \epsilon_i(x), x \in V_i\}$$

constitute a basis for fine open neighborhoods of φ. In this case we say that ϕ in a fine neighborhood of φ is an MC^k fine approximation to φ.

Lemma 3.1. *The fine topology is finer than the topology induced by the Metric* (2.1).

Proof. We must show that if $\mathcal{N}(f, \delta)$ is a δ-neighborhood of f, then we can find $\epsilon > 0$ such that if $D_{g,d}(f, h) < \epsilon$, then $h \in \mathcal{N}(f, \delta)$. Given a map $h \in \mathcal{L}_{g,d}(E, F)$, let

$$\epsilon \coloneqq \min\left\{1, \inf_{x \in E \setminus \{0\}} \frac{\delta(x)}{g(h(x), 0)}\right\}.$$

Now suppose $D_{g,d}(f, h) < \epsilon$, then we can easily see that $d(f, h) < \delta$ and hence $h \in \mathcal{N}(f, \delta)$. □

Remark 3.2. *We know that* (*Proposition 2.1*) $\mathrm{Iso}(E, F)$ *is open in* $\mathcal{L}_{g,d}(E, F)$ *endowed with the topology induced by the metric* (2.1). *By the preceding lemma the fine topology is finer than the metric topology, thereby,* $\mathrm{Iso}(E, F)$ *is open in* $L_{fine}^0(E, F)$.

Definition 3.3. *Let* $f \colon E \to F$ *be a Lipschitz-Fredholm operator of Fréchet spaces. We say that* f *is transversal to a closed subspace* $F_0 \subseteq F$ *and write* $f \pitchfork F_0$ *if*

    (1) $\mathrm{Img}(f) + F_0 = F$, *and*
    (2) *either* F_0 *splits in* F *or* $f^{-1}(F_0)$ *splits in* E.

The following result characterizes the transversality of Lipschitz-Fredholm operators.

Proposition 3.4. *Let $\varphi \in \mathcal{LF}_l(E, F)$. Suppose $F_0 \subseteq F$ is a closed subspace such that $\mathrm{Img}(\varphi) + F_0 = F$. Then $\varphi \pitchfork F_0$ if and only if there are closed subspace $F_1 \subseteq F$ and $E_0 \subseteq E$ with $F = F_0 \oplus F_1$ and $E = E_0 \oplus (E_1 := \varphi^{-1}(F_1))$ such that $\varphi_1 := \varphi|_{E_1} \in \mathrm{Iso}(E_1, F_1)$.*

Proof. Assume that such a closed subspace F_0 is given and $\varphi \pitchfork F_0$. $(\mathrm{Img}(\varphi) \cap F_0)$ splits in F_0 because

$$m = \dim(F_0 / \mathrm{Img}(\varphi)) \leqq \dim(F / \mathrm{Img}(\varphi)) < \infty$$

and hence by Theorem 2.3(2) there exists a subspace $\mathbb{F} \subseteq F_0$ of dimension m such that $F_0 = (\mathrm{Img}(\varphi) \cap F_0) \oplus \mathbb{F}$. Since

$$\mathrm{Img}(\varphi) \cap \mathbb{F} \subseteq \mathrm{Img}(\varphi) \cap F_0,$$

it follows that $\mathrm{Img}(\varphi) \cap \mathbb{F} = \{0\}$. Also,

$$\begin{aligned} \mathrm{Img}(\varphi) + \mathbb{F} &= (\mathrm{Img}(\varphi) + (\mathrm{Img}(\varphi) \cap F_0)) + \mathbb{F} \\ &= \mathrm{Img}(\varphi) + F_0 = F. \end{aligned}$$

Thus, $\mathrm{Img}(\varphi) \oplus \mathbb{F} = F$, therefore,

$$\mathrm{codim}\,\mathrm{Img}(\varphi) = m, \qquad \dim\ker(\varphi) = l + m.$$

Moreover, there exists a closed subspace $\mathbb{E} \subseteq E$ such that

$$E = \ker(\varphi) \oplus \mathbb{E}.$$

The operator $\Phi := \varphi|_{\mathbb{E}} \in \mathcal{L}(\mathbb{E}, \mathrm{Img}(\varphi))$ is injective onto $\mathrm{Img}(\varphi)$, hence, by virtue of open mapping theorem is a homeomorphism and therefore $\Phi \in \mathrm{Iso}(\mathbb{E}, \mathrm{Img}(\varphi))$. Let

$$\mathbb{E}_0 := \Phi^{-1}(\mathrm{Img}(\varphi) \cap F_0) \subseteq \mathbb{E},$$

then $E_0 = \varphi^{-1}(\mathrm{Img}(\varphi) \cap F_0) = \ker(\varphi) \oplus \mathbb{E}_0$.

As $\mathbb{E}_0$ is complemented in E_0, there is a continuous projection Pr_1 of E_0 onto $\mathbb{E}_0$ (see Theorem 2.3(4)). If E_0 is complemented in E, then there exists a continuous projection Pr_2 of E onto E_0. Thus, $\mathrm{Pr}_1 \circ \mathrm{Pr}_2$ is a continuous projection from E to $\mathbb{E}_0$ and its restriction to $\mathbb{E}$ is a again continuous projection onto $\mathbb{E}_0$, thereby, $\mathbb{E}_0$ is complemented in $\mathbb{E}$. This means there is a closed subspace $E_1 \subseteq \mathbb{E}$ (which is also closed in E) such that $\mathbb{E} = E_1 \oplus \mathbb{E}_0$.

By the same argument we have, if F_0 is complemented in F, then $(\mathrm{Img}(\varphi) \cap F_0)$ is complemented in $\mathrm{Img}(\varphi)$ because $(\mathrm{Img}(\varphi) \cap F_0)$ is complemented in F_0. This means there is a closed subspace $F_1 \subseteq \mathrm{Img}(\varphi)$ (which is also closed in F) such that

$$\mathrm{Img}(\varphi) = F_1 \oplus (\mathrm{Img}(\varphi) \cap F_0).$$

Therefore, we have

$$E = \ker(\varphi) \oplus \mathbb{E}_0 \oplus E_1 = E_0 \oplus E_1,$$
$$F = (\mathrm{Img}(\varphi) \cap F_0) \oplus \mathbb{F} \oplus F_1 = F_0 \oplus F_1$$

and $\varphi_1 = \Phi|_{E_1} \in \mathrm{Iso}(E_1, F_1)$. Moreover, $E_1 = \varphi_1^{-1}(F_1)$. The converse is obvious. $\qquad\square$

Proposition 3.5. $\mathcal{LF}_l(E, F)$ *is open in* $\mathcal{L}^0_{fine}(E, F)$.

Proof. Let $\varphi \in \mathcal{LF}_l(E, F)$. We show that there exists $\varepsilon > 0$ such that any $\phi \in \mathcal{L}_{g,d}(E, F)$ which is ε-approximation to φ is a Lipschitz-Fredholm operator of index l.

First we prove for the case $l = 0$, then we show that the general case can be reduced to the case $l = 0$. Let $L \colon E \to F$ (called a corrector) be a linear globally Lipschitz map having finite dimensional range such that $K \coloneqq L + \varphi$ is an isomorphism. Such a linear map always exists. Indeed, L can be any linear globally Lipschitz map from E into F such that

$$\ker(L) \oplus \ker(\varphi) = E, \qquad \mathrm{Img}(L) \oplus \mathrm{Img}(\varphi) = F.$$

Choose $\varepsilon \in (0, 1/2\mathrm{Lip}(K^{-1}))$ small enough and suppose that $\phi, \mathbb{L} \in \mathcal{L}(E, F)$ are ε-approximation to $\varphi \varepsilon$-approximation to φ, and the dimension of the image of $\mathbb{L}$ is finite. Then $\mathbb{K} = \mathbb{L} + \phi$ satisfies

$$d(K(x), \mathbb{K}(x)) < 1/\mathrm{Lip}(K^{-1}),$$

for all $x \in E$, thus $\mathbb{K}$ is an isomorphism (see Remark 3.2) and hence $\phi \in \mathcal{LF}(E, F)$ and $\mathrm{Ind}(\phi) = 0$.

Now suppose $l > 0$, define the linear globally Lipschitz operators $\varphi_l, \phi_l \colon E \to F \times \mathbb{R}^l$ by $\varphi_l(x) \coloneqq (\varphi(x), 0)$ and $\phi_l(x) \coloneqq (\phi(x), 0)$. Then φ_l is a Lipschitz-Fredholm operator of index 0. By the above

argument ϕ_l is a Lipschitz-Fredholm operator of index 0 and hence ϕ is a Lipschitz-Fredholm operator of index l. Likewise, the case $l < 0$ can be proved. □

Theorem 3.6. *Let $\varphi \in \mathcal{LF}_l(E, F)$, and suppose that $F_0 \subseteq F$ is closed and $\varphi \pitchfork F_0$. Then any $\phi \in \mathcal{L}_{g,d}(E, F)$ in some fine neighborhood of φ is transversal to F_0.*

Proof. By Proposition 3.4 there exist closed subsets

$$E_0 \subseteq E, \qquad F_1 \subseteq F, \qquad E_1 := \varphi^{-1}(F_1)$$

such that

$$F = F_0 \oplus F_1, \quad E = E_0 \oplus E_1, \quad \varphi_1 := \varphi|_{E_1} \in \text{Iso}(E_1, F_1).$$

There is a continuous function $\delta(x)$ such that every linear globally Lipschitz map $\psi \colon E_1 \to F_1$ which is δ-approximation to φ_1 is an isomorphism (see Remark 3.2). Let $\pi \colon F \to F_1$ be the projection given by $\pi(f_0 + f_1) = f_1$, and let $\kappa = \text{Id}_F - \pi$. It is immediate that π is linear and globally Lipschitz and $\text{Img}(\kappa) = F_0$. Choose $\varepsilon \in (0, \delta/\text{Lip}(\pi))$ small enough, in view of Proposition 3.5, we can assume that each $\phi \in \mathcal{L}(E, F)$ which is ε-approximation to φ belongs to $\mathcal{LF}_l(E, F)$.

Now we show that each such ϕ is transversal to F_0. Let

$$\Phi := (\pi \circ \varphi)|_{E_1} \in \mathcal{L}(E_1, F_1).$$

Then $d(\Phi, \varphi_1) \leqq \text{Lip}(\pi)\varepsilon < \delta$ and so $\Phi \in \text{Iso}(E_1, F_1)$ (see Remark 3.2). Thus, we only need to prove $F = \text{Img}(\phi) + F_0$. Let $f \in F = F_0 \oplus F_1$ so $f = f_0 + f_1$, where $f_i \in F_i (i = 0, 1)$. We have $\Phi^{-1}(f_1) = e_1 \in E_1 \subseteq E$, $x = \phi(e_1) \in \text{Img}(\phi)$, and $y = f_0 - \kappa(x) \in F_0$. Whence,

$$x + y = \pi(x) + \kappa(x) + f_0 - \kappa(x) = f_0 + \Phi(e_1) = f_0 + f_1 = f,$$

therefore $F = \text{Img}(\phi) + F_0$. □

Now we prove the transversality theorem for MC^k-Lipschitz-Fredholm maps. It is indeed a consequence of the Sard's theorem for these maps, [1, Theorem 4.3]. A careful reading of the proof of

the theorem shows that the minor assumption of endowing manifolds with compatible metrics is superfluous and the theorem remains valid for manifolds without compatible metrics. Thus, the statement of the theorem is as follows:

Theorem 3.7 (Sard's Theorem). *Let M and N be MC^k manifolds, $k \geqq 1$. If $f\colon M \to N$ is an MC^k-Lipschitz-Fredholm map with $k > \max\{\operatorname{Ind} f, 0\}$. Then, the set of regular values of f is residual in N.*

Definition 3.8. *Let $f\colon M \to N$ be a Lipschitz-Fredholm map and let $\imath\colon \mathcal{A} \hookrightarrow N$ be an MC^1 embedding of a finite dimensional manifold $\mathcal{A}$. We say that f is transversal to $\imath$ and write $f \pitchfork \imath$ if $\operatorname{D} f(x)(T_x M) + \operatorname{D} \imath(y)(T_y \mathcal{A}) = T_{f(x)} N$, whenever $f(x) = \imath(y)$. It is also said that the submanifold $\mathbf{A} := \imath(\mathcal{A})$ is transversal to f.*

The following theorem is the analogous of the Smale transversality, [5, Theorem 3.1]. Its proof is just a slight modification of the argument of Smale.

Theorem 3.9. *Let M and N be MC^k manifolds modelled on spaces (F, d) and (E, g), respectively. Let $f\colon M \to N$ be an MC^k-Lipschitz-Fredholm map and let $\imath\colon \mathcal{A} \hookrightarrow N$ be an MC^1-embedding of a finite dimension manifold $\mathcal{A}$ with $k > \max\{\operatorname{Ind} f + \dim \mathcal{A}, 0\}$. Then there exists an MC^1 fine approximation $\mathbf{g}$ of $\imath$ such that $\mathbf{g}$ is embedding and $f \pitchfork \mathbf{g}$. Furthermore, Suppose S is a closed subset of $\mathcal{A}$ and $f \pitchfork \imath(S)$, then $\mathbf{g}$ can be chosen so that $\imath = \mathbf{g}$ on S.*

Proof. Since manifolds are second countable we only need to work in local coordinates. Assume that $y \in \mathcal{A}$ and $n = \dim \imath(\mathcal{A})$. Since $\imath(\mathcal{A})$ is an embedded submanifold of finite type of N, we may find an open neighborhood $U \subset \mathbb{R}^n$ about y, a chart about $\imath(y)$ and a splitting $E = \mathbb{R}^n \oplus E_1$ such that $\imath(y) = \imath(y, 0)$ in the neighborhoods. Let $\pi_2\colon E \to E_1$ be the projection onto E_1. Let $\overline{V} \subset U$ be a neighborhood of y, and h a smooth real valued function which is 1 on $\overline{V}$ and 0 outside of U. Since $\pi_1 \circ f$ is locally Fredholm-Lipschitz it follows by Sard's Theorem (Theorem 3.7) that there is a regular

value z for $\pi_1 \circ f$ which is close to 0. Now define

$$\mathbf{g}(y) = h(y)(z, y) + [1 - h(y)]\imath(y).$$

It is immediate that $f \pitchfork \mathbf{g}$ on V, and for z sufficiently close to 0, $\mathbf{g}$ is MC^1 fine approximation to $\imath$. The second statement follows by our definition of $\mathbf{g}$. $\qquad\square$

4. Transversal submanifolds

We will need the following inverse function theorem.

Theorem 4.1. [2, Theorem 4.7], Inverse Function Theorem for MC^k-maps. *Let (E, g) be a Fréchet space with standard metric g. Let $U \subset E$ be open, $x_0 \in U$ and $f\colon U \subset E \to E$ an MC^k-map, $k \geq 1$. If $f'(x_0) \in \mathrm{Aut}\,(E)$, then there exists an open neighborhood $V \subseteq U$ of x_0 such that $f(V)$ is open in E and $f|_V\colon V \to f(V)$ is an MC^k- diffeomorphism.*

To avoid some technical complications we consider only manifolds without boundary in the sequel.

Theorem 4.2. *Let M and N be MC^k manifolds modelled on spaces (F, d) and (E, g), respectively. Suppose that $f\colon M \to N$ is an MC^k-Lipschitz-Fredholm map of index l. Let $\mathcal{A}$ be a submanifold of N with dimension m and let $\imath\colon \mathcal{A} \hookrightarrow N$ be the inclusion. If f is transversal to $\mathcal{A}$, then $f^{-1}(\mathcal{A})$ is a submanifold of M of dimension $l + m$. For all $x \in f^{-1}(\mathcal{A})$ we have $T_x(f^{-1}(\mathcal{A})) = (T_x f)^{-1}(T_{f(x)}\mathcal{A})$.*

Proof. If $f^{-1}(\mathcal{A}) = \varnothing$ the theorem is clearly valid so let us assume that $f^{-1}(\mathcal{A}) \neq \varnothing$. Let (ψ, U) be a chart at $f(x_0) \in \mathcal{A}$ in N with the submanifold property for $\mathcal{A}$. Let U_1, U_2 be open subsets of $E, \mathbb{R}^m$ such that $\psi(U) = U_1 \oplus U_2, \psi(U \cap \mathcal{A}) = U_1 \oplus \{0\}$, and $\psi(f(x_0)) = (0, 0)$. Let (V, φ) be a chart at x_0 in M such that $\varphi(x_0) = 0$, $\varphi\colon V \to \varphi(V) \subset F$ and $f(V) \subset U$. Let

$$\mathbf{f} := \psi \circ f \circ \varphi^{-1}\colon \varphi(V) \to \psi(U)$$

be the local representative of f. Then $\mathbf{f}(0) = (0,0)$ and by hypothesis $\mathbf{f}$ is a Lipschitz-Fredholm map, in particular, $D\mathbf{f}(0) \in \mathcal{LF}_l(F, E)$. The tangent map $T_{f(x_0)}\imath\colon T_{f(x_0)}\mathcal{A} \to T_{f(x_0)}N$ is injective with closed split image. Hence $T_{f(x_0)}\mathcal{A}$ can be identified with a closed split subspace of $T_{f(x_0)}N$. Thus $D f(x_0)$ is transversal to $T_{f(x_0)}\mathcal{A}$. Therefore, keeping in the mind the definition of the differential in terms of tangent maps, $D\mathbf{f}(0)$ is transversal to $T\psi(T_{f(x_0)}\mathcal{A}) = U_1 \oplus \{0\} =: E_1$. Then, by virtue of Proposition 3.4 there are closed subsets $F_1 \subset F$, $E_0 \subset E$ such that $F = F_1 \oplus (F_0 := D\mathbf{f}(0)(E_0))$, $E = E_1 \oplus E_0$, $\Delta := D\mathbf{f}(0)\mid_{F_0} \in \mathrm{Iso}(F_0, E_0)$ and $\Delta_1 := D\mathbf{f}(0)\mid_{F_1} \in \mathrm{Iso}(F_1, E_1)$. Moreover, $\dim F_0 = m + l$.

Consider the projection $\pi\colon F \to F_1$ given by

$$\pi(f_0 + f_1) = f_1.$$

Since F_1 and F_0 are closed and complementary it follows that obviously the map $\kappa = \mathrm{Id}_F - \pi$ is the unique projection with $\mathrm{Img}\,(\kappa) = F_0$ and $\ker(\kappa) = F_1$. Let $\pi_1\colon E \to E_0$ be the projection given by $\pi_1(e_0 + e_1) = e_0$. Then, $\Pi := \Delta^{-1} \circ \pi_1 \circ D\mathbf{f}(0)$ is a projection with $\mathrm{Img}\,(\Pi) = F_0$ and $F_1 \subseteq \ker(\Pi)$. Since $F = F_0 \oplus F_1$, it follows that $F_1 = \ker(\Pi)$ and therefore $\Pi = \kappa$.

Now define the map $H\colon \varphi(V) \to F$ of class MC^k by

$$H(x) = \pi(x) + \Delta^{-1} \circ \pi_1 \circ \mathbf{f}(x).$$

We obtain that $H(0) = 0$ and

$$DH(0) = \pi + \Delta^{-1} \circ \pi_1 \circ D\mathbf{f}(0) = \pi + \kappa = \mathrm{Id}_F.$$

If we choose V small enough, then by Theorem 4.1 H is an MC^k-diffeomorphism onto an open neighborhood $\mathcal{U} \subseteq \psi(U)$ of $\psi(f(x_0)) = (0,0)$. Let $\Phi = H \circ \varphi^{-1}$, then (Φ, F_0) is a chart for x_0 on V with the submanifold property. Because we have

$$\begin{aligned} x \in f^{-1}(\mathcal{A}) \;\Leftrightarrow\; \psi(f(x)) \in U_1 \oplus \{0\} \;&\Leftrightarrow\; \mathbf{f}(\varphi(x)) \in U_1 \oplus \{0\} \\ &\Leftrightarrow\; H(\varphi(x)) \in F_0. \end{aligned}$$

Let $p \in \mathcal{A}$, $\gamma \colon \mathbb{R} \to M$ a smooth curve sending zero to p, and $j_p^1 \gamma$ the 1-jet of γ at p.

$$
\begin{aligned}
j_p^1 \gamma \in T_p \mathcal{A} &\Leftrightarrow j_{\varphi(p)}^1 (\varphi \circ \gamma) = T\varphi(j_p^1 \gamma) \in \varphi(V) \times F, \ \varphi \circ \gamma \subset \varphi(V) \\
&\Leftrightarrow T\mathbf{f}(j_{\varphi(p)}^1 (\varphi \circ \gamma)) \in \psi(U) \times E \\
&\Leftrightarrow \mathbf{f}(\varphi \circ \gamma) = \psi(f \circ \gamma) \subset \psi(U) \\
&\Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t} \psi(f \circ \gamma) \mid_{t=0} = v = \frac{\mathrm{d}}{\mathrm{d}t} \psi([\psi^{-1}(\psi(f(x)) + tv)]), \\
&\quad \ \ \psi(f \circ \gamma) \subset \psi(U) \\
&\Leftrightarrow j_{f(p)}^1 (f \circ \gamma) = j_p^1 [\psi^{-1}(\psi(f(p)) + tv)] \in T_{f(p)} \mathcal{A} \\
&\Leftrightarrow j_p^1 \gamma \in (T_p f)^{-1}(T_{f(p)} \mathcal{A})
\end{aligned}
$$

This proves the second assertion. $\qquad\square$

If manifolds have nonempty boundary we just need to modify the proof by extending the considered maps.

Corollary 4.3. *Let $f \colon M \to N$ be an MC^k-Lipschitz-Fredholm map of index l. If y is a regular value of f, then the level set $f^{-1}(y)$ is a submanifold of dimension l and its tangent space at x is $\ker T_x f$.*

Proof. The set $\{y\}$ is transversal to f so the result follows from the theorem. $\qquad\square$

Corollary 4.4. *Let $f \colon M \times N \to O$ be a smooth Lipschitz-Fredholm map of manifolds, we write $f_x := f(\cdot, x)$, and let $\mathcal{A}$ be a closed finite dimension submanifold of O. Assume that $f \pitchfork \mathcal{A}$ and for all $(m, n) \in f_n^{-1}(\mathcal{A})$ the composition*

$$
(T_m M \xrightarrow{\mathrm{D} f_n(m)} T_{f_n(m)} O \xrightarrow{Q} T_{f_n(m)} O / T_n S)
$$

is Lipschitz-Fredholm. Then there is a residual set of n in O for which the map $f_n \colon M \to O$ is transversal to $\mathcal{A}$.

Proof. By hypothesis the kernel of $Q \circ \mathrm{D} f(x)$ is complemented for all $x \in f^{-1}(\mathcal{A})$. By the preceding theorem $B := f^{-1}(\mathcal{A})$ is a Fréchet submanifold. The map $f \mid_B$ is smooth Lipschitz-Fredholm,

therefore by Sard's theorem there is a residual set of regular values of it in O. If $n \in N$ is a regular value of $f \mid_B$, then f_n is transversal to $\mathcal{A}$. $\square$

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