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Two-point boundary value problems for differential-operator equations

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We study general two-point boundary value problems for a non-homogeneous differential-operator equation of the second order with an unbounded linear operator in a Banach space. The main classical solvability condition is given in terms of the property of the resolvent of the operator at the points, which are opposite to the eigenvalues of the corresponding ordinary differential operator. At the end of the paper, two particular types of boundary value conditions are treated: periodic and Dirichlet.

1. Introduction

In this paper, we study general two-point boundary value problems for a non-homogeneous differential-operator equation of the second order with an unbounded linear operator in a Banach space. We impose some restrictions on the right-hand side of the equation. Then, we formulate the conditions of the unique solvability of the problems in terms of the property of the resolvent of the operator at the points, which are opposite to the eigenvalues of the corresponding ordinary differential operator. In fact, we find sufficient conditions for the unique solvability, but some of them are also necessary. At the end of the paper, two particular types of boundary value conditions are treated: periodic and Dirichlet.

We represent the solutions of the problems as a series of vector-valued functions which include the eigenfunctions of the correspondi-ng ordinary boundary value problem. To obtain the convergence of the series, we essentially use the Abel transform of the series. Here we use an approach suggested by A. V. Knyazyuk in the paper [8] for the study of the model Dirichlet problem.

In the paper, we study classical solutions of the problems. The solutions from the L_p spaces and from the Hölder spaces have been studied by V. Arendt and S. Bu in [1] by using of the technique of Fourier series and Marcinkiewicz multipliers. The types of boundary value conditions, which are covered by our results, are essentially wider than those in the above mentioned papers. In particular, our paper contains a generalization of the main result in [8].

Problems in L_p spaces, with rather general non-local boundary value conditions (multipoint, integro-differential, functional), and for higher order abstract differential-operator equations, have been studied in a series of papers by A. Favini and Ya. Yakubov [2]-[5] (paper [2] is joint with V. Shakhmurov). For the results in the framework of Hilbert spaces, we refer the reader to the monograph by S. Yakubov and Ya. Yakubov [9] and reference therein.

Solvability of problems of the form (1)-(2), in some weighted Hölder spaces, has been studied by L. M. Gershtein and P. E. Sobolevskii in [6]. The main solvability condition is given in some implicit form. Moreover, in contrast to our case, the operator A in [6] is assumed to be bounded weakly positive with a compact inverse operator A^{-1}. In [7], the authors continue their study for problems from [6] but with a non-constant operator $A = A(t)$. Some additional restrictions on $A(t)$ are applied.

2. The statement of the problem

In a Banach space X, we consider the differential-operator equation

$$\frac{d^2v}{dt^2} = Av + f(t), \quad 0 < t < T \tag{1}$$

with the boundary value conditions

$$\begin{cases} L_1(v) := \alpha_{11}v(0) + \alpha_{12}v'(0) + \beta_{11}v(T) + \beta_{12}v'(T) = 0, \\ L_2(v) := \alpha_{21}v(0) + \alpha_{22}v'(0) + \beta_{21}v(T) + \beta_{22}v'(T) = 0. \end{cases} \tag{2}$$

Here, A ia a closed linear unbounded operator with domain $D(A)$, $f(t)$ is a continuous on $[0, T]$ vector valued function, the coefficients α_{ij} and β_{ij} are complex numbers. It is assumed that the forms $L_1(v), L_2(v)$ are linearly independent. By a solution of the problem (1), (2) we mean a continuously differentiable on $[0, T]$ function $v(t)$ which takes the values in $D(A)$, has a continuous on $(0, T)$ second order derivative and satisfies (1), (2).

We use the eigenfunctions and eigenvalues of the ordinary differential operator of the second order defined by

$$\begin{gathered} L(y)(t) = -\frac{d^2y}{dt^2}, \quad 0 < t < T, \\ L_1(y) = 0, \quad L_2(y) = 0. \end{gathered} \tag{3}$$

The operator L is defined on continuously differentiable on $[0, T]$ functions $y(t)$, which have a continuous on $(0, T)$ second order derivative $y''(t)$, with $y''(t) \in L_2(0, T)$, satisfying the boundary value conditions (2).

3. The conditions

Here, we present the main conditions on the data of the problem which are used in the paper. Below, C means a positive constant. The conditions on the ordinary differential operator L in (3) are the following:

(α_1) The operator L, treated as an operator in $L_2(0,T)$, is a symmetric operator.

(α_2) The operator L has a complete, in $L_2(0,T)$, orthonormal system $\varphi_n(t),\ n=1,2,\dots$ of eigenfunctions.

(α_3) The eigenvalues $\lambda_n,\ n=1,2,\dots$ of the operator L satisfy the relation

$$\liminf_{n\to\infty}\frac{|\lambda_n|}{n^2}>0.$$

(α_4) There is a sequence of numbers $s_n,\ n=1,2,\dots$ such that $\lim_{n\to\infty}s_n=\infty$ and

$$|s_n-s_{n+1}|\le C,\ \left|\frac{\lambda_n}{s_n}\right|\le Cn,\ \left|\frac{\lambda_{n+1}}{s_{n+1}}-\frac{\lambda_n}{s_n}\right|\le C,\quad n=1,2,\dots.$$

Note that the typical cases for condition (α_4) are $\lambda_n=s_n^2$ or $\lambda_n=-s_n^2$.

The common property of the operators $-A$ and L is that their spectra are disjoint with the following additional condition:

(β) All the numbers $-\lambda_n,\ n=1,2,\dots$ are regular points of the operator A and, moreover, there exists a constant $M>0$ such that the inequalities

$$\|\lambda_n(A+\lambda_nI)^{-1}\|\le M,\quad n=1,2,\dots$$

hold.

To formulate the conditions on the function $f(t)$ in the right hand side in (1), we define the "Fourier coefficients"

$$f_n=\int_0^T f(t)\overline{\varphi_n(t)}\,dt,\quad n=1,2,\dots. \tag{1}$$

The conditions are the following:

(γ) The inequalities

$$\|f_n\varphi_n(t)\|\le C,\ \|f_n\varphi_n'(t)\|\le C,\quad n=1,2,\dots,\ 0\le t\le T \tag{2}$$

and

$$\| \sum_{k=1}^{n} s_k f_k \varphi_k(t) \| \le C(\delta, \gamma), \quad n = 1, 2, \ldots, \ \delta \le t \le \gamma \tag{3}$$

for any δ, γ, with $0 < \delta < \gamma < T$, and with s_i defined in the condition (α_4), hold.

In particular cases, presented in two last sections, the condition (γ) turns out to be valid if a more explicit condition holds:

(γ_0) The function $f(t)$ has, on $[0, T]$, the derivative which satisfies the Hölder condition, i.e., for some $C > 0$, $0 < \alpha \le 1$,

$$\|f'(t) - f'(s)\| \le C|t - s|^{\alpha}, \ \forall t, s \in [0, T].$$

Let $v(t)$ be a solution of the problem (1), (2) from section 2, if it exists. Set

$$v_n = \int_0^T v(t) \varphi_n(t) \, dt, \quad n = 1, 2, \ldots \tag{4}$$

The following lemma yields the connection between the "Fourier coefficients" of the function $f(t)$ in the right hand side of the equation and of the solution $v(t)$.

Lemma 1. *Assume that the condition (α_1) holds and that $v(t)$ is a solution of the problem (1), (2) from section 2 with the given function $f(t)$.*

Then, the equalities

$$(\lambda_n I + A) v_n = -f_n, \quad n = 1, 2, \ldots \tag{5}$$

hold with v_n, f_n as in (1), (4), and λ_n to be the eigenvalues of the operator L.

Proof. Multiplying the equality (1) from section 2 by $\varphi_n(t)$ and integrating from 0 to T we get

$$\int_0^T \frac{d^2 v}{dt^2} \varphi_n(t) \, dt = A v_n + f_n, \quad n = 1, 2, \ldots.$$

Integrating by parts in the left hand side and using the condition (α_1) and the fact that $\varphi_n(t)$ is an eigenfunction of the operator L with the eigenvalue λ_n, we obtain (5). ⋄

4. The uniqueness theorem

At first, we consider the uniqueness criteria for the problem (1), (2) from section 2, i.e., for the homogeneous problem

$$\begin{aligned} &\frac{d^2v}{dt^2} = Av, \quad 0 < t < T, \\ &L_1(v) = 0, \quad L_2(v) = 0. \end{aligned} \tag{1}$$

Theorem 2. *Assume that the conditions (α_1), (α_2) hold.*

The problem (1) has only a trivial solution if and only if each eigenvalue of the operator L in (3) from section 2 is not an eigenvalue of the operator $-A$.

Proof. Assume that λ is a common eigenvalue of the operators $-A$ and L with the corresponding eigenvector $g \in X$ of A and eigenfunction $\varphi(t)$ of L. Then, the function $v(t) = \varphi(t)g$ is a nontrivial solution of the problem (1).

Assume now that the sets of eigenvalues of the operators L and $-A$ are disjoint. Let $v(t)$ be a solution of the problem (1). Using Lemma 1, we get

$$-\lambda_n v_n = Av_n, \quad n = 1, 2, \dots .$$

Since the operators $\lambda_n I + A$, $n = 1, 2, \dots$ are injective, we get $v_n = 0$, $n = 1, 2, \dots$ and, therefore, since the system $\{\varphi_n(t)\}$ is complete, we conclude that $v(t) = 0$, $0 \le t \le T$. ⋄

5. The existence and uniqueness theorems

In this section, we present the basic results of the paper. First, we show that the existence of the unique solution of the problem for any admissible function $f(t)$ implies that all the points λ_n, defined above, are regular points of the operator $-A$.

Theorem 3. *Assume that the conditions (α_1), (α_2) hold and the problem (1), (2) from section 2 has a unique solution for any continuous $f(t)$ satisfying the condition (γ).*

Then, every eigenvalue $\lambda_n,\ n = 1, 2, \dots$ of the operator L is a regular point of the operator $-A$.

Proof. For any positive integer m and for any $x \in X$ take $f(t) = \varphi_m(t)x$. Let $v(t)$ be the corresponding solution of the problem. Consider the expantion of $v(t)$ in the form

$$v(t) = \sum_{n=1}^{\infty} v_n \varphi_n(t) \tag{1}$$

with coefficients v_n defined by (4) from section 3. Using Lemma 1, we obtain the equalities (5) from section 3 with $f_m = x$ and $f_n = 0,\ n \neq m$. From the uniqueness of the solution, using Theorem 2, we conclude that all the numbers $\lambda_n,\ n = 1, 2, \dots$ are not eigenvalues of the operator $-A$. Hence, we get $v_n = 0,\ n \neq m$ and, moreover, the equation

$$(\lambda_m I + A)v_m = -x$$

has a unique solution v_m. Since this holds for any $x \in X$ we conclude that $-\lambda_m$ is a regular point of the operator A and $v_m = -(\lambda_m I + A)^{-1}x$. Hence, it follows that $v(t) = v_m\varphi_m(t) = -(\lambda_m I + A)^{-1}x\varphi_m(t)$. ◇

We prove now, under some additional conditions on the resolvent of the operator A at the points $-\lambda_n$, the unique solvability of the problem.

Theorem 4. *Assume that the conditions (α_1)-(α_4) and (β) hold.*

Then, for any continuous $f(t)$, satisfying the condition (γ), the problem (1), (2) from section 2 has a unique solution.

Proof. Take any $f(t)$ satisfying (γ). We consider the expression

$$v(t) = -\sum_{n=1}^{\infty} (\lambda_n I + A)^{-1} f_n \varphi_n(t), \quad 0 \leq t \leq T \tag{2}$$

with f_n defined in (1) from section 3, and prove that $v(t)$ is the unique solution of the problem (1), (2) from section 2. Formally differentiating, we get

$$v'(t) = -\sum_{n=1}^{\infty} (\lambda_n I + A)^{-1} f_n \varphi_n'(t), \quad 0 \leq t \leq T. \tag{3}$$

The conditions (α_3) and (β) and the inequalities in (2) from section 3 imply that the series in (2), (3) converge uniformly on $[0, T]$. Hence, it follows that $v(t)$ is a continuously differentiable on $[0, T]$ function with the derivative defined in (3). Moreover, since $\varphi_n(t)$ satisfy the boundary value conditions in (3) from section 2, the formulas (2), (3) imply that the function $v(t)$ satisfy the boundary value conditions (2) from section 2.

Formally differentiating (2) twice and using the equality $\varphi_n''(t) = -\lambda_n \varphi_n(t)$, we get

$$v''(t) = \sum_{n=1}^{\infty} \lambda_n (\lambda_n I + A)^{-1} f_n \varphi_n(t). \tag{4}$$

We now prove that the series in (4) converges uniformly on $[\delta, \gamma]$ for any $\delta, \gamma \in (0, T),\ \delta < \gamma$. This will imply that the formula (4) yields the second derivative of the function $v(t)$ in the interval $(0, T)$.

Set

$$a_n = \frac{\lambda_n}{s_n} (\lambda_n I + A)^{-1}, \quad n = 1, 2, \ldots.$$

Then, the series in (4) has the form $\sum_{n=1}^{\infty} a_n s_n f_n \varphi_n(t)$. For any $\delta, \gamma \in (0, T),\ \delta < \gamma$ we prove the uniform convergence of this series, for $\delta \leq t \leq \gamma$, using the Abel transform. To this end, we should check that

$$\lim_{n\to\infty}(s_1 f_1\varphi_1(t) + s_2 f_2\varphi_2(t) + \cdots + s_n f_n\varphi_n(t))a_n = 0, \quad \delta \le t \le \gamma. \tag{5}$$

Indeed, by virtue of the condition (β), we get

$$\|a_n\| \le \frac{C}{|s_n|}, \quad n = 1, 2, \dots,$$

which, together with the condition (3) from section 3 and the condition $\lim_{n\to\infty} s_n = \infty$, implies (5). Thus, the Abel transform implies that the uniform, on $[\delta, \gamma]$, convergence of the series

$$\sum_{n=1}^{\infty} a_n s_n f_n \varphi_n(t)$$

follows from the uniform convergence of the series

$$\sum_{n=1}^{\infty}(s_1 f_1\varphi_1(t) + s_2 f_2\varphi_2(t) + \cdots + s_n f_n\varphi_n(t))(a_n - a_{n+1}) \tag{6}$$

on the same segment. So, check the uniform convergence of (6). We have

$$a_n - a_{n+1} = \frac{\lambda_n}{s_n}(\lambda_n I + A)^{-1} - \frac{\lambda_{n+1}}{s_{n+1}}(\lambda_{n+1} I + A)^{-1}.$$

Hence, it follows that

$$\begin{aligned} a_n - a_{n+1} &= (\lambda_n I + A)^{-1}\left(\frac{\lambda_n}{s_n} - \frac{\lambda_{n+1}}{s_{n+1}}\right) \\ &\quad + \frac{\lambda_{n+1}}{s_{n+1}}((\lambda_n I + A)^{-1} - (\lambda_{n+1} I + A)^{-1}). \end{aligned}$$

The resolvent identity yields

$$(\lambda_n I + A)^{-1} - (\lambda_{n+1} I + A)^{-1} = (\lambda_{n+1} - \lambda_n)(\lambda_n I + A)^{-1}(\lambda_{n+1} I + A)^{-1}.$$

Substituting this into the previous equality, we get

$$a_n - a_{n+1} = (\lambda_n I + A)^{-1}\left(\frac{\lambda_n}{s_n} - \frac{\lambda_{n+1}}{s_{n+1}}\right) + (\lambda_{n+1} - \lambda_n)(\lambda_n I + A)^{-1}(\lambda_{n+1} I + A)^{-1}\frac{\lambda_{n+1}}{s_{n+1}}. \qquad (7)$$

Using (β) and (α_3), (α_4), and also the equality

$$\lambda_{n+1} - \lambda_n = s_n\left(\frac{\lambda_{n+1}}{s_{n+1}} - \frac{\lambda_n}{s_n}\right) + \frac{\lambda_{n+1}}{s_{n+1}}(s_{n+1} - s_n),$$

we get, from (7),

$$\|a_n - a_{n+1}\| \leq \frac{C}{n^2}.$$

Hence, using (3) from section 3, we conclude that the series (6) converges uniformly on t in $[\delta, \gamma]$. Thus, $v(t)$ has a continuous, on $(0, T)$, second derivative which is defined by (4).

Now, using the formula

$$A(\lambda_n I + A)^{-1} = I - \lambda_n(\lambda_n I + A)^{-1},$$

we get, from (2), for $0 < t < T$,

$$Av(t) = \sum_{n=1}^{\infty} \lambda_n(\lambda_n I + A)^{-1} f_n \varphi_n(t) - \sum_{n=1}^{\infty} f_n \varphi_n(t) = v''(t) - f(t),$$

which implies that $v(t)$ is a solution of the equation (1) from section 2. ◇

6. The periodic boundary value conditions

Consider a problem of the form (1), (2) from section 2 with periodic boundary value conditions

$$\begin{gathered} \frac{d^2 v}{dt^2} = Av + f(t), \quad 0 < t < 2\pi, \\ v(0) - v(2\pi) = 0, \quad v'(0) - v'(2\pi) = 0. \end{gathered} \qquad (1)$$

The corresponding operator L in (3) from section 2 has a complete, in $L_2(0, 2\pi)$, orthonormal system of eigenfunctions $\varphi_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$, $n = 0, \pm 1, \pm 2, \dots$ with eigenvalues $\lambda_n = n^2$. The sequence s_n in (α_4) is defined by $s_n = in$, $n = 0, \pm 1, \pm 2, \dots$.

Assume that the operator A satisfies the condition (β). This means that the numbers $-n^2$, $n = 0, 1, 2, \dots$ are regular points of the operator A and the inequalities

$$\|n^2 (A + n^2 I)^{-1}\| \le M, \quad n = 0, 1, 2, \dots \tag{2}$$

hold.

Assume that the function $f(t)$ in (1) satisfies the condition (γ_0). We check that the condition (γ) holds. The inequalities

$$\|f_n \varphi_n(t)\| \le C, \quad n = 0, \pm 1, \pm 2, \dots, \ 0 \le t \le 2\pi$$

are obvious. Integrating by parts in (1) from section 3, we get

$$f_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t) e^{-int}\, dt = \frac{1}{\sqrt{2\pi}} \Big(i \frac{f(2\pi) - f(0)}{n} + \frac{1}{in} \int_0^{2\pi} f'(t) e^{-int}\, dt \Big). \tag{3}$$

$>$From here, using the fact that the functions $f(t)$ and $f'(t)$ are bounded, we obtain the inequalities

$$\|f_n \varphi_n'(t)\| \le C, \quad n = 0, \pm 1, \pm 2, \dots, \ 0 \le t \le 2\pi.$$

So, (2) from section 3 has been proved. It remains to check (3) from section 3, i.e.,

$$\Big\| \sum_{k=-n}^{n} ik f_k e^{ikt} \Big\| \le C, \quad n = 0, 1, 2, \dots, \quad 0 < \delta \le t \le \gamma < 2\pi. \tag{4}$$

Set $g(t) = f'(t)$. The Fourier coefficients of the function $g(t)$ are

$$g_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f'(t) e^{-ikt}\, dt, \quad k = 0, \pm 1, \pm 2, \dots.$$

The formula (3) implies

$$ikf_k = g_k - \frac{f(2\pi) - f(0)}{\sqrt{2\pi}}, \quad k = 0, \pm 1, \pm 2, \ldots$$

and, therefore,

$$\sum_{k=-n}^{n} ikf_k e^{ikt} = \sum_{k=-n}^{n} g_k e^{ikt} + \frac{f(0) - f(2\pi)}{\sqrt{2\pi}} \sum_{k=-n}^{n} e^{ikt}, \quad n = 0, 1, 2, \ldots.$$

or

$$\| \sum_{k=-n}^{n} ikf_k e^{ikt} \| \leq \| \sum_{k=-n}^{n} g_k e^{ikt} \| + \frac{\|f(0) - f(2\pi)\|}{\sqrt{2\pi}} \left| \sum_{k=-n}^{n} e^{ikt} \right|. \quad (5)$$

The condition (γ_0) implies that the Fourier series of the function $g(t)$ converges to $g(t)$, uniformly on any $[\delta, \gamma] \subset (0, 2\pi)$. Hence,

$$\left\| \sum_{k=-n}^{n} g_k e^{ikt} \right\| \leq C, \quad 0 < \delta \leq t \leq \gamma < 2\pi, \; n = 0, 1, 2, \ldots. \quad (6)$$

Further, we have

$$\sum_{k=-n}^{n} e^{ikt} = \frac{\sin \frac{(2n+1)t}{2}}{\sin \frac{t}{2}}$$

and, therefore,

$$\left| \sum_{k=-n}^{n} e^{ikt} \right| \leq C, \quad 0 < \delta \leq t \leq \gamma < 2\pi. \quad (7)$$

>From the relations (5)-(7), the relations in (4) follow.

Thus, for any operator A, satisfying the condition (2), and for any $f(t)$, satisfying the condition (γ_0), the problem (1), by Theorem 4, has a unique solution.

7. The Dirichlet boundary value conditions

Consider now a problem of the form (1), (2) from section 2 with Dirichlet boundary value conditions

$$\begin{gathered}\frac{d^2v}{dt^2} = Av + f(t), \quad 0 < t < \pi, \\ v(0) = 0, \quad v(\pi) = 0.\end{gathered} \tag{1}$$

The corresponding operator L in (3) from section 2 has a complete, in $L_2(0,\pi)$, orthonormal system of eigenfunctions $\varphi_n(t) = \sqrt{\frac{2}{\pi}}\sin nt,\ n = 1, 2, \ldots$ with eigenvalues $\lambda_n = n^2$. The sequence s_n in (α_4) is defined by $s_n = n,\ n = 1, 2, \ldots$.

Assume that the operator A satisfies the condition (β). This means that the numbers $-n^2,\ n = 1, 2, \ldots$ are regular points of the operator A and the inequalities

$$\|n^2(A + n^2 I)^{-1}\| \le M, \quad n = 1, 2, \ldots \tag{2}$$

hold.

Assume that the function $f(t)$ in (1) satisfies the condition (γ_0). We check that the condition (γ) holds. Integrating by parts in (1) from section 3, we get

$$\begin{aligned}f_n = \sqrt{\frac{2}{\pi}}\int_0^\pi f(t)\sin nt\,dt = \sqrt{\frac{2}{\pi}}\Big(&\frac{-f(\pi)\cos(n\pi) + f(0)}{n} \\ &+ \frac{1}{n}\int_0^\pi f'(t)\cos nt\,dt\Big).\end{aligned} \tag{3}$$

>From (3), using the fact that the functions $f(t)$ and $f'(t)$ are bounded, we obtain (2) from section 3. It remains to check (3) from section 3, i.e.,

$$\left\|\sum_{k=1}^n k f_k \varphi_k(t)\right\| \le C, \quad n = 1, 2, \ldots, \quad 0 < \delta \le t \le \gamma < \pi. \tag{4}$$

The relation (3) implies that

$$\sum_{k=1}^{n} k f_k \varphi_k(t) = \frac{2}{\pi} \sum_{k=1}^{n} \Big(f(0) - f(\pi) \cos k\pi + \int_0^{\pi} f'(s) \cos ks \; ds \Big) \sin kt,$$

i.e.,

$$\sum_{k=1}^{n} k f_k \varphi_k(t) = \frac{2}{\pi} \sum_{k=1}^{n} f(0) \sin kt + \frac{2}{\pi} \sum_{k=1}^{n} f(\pi)(-1)^{k+1} \sin kt + \frac{2}{\pi} \int_0^{\pi} f'(s) \sum_{k=1}^{n} \sin kt \cos ks \; ds. \tag{5}$$

Set

$$D_n(t) = \sum_{k=1}^{n} \sin kt. \tag{6}$$

The formula

$$\sum_{k=1}^{n} \sin kt = \frac{h_n(t)}{\sin \frac{t}{2}}, \tag{7}$$

with $h_n(t) = \sin \frac{n+1}{2} t \sin \frac{nt}{2}$, implies

$$|D_n(t)| \leq \frac{C}{|\sin \frac{t}{2}|}. \tag{8}$$

The first entry in (5) has the form

$$\frac{2}{\pi} \sum_{k=1}^{n} f(0) \sin kt = \frac{2}{\pi} f(0) D_n(t)$$

and, hence, the uniform boundedness of this entry on $t \in [\delta, \gamma] \subset (0, \pi)$, $n = 1, 2, \ldots$ follows from (8). The second entry in (5) has the form

$$\frac{2}{\pi}\sum_{k=1}^{n} f(\pi)(-1)^{k+1}\sin kt = -\frac{2}{\pi}f(\pi)\sum_{k=1}^{n}\sin(kt+k\pi)$$
$$= -\frac{2}{\pi}f(\pi)D_n(t+\pi)$$

and, using (8), we get

$$\|\frac{2}{\pi}\sum_{k=1}^{n} f(\pi)(-1)^{k+1}\sin kt\| \leq \frac{C}{|\sin\frac{t+\pi}{2}|} = \frac{C}{|\cos\frac{t}{2}|}$$

and, hence, the uniform boundedness of this entry on $t \in [\delta, \gamma],\ n = 1, 2, \dots$ follows. Finally, consider the third term in (5). We have

$$\frac{2}{\pi}\int_0^\pi f'(s)\sum_{k=1}^{n}\sin kt\cos ks\ ds = \frac{1}{\pi}\int_0^\pi f'(s)D_n(t+s)\ ds$$
$$+\frac{1}{\pi}\int_0^\pi f'(s)D_n(t-s)\ ds. \qquad (9)$$

Using formula (7), we get

$$D_n(t+s) = \frac{h_n(t+s)}{\sin\frac{t+s}{2}}.$$

Here, it is clear that $|h_n(t+s)| \leq 1$ and that

$$1/|\sin\frac{t+s}{2}| \leq C, \quad 0 \leq s \leq \pi,\ 0 < \delta \leq t \leq \gamma < \pi.$$

Since $f'(t)$ is a bounded function, we conclude that the first term in (9) is uniformly bounded on $t \in [\delta, \gamma],\ n = 1, 2, \dots$. Now, consider the second term. We have

$$\frac{1}{\pi}\int_0^\pi f'(s)D_n(t-s)\ ds = \frac{1}{\pi}\int_0^\pi (f'(s)-f'(t))D_n(t-s)\ ds$$
$$+\frac{1}{\pi}f'(t)\int_0^\pi D_n(t-s)\ ds. \qquad (10)$$

By the condition (γ_0), we have

$$\|f'(s) - f'(t)\| \leq C|t-s|^{\alpha}, \ \forall t, s \in [0, \pi].$$

Using (8), we get

$$|D_n(t-s)| \leq \frac{C}{|\frac{t-s}{2}|}, \quad t \neq s.$$

Thus, we conclude that

$$\|\frac{1}{\pi}\int_0^{\pi}(f'(s) - f'(t))D_n(t-s)\, ds\| \leq C.$$

Since $f'(t)$ is bounded, it is enough to check the boundedness of the last integral in (10). Using (6), we get

$$\begin{aligned}\int_0^{\pi} D_n(t-s)\, ds &= \sum_{k=1}^{n}\int_0^{\pi}\sin k(t-s)\, ds \\ &= \sum_{k=1}^{n}\frac{\cos(kt-k\pi) - \cos kt}{k} \\ &= -2\sum_{m=0}^{p}\frac{\cos(2m+1)t}{2m+1}.\end{aligned}$$

On the other hand, $\sum_{m=0}^{n-1}\cos(2m+1)t = \frac{\sin 2nt}{2\sin t}$. Then, $\lim_{n\to\infty}\sum_{m=0}^{n-1}\cos(2m+1)t\cdot\frac{1}{2n-1} = 0$, $0 < \delta \leq t \leq \gamma < \pi$. Therefore, by the Abel transform, the uniform convergence of $\sum_{m=0}^{\infty}\frac{\cos(2m+1)t}{2m+1}$, $0 < \delta \leq t \leq \gamma < \pi$, follows from uniform convergence (on the same segment) of

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\cos(2m+1)t\cdot\Big(\frac{1}{2n-1} - \frac{1}{2n+1}\Big) = \sum_{n=1}^{\infty}\frac{\sin 2nt}{(2n-1)(2n+1)\sin t}$$

which is true by the Weierstrass M-test. Thus, we conclude that

$$\|\int_0^{\pi} D_n(t-s)\, ds\| \leq C, \quad n = 1, 2, \ldots, \ 0 < \delta \leq t \leq \gamma < \pi,$$

which completes the proof of (4).

So, for any operator A, satisfying the condition (2), and for any $f(t)$, satisfying the condition (γ_0), the problem (1), by Theorem 4, has a unique solution.

Consider now the Dirichlet problem for the homogeneous equation with non-homogeneous boundary value conditions

$$\begin{aligned} &\frac{d^2v}{dt^2} = Av, \quad 0 < t < \pi, \\ &v(0) = x_0, \quad v(\pi) = x_1 \end{aligned} \tag{11}$$

with $x_0, x_1 \in D(A)$. By a standard way, the problem (11) is reduced to the problem (1) with $f(t) = \frac{\pi-t}{\pi}Ax_0 + \frac{t}{\pi}Ax_1$. Obviously, the linear function $f(t)$ satisfies condition (γ_0).

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