Збірник праць Інституту математики НАН України 2016, т. 13, № 3, 293–312

УДК 517.983.27

Nonlinear approximation of the classes $\mathcal{F}^{\psi}_{q,r}$ of functions of several variables in the integral metrics

A. L. Shidlich[1*]

[1] *Institute of Mathematics of NAS of Ukraine, Kyiv; shidlich@gmail.com, andy709@list.ru*

У роботі знайдено точні порядкові оцінки нелінійних апроксимативних характеристик (таких як найкраще m-членне тригонометричне наближення, найкраще m-членне ортогональне тригонометричне наближення, наближення m-членними гріді поліномами) класів $\mathcal{F}^{\psi}_{q,r}$ функцій багатьох змінних у інтегральній метриці.

В работе найдены точные порядковые оценки нелинейных аппроксимативных характеристик (таких как лучше m–членне тригонометрическое приближения, лучше m–членне ортогональное тригонометрическое приближения, приближение m –членнимы гриди полиномами) классов $\mathcal{F}^{\psi}_{q,r}$ функций многих переменных в интегральной метрике.

Introduction

Let d be a fixed natural number, let $\mathbb{R}^d$ and $\mathbb{Z}^d$ be the sets of all ordered collections $k := (k_1, \ldots, k_d)$ of d real and integer numbers correspondingly. Let also $\mathbb{T}^d := [0, 2\pi]^d$ denote d-dimensional torus.

Further, let $L_p := L_p(\mathbb{T}^d)$, $1 \le p \le \infty$, be the space of all Lebesgue-measurable on $\mathbb{R}^d$ 2π-periodic in each variable functions f with finite norm

$$||f||_{L_p} := \begin{cases} \left((2\pi)^{-d}\int_{\mathbb{T}^d} |f(x)|^p dx\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{T}^d} |f(x)|, & p = \infty. \end{cases}$$

*The present investigation was supported, in part, by the FP7-People-2011-IRSES project number 295164 (EUMLS: EUUkrainian Mathematicians for Life Sciences).

Set $(k,x) := k_1x_1+k_2x_2+\ldots+k_dx_d$, $e_k(x) := e^{i(k,x)}$ and for any $f\in L_1$, we denote the Fourier coefficients of f by

$$\widehat{f}(k) := (2\pi)^{-d}\int_{\mathbb{T}^d} f(x)\overline{e}_k(x)dx, \quad k\in\mathbb{Z}^d,$$

where $\overline{z}$ is the complex conjugate of z.

The space $S^p := S^p(\mathbb{T}^d)$, $0<p<\infty$, (see, for example, [5] (Ch. XI)) is the space of all functions $f\in L_1$ such that

$$||f||_{S^p} := ||\{|\widehat{f}(k)|\}_{k\in\mathbb{Z}^d}||_{l_p(\mathbb{Z}^d)} = \bigg(\sum_{k\in\mathbb{Z}^d}|\widehat{f}(k)|^p\bigg)^{\frac{1}{p}} < \infty. \qquad (1)$$

The functions $f\in L_1$ and $g\in L_1$ are equivalent in the space S^p, when $\|f-g\|_{S^p}=0$.

We denote by l_p^d, $0<p\le\infty$, the space $\mathbb{R}^d$ equipped with l_p-(quasi-)norm that is defined for $x=\{x_i\}_{i=1}^d\in\mathbb{R}^d$ by

$$|x|_p := ||x||_{l_p} = \begin{cases} \Big(\sum_{i=1}^d |x_i|^p\Big)^{\frac{1}{p}}, & 0<p<\infty,\\ \sup_{1\le i\le d}|x_i|, & p=\infty. \end{cases}$$

Let also $\psi=\psi(t)$, $t\ge 1$, be a positive decreasing function, $\psi(0):=\psi(1)$ and $0<q,r\le\infty$.

We investigate asymptotical behavior of some important approximative characteristics (in the sense of order estimates) of the classes of functions of several variables $\mathcal{F}^{\psi}_{q,r}$, defined by the following equality:

$$\mathcal{F}^{\psi}_{q,r} := \bigg\{ f\in L_1 \;\;:\;\; ||\{|\widehat{f}(k)|/\psi(|k|_r)\}_{k\in\mathbb{Z}^d}||_{l_q(\mathbb{Z}^d)} \le 1 \bigg\}.$$

If $\psi(t)=t^{-s}$, $s\in\mathbb{N}$ and $r=\infty$, then $\mathcal{F}^{\psi}_{q,\infty} =: \mathcal{F}^{s}_{q,\infty}$ is a set of functions whose sth partial derivatives have absolutely convergent Fourier series. When $q=2$, $\mathcal{F}^{s}_{q,\infty}$ is equivalent (modulo constants) to the unit ball of the Sobolev class W^s_2.

Approximative characteristics of the classes $\mathcal{F}^{\psi}_{q,r}$ for different $r\in(0,\infty]$ and for the various functions ψ were investigated by many authors (see, for example, [1]– [5]). In particular, in [1], the authors found the exact order estimates of the quantities of the best m-term trigonometric approximations of the classes $\mathcal{F}^{s}_{q,\infty}$, $s>0$, in the spaces L_p. Temlyakov [2] obtained the exact order estimates of approximations of these classes by

m-term greedy polynomials in L_p. In the case where $\psi(t)$ is a positive function that decreases to zero no faster than some power function, the quantities of the best m-term one-sided trigonometric approximations and the quantities of approximations by m-term one-sided Greedy-liked polynomials of the classes $\mathcal{F}^{\psi}_{q,\infty}$ were studied in [3].

It should be noted that in [4], [5] (Ch. XI) Stepanets got the exact values the best m-term trigonometric approximations of the classes $\mathcal{F}^{\psi}_{q,r}$ in the spaces S^p. These results are used in the proof and presented in section 4.

1 Approximative characteristics

In this section, we give the definition of the approximation quantities for the functions of the classes $\mathcal{F}^{\psi}_{q,r}$, which are considered in this paper.

Further, for $f \in L_1$, let $\{k_l\}_{l=1}^{\infty} = \{k_l(f)\}_{l=1}^{\infty}$ denote the rearrangement of vectors of $\mathbb{Z}^d$ such that

$$|\widehat{f}(k_1)| \geq |\widehat{f}(k_2)| \geq \dots . \tag{2}$$

In general case, this rearrangement is not unique. In such case, we take any rearrangement satisfying (2).

We define Σ_m to be the class of all complex trigonometric polynomials of the form $T = \sum_{k\in\gamma_m} c_k e_k$, where γ_m is any collection of m different vectors from the set $\mathbb{Z}^d$.

For $f \in \mathcal{F}^{\psi}_{q,r}$, we consider the following quantities:

$$||f - G_m(f)||_X := ||f(\cdot) - \sum_{l=1}^{m} \widehat{f}(k_l) e_{k_l}||_X, \tag{3}$$

$$\sigma^{\perp}_m(f)_X := \inf_{\gamma_m} ||f - \sum_{k\in\gamma_m} \widehat{f}(k) e_k||_X, \tag{4}$$

and

$$\sigma_m(f)_X := \inf_{T\in\Sigma_m} ||f - T||_X = \inf_{\gamma_m, c_k} ||f - \sum_{k\in\gamma_m} c_k e_k||_X, \tag{5}$$

where X is one of the spaces L_p, $1 \leq p \leq \infty$, or S^p, $0 < p < \infty$, c_k are any complex numbers. Here, it is assumed that the embedding $\mathcal{F}^{\psi}_{q,r} \subset X$ is true.

The quantities (5) and (4) are respectively called the best m-term trigonometric and the best m-term orthogonal trigonometric approximations of the function f in the space X. The quantity (3) is called the approximation of the function f by m-term greedy polynomials in the space X.

For a set $\mathfrak{N} \subset X$, we put

$$\sigma_m^{\perp}(\mathfrak{N})_X := \sup_{f\in\mathfrak{N}} \sigma_m^{\perp}(f)_X \quad \text{and} \quad \sigma_m(\mathfrak{N})_X := \sup_{f\in\mathfrak{N}} \sigma_m(f)_X.$$

In general case, the quantities (3) depend on the choice of the rearrangement satisfying (2). So, for the unique definition, we put

$$G_m(\mathfrak{N})_X := \sup_{f\in\mathfrak{N}} \inf_{\{k_l(f)\}_{l=1}^{\infty}} ||f(\cdot) - \sum_{l=1}^{m} \widehat{f}(k_l(f))e_{k_l(f)}||_X. \tag{6}$$

In (6), for any function $f \in \mathfrak{N}$, we consider the infimum on all rearrangements, satisfying (2), but it should be noted that results, formulated in this paper, are also true for any other rearrangements, satisfying (2).

Research of the quantities of the form (3)–(5) goes back to the paper of S.B. Stechkin [6]. Order estimates of these quantities on different classes of functions of one and several variables were obtained by many authors. In particular, the bibliography of papers with the similar results can be found in [7], [8], [9].

Note that for $f \in L_p$,

$$\sigma_m(f)_{L_p} \le \sigma_m^{\perp}(f)_{L_p} \le ||f - G_m(f)||_{L_p}. \tag{7}$$

and by virtue of (1), for $f \in S^p$,

$$\sigma_m(f)_{S^p} = \sigma_m^{\perp}(f)_{S^p} = ||f - G_m(f)||_{S^p}. \tag{8}$$

2 Main results

The main purpose of this work is to find the dependence of the choice of the parameters r, ψ and q on the rate of convergence to zero, as $m \to \infty$, of the approximative characteristics of the classes $\mathcal{F}_{q,r}^{\psi}$.

For a real number a, we denote $(a)_+ = \max\{0, a\}$. As mentioned above, in the case where $\psi(t)$ is a power function, i.e., $\psi(t) = t^{-s}$, $s > 0$, for all $1 \le p \le \infty$, the exact order estimates of the quantities

$\sigma_m(\mathcal{F}^{\psi}_{q,\infty})_{L_p}$ and $G_m(\mathcal{F}^{\psi}_{q,\infty})_{L_p}$ were obtained in [1] and [2], correspondingly. In particular, from Theorems 6.1 [1] and 3.1 [2], it follows that for all $s > d(1-\frac{1}{q})_+$,

$$\sigma_m(\mathcal{F}^{s}_{q,\infty})_{L_p} \asymp m^{-\frac{s}{d}-\frac{1}{q}+\frac{1}{2}}, \quad 1 \le p \le \infty, \tag{9}$$

and

$$G_m(\mathcal{F}^{s}_{q,\infty})_{L_p} \asymp \begin{cases} m^{-\frac{s}{d}-\frac{1}{q}+\frac{1}{2}}, & 1 \le p < 2, \\ m^{-\frac{s}{d}-\frac{1}{q}+1-\frac{1}{p}}, & 2 \le p < \infty. \end{cases} \tag{10}$$

For positive sequences $\alpha(m)$ and $\beta(m)$, the expression '$a(m) \asymp b(m)$' means that there are constants $0 < K_1 < K_2$ such that for any $m \in \mathbb{N}$, $\alpha(m) \le K_2\beta(m)$ (in this case, we write '$\alpha(m) \ll \beta(m)$') and $\alpha(m) \ge K_1\beta(m)$ (in this case, we write '$\alpha(m) \gg \beta(m)$').

From the following Theorem 2.6, in particular, it follows that for the quantities $\sigma_m(\mathcal{F}^{\psi}_{q,\infty})_{L_p}$ and $G_m(\mathcal{F}^{\psi}_{q,\infty})_{L_p}$, the estimates of forms (9) and (10) are satisfied for a wider set of the functions ψ. To formulate this statement, we use the following notation: let B denote the set of all positive decreasing functions such that

$$\lim_{t\to\infty} \psi(t) = 0, \tag{11}$$

and for all $t \ge 1$, the following relation is true:

$$1 < \psi(t)/\psi(2t) \le K. \tag{12}$$

Here and in what follows, K, $K_0, \ldots$ are positive constants which are independent of the variable t.

Теорема 2.6. *Assume that $1 \le r \le \infty$, $1 \le p < \infty$, $0 < q < \infty$, $\psi \in B$ and in the case $p/(p-1) < q$, moreover, for all t, larger than a certain number t_0, $\psi(t)$ is convex and satisfies the condition*

$$t|\psi'(t)|/\psi(t) \ge K_0 > \beta, \quad \psi'(t) := \psi'(t+), \tag{13}$$

where $\beta := d(\frac{1}{2}-\frac{1}{q})$, when $1 < p \le 2$ and $\beta := d(1-\frac{1}{p}-\frac{1}{q})$, when $2 \le p < \infty$. Then

$$G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp \sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp \begin{cases} \psi(m^{\frac{1}{d}})m^{\frac{1}{2}-\frac{1}{q}}, & 1 \le p \le 2, \\ \psi(m^{\frac{1}{d}})m^{1-\frac{1}{p}-\frac{1}{q}}, & 2 \le p < \infty \end{cases},$$

for all $1 \le p \le 2$,

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp \psi(m^{\frac{1}{d}}) m^{\frac{1}{2}-\frac{1}{q}}, \tag{14}$$

and for all $2 < p < \infty$,

$$\psi(m^{\frac{1}{d}}) m^{\frac{1}{2}-\frac{1}{q}} \ll \sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll \psi(m^{\frac{1}{d}}) m^{1-\frac{1}{p}-\frac{1}{q}}.$$

In the case $2 < p \le \infty$, the following theorem is true.

Теорема 2.7. *Assume that* $1 \le r \le \infty$, $2 < p \le \infty$, $0 < q < \infty$, *the function* ψ *belongs to the set* B *and for all* t, *larger than a certain number* t_0, $\psi(t)$ *is convex and satisfies condition (13) with* $\beta = d(1-\frac{1}{q})_+$. *Then relation (14) holds.*

Note that conditions in Theorems 2.6 and 2.7 guarantee the embedding $\mathcal{F}^{\psi}_{q,r} \subset L_p$.

Putting $r = \infty$ and $\psi(t) = t^{-s}$, $s > 0$, from Theorems 2.6 and 2.7 we obtain the following corollary:

Наслідок 2.2. *Assume that* $1 \le p < \infty$, $0 < q < \infty$, s *is a positive number, which in the case* $p/(p-1) < q$, *satisfies the inequality* $s > \beta$, *where* β *is defined in Theorem 2.6. Then for all* $1 \le p < \infty$, *relation relation (10) holds and for all* $1 \le p \le 2$, *relation (9) holds. If* $s > d(1-\frac{1}{q})_+$, *then relation (9) holds for all* $1 \le p \le \infty$.

This statement complements the results mentioned above of [1] and [2] in the following sense:

- from Corollary 2.2, in particular, it follows that in the case $1 < q \le p/(p-1)$, relation (9) (for $1 \le p \le 2$) and relation (10) (for $1 \le p < \infty$) also hold for all $s > 0$,
- if $1 < p \le 2$ and $q > p/(p-1)$, then relations (9) and (10) also hold for all s such that $d(\frac{1}{2}-\frac{1}{q}) < s \le d(1-\frac{1}{q})$,
- if $2 < p < \infty$ and $q > p/(p-1)$, then relation (10) also holds for all s such that $d(1-\frac{1}{p}-\frac{1}{q}) < s \le d(1-\frac{1}{q})$,
- in the case $2 < p \le \infty$, conditions on s in Corollary 2.2 (for validity of relation (9)) are the same as in Theorem 6.1 [1].

Note also that if $0 < q \le p/(p-1)$, then the conditions of Theorem 2.6 are satisfied, for example, for the function $\psi(t) = t^{-s}\ln^{\varepsilon}(t+e)$, where $s > 0$, $\varepsilon \in \mathbb{R}$, as well as for the function $\psi(t) = \ln^{\varepsilon}(t+e)$, $\varepsilon < 0$. If $1 < p/(p-1) < q$ and $1 < p \le 2$, then the conditions of Theorem 2.6 are satisfied for the function $\psi(t) = t^{-s}\ln^{\varepsilon}(t+e)$, where $\varepsilon \in \mathbb{R}$ and $s > d(\frac{1}{2} - \frac{1}{q})$. If $1 < p/(p-1) < q$ and $2 < p < \infty$, then the conditions of Theorem 2.6 are satisfied for the function $\psi(t) = t^{-s}\ln^{\varepsilon}(t+e)$, where $\varepsilon \in \mathbb{R}$ and $s > d(1 - \frac{1}{p} - \frac{1}{q})$. The conditions of Theorem 2.7 are satisfied for the function $\psi(t) = t^{-s}\ln^{\varepsilon}(t+e)$, where $\varepsilon \in \mathbb{R}$ and $s > d(1-\frac{1}{q})_+$.

The proof of Theorems 2.6 and 2.7 will be given in Section 5.

3 Order estimates for some functionals and their applications

4.1. Let $\Psi = \{\Psi(j)\}_{j=1}^{\infty}$ be a nonincreasing positive sequence such that

$$\lim_{j\to+\infty} \Psi(j) = 0. \tag{15}$$

The following Lemma 3.2 is essentially used for proving upper estimates in Theorem 2.6. This lemma gives exact order estimates for the following functionals $H_m(\Psi, s)$, which in the case $s \in (0,1]$, are defined by the equality

$$H_m(\Psi, s) := \sup_{l>m}(l-m)\bigg(\sum_{j=1}^{l} \Psi^{-s}(j)\bigg)^{-\frac{1}{s}}, \tag{16}$$

and for $s \in (1,\infty)$, they are defined by the equality

$$H_m(\Psi, s) := \Bigg((l_m - m)^{s'}\bigg(\sum_{j=1}^{l} \Psi^{-s}(j)\bigg)^{-\frac{s'}{s}} + \sum_{j=l_m+1}^{\infty} \Psi^{s'}(j)\Bigg)^{\frac{1}{s'}}, \tag{17}$$

where $1/s + 1/s' = 1$,

$$\sum_{j=1}^{\infty} \Psi^{s'}(j) < \infty, \tag{18}$$

and the number l_m is given by relation

$$\Psi^{-s}(l_m) \le \frac{1}{l_m - m}\sum_{j=1}^{l_m} \Psi^{-s}(j) < \Psi^{-s}(l_m+1). \tag{19}$$

Note that in the terms of similar functionals, solutions of many problems of approximation theory are formulated (see, eg, [4], [5] (Ch. XI), [10] (Ch.VI), [11], [12], [13]). Therefore, the problem of finding such estimates is interesting.

Let $d \in \mathbb{N}$, M_0, c_1 and c_2 be fixed positive numbers. Let also $\nu = \{\nu_i\}_{i=0}^{\infty}$ be an increasing sequence of natural numbers such that $\nu_0 := 1$ and for all n, greater than a certain number n_0,

$$M_0(n-c_1)^d < V_n := \sum_{k=0}^{n} \nu_k \le M_0(n+c_2)^d. \tag{20}$$

Further, let $\mathcal{S}_d(M_0) = \mathcal{S}_d(M_0, c_1, c_2)$ denote the set of all positive nonincreasing step sequences Ψ, satisfying condition (15), which are represented as

$$\Psi(t) = \psi(n), \quad t \in (V_{n-1}, V_n], \quad n = 1, 2, \dots, \tag{21}$$

where ψ is the decreasing sequence of different values of the sequence Ψ.

Without loss of generality, we assume that the sequences ψ are restrictions of certain positive continuous functions $\psi(t)$ of continuous argument $t \ge 1$ on the set of natural numbers $\mathbb{N}$.

Лема 3.2. *Let $s \in (0, \infty)$, $d \in \mathbb{N}$, the sequence Ψ belongs to the set $\mathcal{S}_d(M_0)$ and the sequence of its different values is a restriction of a certain function $\psi \in B$ on the set $\mathbb{N}$. Furthermore, in case $s > 1$, we also assume that for all t, greater than a certain number t_0, the function $\psi(t)$ is convex and satisfies condition (13) with $\beta = d/s'$. Then the following relation is true:*

$$H_m(\Psi; s) \asymp \psi(m^{\frac{1}{d}}) m^{1-\frac{1}{s}}. \tag{22}$$

Let us note that for any $\Psi \in \mathcal{S}_d(M_0)$, condition (13) (with $\beta = d/s'$) guarantees convergence of the series in (18), when $s > 1$. Indeed, in this case, for all $\tau \ge t_0$,

$$|\psi'(\tau)|/\psi(\tau) \ge K_0/\tau. \tag{23}$$

Integrating each part of this relation in the range from t_0 to t, $t > t_0$, we obtain $\psi(t) \ll t^{-K_0} \ll t^{-d/s'}$. Therefore, in view of (21) and (20), we conclude

$$\sum_{j=1}^{\infty} \Psi^{s'}(j) = \sum_{n=1}^{\infty} \nu_n \psi^{s'}(n) \ll \sum_{n=1}^{\infty} n^{d-1} \psi^{s'}(n) \ll \sum_{n=1}^{\infty} n^{d-1} n^{-s' K_0} < \infty.$$

Also note that in the case where the sequences Ψ are restrictions of certain positive convex functions $\psi(t)$ of continuous argument $t \ge 1$ on the set $\mathbb{N}$, the functionals $H_n(\Psi, s)$ were considered in [13].

4.2. Proof of Lemma 3.2. First, consider the case $s \in (0,1]$. In view of (16) and (21), the functionals $H_m(\Psi; s)$ can be represented as

$$H_m(\Psi, s) = \sup_{l>m}(l-m)\bigg(\sum_{n=1}^{n_l-1} \frac{\nu_n}{\psi^s(n)} + \frac{l - V_{n_l-1}}{\psi^s(n_l)}\bigg)^{-\frac{1}{s}} =: \widetilde{H}_m(\psi, s),$$

where n_l denote a number such that

$$V_{n_l-1} < l \le V_{n_l}. \tag{24}$$

By virtue of (20), we see that

$$\nu_n \asymp n^{d-1}. \tag{25}$$

and for all $l > m \ge n_0$,

$$(l/M_0)^{\frac{1}{d}} - c_2 \le n_l < (l/M_0)^{\frac{1}{d}} + c_1 + 1. \tag{26}$$

If $\psi \in B$, then for any $s > 0$ and $l = 2, 3, \ldots$,

$$\frac{l^d}{\psi^s(l)} \ll \frac{(l/2)^d}{\psi^s(l/2)} \ll \sum_{l/2 \le n \le l} \frac{n^{d-1}}{\psi^s(n)} \ll \sum_{k=1}^{l} \frac{n^{d-1}}{\psi^s(n)} \ll \frac{l^d}{\psi^s(l)}.$$

Therefore,

$$\sum_{n=1}^{l} \frac{\nu_n}{\psi^s(n)} \asymp \sum_{n=1}^{l} \frac{n^{d-1}}{\psi^s(n)} \asymp \frac{l^d}{\psi^s(l)}. \tag{27}$$

Further, by virtue of (26) and the definition of the set B, we see that $\psi(n_l) \asymp \psi((l/M_0)^{\frac{1}{d}}) \asymp \psi(l^{\frac{1}{d}})$. In view of (27), we conclude that

$$\widetilde{H}_m(\psi, s) \asymp \sup_{l>m}(l-m)\bigg(\frac{n_l^d}{\psi^s(n_l)}\bigg)^{-\frac{1}{s}} \asymp$$

$$\asymp \sup_{l>m} \psi(l^{\frac{1}{d}})(l-m)/l^{\frac{1}{s}} \ll \psi(m^{\frac{1}{d}}) \sup_{l>m}(l-m)/l^{\frac{1}{s}}. \tag{28}$$

For $t > 0$, $m \in \mathbb{N}$ and $s \in (0,1)$, the function $h(t) = h(t,s) = (t-m)/t^{\frac{1}{s}}$ attains its maximal value at the point $t_* = m/(1-s)$, and

$$h(t_*, s) = s(m/(1-s))^{1-\frac{1}{s}}. \tag{29}$$

If $s=1$, then the function $h(t)=h(t;1)$ is non-decreasing and tends to 1 as t increases. Therefore,

$$\sup_{t>0} h(t;1) = \sup_{t>0}(t-m)/t = \lim_{t\to+\infty}(t-m)/t = 1. \tag{30}$$

Combining (3)–(30), we obtain necessary upper estimates:

$$H_m(\Psi,s) = \widetilde{H}_m(\psi,s) \asymp \sup_{l>m} \psi(l^{\frac{1}{d}})(l-m)/l^{\frac{1}{s}} \ll \psi(m^{\frac{1}{d}})m^{1-\frac{1}{s}}.$$

Taking into account (3) and the inclusion $\psi\in B$, we also obtain the lower estimates

$$H_m(\Psi,s) = \widetilde{H}_m(\psi,s) \gg \psi((2m)^{\frac{1}{d}})(2m-m)/(2m)^{\frac{1}{s}} \asymp \psi(m^{\frac{1}{d}})m^{1-\frac{1}{s}}.$$

Now, we consider the case $s>1$. To simplify the notes, we set

$$Q_m(\Psi,l) := (l-m)\bigg(\sum_{j=1}^{l}\Psi^{-s}(j)\bigg)^{-1}, \quad l\ge m,\ l\in\mathbb{N}.$$

For any $l>m$, we have

$$Q_m(\Psi,l+1) = Q_m(\Psi,l)+$$
$$+\Big(\Psi^s(l+1) - Q_m(\Psi,l)\Big)\Psi^{-s}(l+1)\bigg(\sum_{i=1}^{l+1}\Psi^{-s}(i)\bigg)^{-1}$$

and

$$\Psi^s(l+1) = Q_m(\Psi,l+1)+$$
$$+\Big(\Psi^s(l+1) - Q_m(\Psi,l)\Big)\sum_{j=1}^{l}\Psi^{-s}(j)\bigg(\sum_{i=1}^{l+1}\Psi^{-s}(i)\bigg)^{-1},$$

Therefore, in view of monotonicity of the function Ψ and relation (19), we conclude that for all $l\ge l_m$, $Q_m(\Psi,l) > Q_m(\Psi,l+1) > \Psi^s(l+1)$ and for all $l\in[m,l_m)$, $Q_m(\Psi,l)\le Q_m(\Psi,l+1)\le\Psi^s(l+1)$. This yields that

$$\sup_{l>m} Q_n(\Psi,l) = Q_m(\Psi,l_m). \tag{31}$$

According to (19), we get $\Psi(l_m+1) > \Psi(l_m)$. Hence, if the function $\Psi(t)$ is represented in the form (21), then

$$l_m = V_{k_{l_m}} = \sum_{i=0}^{n_{l_m}} \nu_i \tag{32}$$

where n_{l_m} is defined in (24) for $l = l_m$. In this case, the functionals $H_m(\Psi, s)$, $s \in (1, \infty)$ can be represented as

$$H_m(\Psi, s) = \Bigg((l_m - n)^{s'} \Bigg(\sum_{n=1}^{n_{l_m}} \frac{\nu_n}{\psi^s(n)} \Bigg)^{-\frac{s'}{s}} + \\ + \sum_{n=n_{l_m}+1}^{\infty} \nu_n \psi^{s'}(n) \Bigg)^{\frac{1}{s'}} := \widetilde{H}_m(\psi, s), \quad (33)$$

where

$$\psi^{-s}(n_{l_m}) \leq \frac{1}{l_m - m} \sum_{j=1}^{n_{l_m}} \frac{\nu_n}{\psi^s(n)} < \psi^{-s}(n_{l_m} + 1). \quad (34)$$

By virtue of (31), for the function

$$\widetilde{Q}_m(\psi, l) := (l - m) \Bigg(\sum_{n=1}^{n_l - 1} \frac{\nu_n}{\psi^s(n)} + \frac{l - V_{n_l - 1}}{\psi^s(n_l)} \Bigg)^{-1},$$

where n_l is defined in (24), the following relation is satisfied:

$$\sup_{l>m} \widetilde{Q}_m(\psi, l) = \widetilde{Q}_m(\psi, l_m) = (l_m - m) \Bigg(\sum_{n=1}^{n_{l_m}} \frac{\nu_n}{\psi^s(n)} \Bigg)^{-1}. \quad (35)$$

Then similarly to the case $s \in (0, 1]$, we show that

$$\widetilde{Q}_m(\psi, l_m) = \sup_{l>m} \widetilde{Q}_m(\psi, l) \asymp \psi^s(m^{\frac{1}{d}}). \quad (36)$$

Taking into account (34)–(36) and the definition of the set B, we see that

$$\psi(n_{l_m}) \asymp \psi(m^{\frac{1}{d}}). \quad (37)$$

Since $\psi \in B$, then in view of (25), we conclude that for any $l \in \mathbb{N}$,

$$\sum_{n=l+1}^{\infty} \nu_n \psi^{s'}(n) \gg \sum_{n=l+1}^{2l} n^{d-1} \psi^{s'}(n) \gg l^d \psi^{s'}(l). \quad (38)$$

By virtue of (13), the function $t^d \psi^{s'}(t)$ decreases to zero at $t > t_0$. Therefore,

$$\sum_{n=l+1}^{\infty} \nu_n \psi^{s'}(n) \ll \sum_{n=l+1}^{\infty} n^{d-1} \psi^{s'}(n) \ll \int_{l}^{\infty} t^{d-1} \psi^{s'}(t) dt =: \mathcal{J}_l.$$

Integrating by parts, we obtain

$$\mathcal{J}_l \le \frac{1}{K_0}\int\limits_l^\infty t^d\psi^{s'-1}(t)|\psi'(t)|dt = \frac{l^d\psi^{s'}(l)}{K_0 s'} + \frac{d}{K_0 s'}\mathcal{J}_l.$$

Then in view of (13), we see that $\mathcal{J}_l \ll l^d\psi^{s'}(l)$. Hence, we get the estimate

$$\sum_{n=l+1}^{\infty}\nu_n\psi^{s'}(n) \ll l^d\psi^{s'}(l), \tag{39}$$

that together with (38) proves the relation

$$\sum_{n=l+1}^{\infty}\nu_n\psi^{s'}(n) \asymp l^d\psi^{s'}(l). \tag{40}$$

In the end of the proof, let us show that

$$n_{l_m} \asymp m^{\frac{1}{d}}. \tag{41}$$

Indeed, by virtue of (32) and (26), we see that $\widetilde{m} := (m/M_0)^{\frac{1}{d}} - c_2 \le (l_m/M_0)^{\frac{1}{d}} - c_2 \le n_{l_m}$. On the other hand, integrating each part of (23) in the range from $\widetilde{m}$ to n_{l_m}, $\widetilde{m} > t_0$, we obtain $\psi(\widetilde{m})/\psi(n_{l_m}) \ge (n_{l_m}/\widetilde{m})^{K_0}$.Therefore, in view of (37) and (12), we see that $\widetilde{m} \gg n_{l_m}$ and relation (41) is true.

Thus, combining the relations (3), (27), (36), (37), (40) and (41) we obtain the estimate (22), i.e.,

$$H_m(\Psi,s) = \widetilde{H}_m(\psi,s) \asymp \left(\psi^{ss'}(m^{\frac{1}{d}})\cdot\left(m/\psi^s(m^{\frac{1}{d}})\right)^{(1-\frac{1}{s})s'} + \right.$$

$$\left. + m\psi^{s'}(m^{\frac{1}{d}})\right)^{1/s'} \asymp \psi(m^{\frac{1}{d}})m^{\frac{1}{s'}} \asymp \psi(m^{\frac{1}{d}})m^{1-\frac{1}{s}}.$$

4.3. In this section, we apply Lemma 3.2 for estimation of the exact upper bounds of the quantities (3)–(5) on the classes $\mathcal{F}^{\psi}_{q,r}$ in the spaces $S^p(\mathbb{T}^d)$. For all $0<p,\ q<\infty$, the exact values of the quantities $\sigma_m(\mathcal{F}^{\psi}_{q,r})_{S^p}$, as well as the exact values of the quantities $G_m(\mathcal{F}^{\psi}_{q,r})_{S^p}$ and $\sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{S^p}$ (due to (8)) were obtained by A.I. Stepanets ([4], [5] (Ch. XI)). In particular, from Theorem 9.1 of [5] (Ch. XI), it follows that for all $m \in \mathbb{N}$,

$0 < q \le p < \infty$ and for any positive function $\psi = \psi(t)$, $t \ge 0$, satisfying condition (11),

$$\sigma_m^p(\mathcal{F}_{q,r}^{\psi})_{S^p} = \sup_{l>m}(l-m)\Big(\sum_{j=1}^{l} \bar{\psi}^{-q}(j)\Big)^{-\frac{p}{q}}, \qquad (42)$$

where $\bar{\psi} = \bar{\psi}(j)$, $j = 1, 2, \ldots$, is the decreasing rearrangement of the system of numbers $\psi(|k|_r)$, $k \in \mathbb{Z}^d$. If $0 < p < q < \infty$ and the positive function $\psi = \psi(t)$, $t \ge 0$, satisfies the condition

$$\sum_{k\in\mathbb{Z}^d} \psi^{\frac{pq}{q-p}}(|k|_r) < \infty, \qquad (43)$$

then from Theorem 9.4 of [5] (Ch. XI) it follows that

$$\sigma_m^p(\mathcal{F}_{q,r}^{\psi})_{S^p} = \left((l_m - m)^{\frac{q}{q-p}} \left(\sum_{j=1}^{l_m} \bar{\psi}^{-q}(j)\right)^{\frac{p}{q-p}} + \sum_{j=l_m+1}^{\infty} \bar{\psi}^{\frac{pq}{q-p}}(j)\right)^{\frac{q-p}{q}}, \quad (44)$$

where $\bar{\psi} = \bar{\psi}(j)$, $j = 1, 2, \ldots$, is the decreasing rearrangement of the system of numbers $\psi(|k|_r)$, $k \in \mathbb{Z}^d$, and the number l_m is defined by

$$\bar{\psi}^{-q}(l_m) \le \frac{1}{l_m - m} \sum_{j=1}^{l_m} \bar{\psi}^{-q}(j) < \bar{\psi}^{-q}(l_m + 1).$$

Taking into account notation (16) and (17), we can write relations (42) and (44) as

$$\sigma_m^p(\mathcal{F}_{q,r}^{\psi})_{S^p} = H_m(\bar{\psi}^p, q/p), \quad 0 < p,\, q < \infty.$$

Furthermore, if the number $V_n := |\Delta_{n,r}^d|$ of elements of the set

$$\Delta_{n,r}^d := \{k \in \mathbb{Z}^d : |k|_r \le n, \quad n = 0, 1, \ldots\}.$$

for all sufficiently large $n \in \mathbb{N}$ (n is greater than some positive number n_0) satisfies the following condition:

$$M_r(n - c_1)^d < V_n = |\Delta_{n,r}^d| \le M_r(n + c_2)^d, \qquad (45)$$

where M_r, c_1 and c_2 are certain positive constants, then the sequence $\bar{\psi} = \bar{\psi}(j)$, $j = 1, 2, \ldots$, belongs to the set $\mathcal{S}_d(M_r) = \mathcal{S}_d(M_r, c_1, c_2)$. Thus, by virtue of Lemma 3.2, we can formulate the following statement:

Assertion 4.1. *Assume that* $0 < r \le \infty$, $0 < p < \infty$, $0 < q < \infty$, *condition (45) holds,* $\psi \in B$ *and in the case* $p < q$, *moreover, for all* t, *larger than a certain number* t_0, ψ^p *is convex and condition (13) holds with* $\beta = d(\frac{1}{p} - \frac{1}{q})$. *Then*

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{S^p} \asymp \psi(m^{\frac{1}{d}}) m^{\frac{1}{p} - \frac{1}{q}}. \tag{46}$$

It is clear that in the case $r = \infty$, condition (45) is satisfied and $M_\infty = vol\{k \in \mathbb{R}^d : |k|_\infty \le 1\} = 2^d$. If $r = 1$, then $M_1 = vol\{k \in \mathbb{R}^d : |k|_1 \le 1\} = 2^d/d!$. Unfortunately, we do not know whether a similar relation for other r is valid. However, one can formulate the following corollary:

Наслідок 3.3. *Assume that* $0 < p < \infty$, $0 < q < \infty$, *condition (45) holds,* $\psi \in B$ *and in the case* $p < q$, *moreover, for all* t, *larger than a certain number* t_0, ψ^p *is convex and condition (13) holds with* $\beta = d(\frac{1}{p} - \frac{1}{q})$. *Then for all* $1 \le r \le \infty$, *relation (46) is true.*

Indeed, for any numbers $r \in [1, \infty]$, $0 < q < \infty$ and for any positive decreasing function ψ

$$\mathcal{F}^{\psi}_{q,1} \subset \mathcal{F}^{\psi}_{q,r} \subset \mathcal{F}^{\psi}_{q,\infty}. \tag{47}$$

Therefore, if conditions of Corollary 3.3 are satisfied, then for all $r \in [1, \infty]$,

$$\frac{\psi(m^{\frac{1}{d}})}{m^{\frac{1}{q} - \frac{1}{p}}} \ll \sigma_m(\mathcal{F}^{\psi}_{q,1})_{S^p} \ll \sigma_m(\mathcal{F}^{\psi}_{q,r})_{S^p} \ll \sigma_m(\mathcal{F}^{\psi}_{q,\infty})_{S^p} \ll \frac{\psi(m^{\frac{1}{d}})}{m^{\frac{1}{q} - \frac{1}{p}}}.$$

4 Proof of Theorems 2.6 and 2.7.

5.1. Proof of Theorem 2.6.

Upper estimates. In the case $1 \le p \le 2$, by virtue of (7) and (8), we have

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll G_m(\mathcal{F}^{\psi}_{q,r})_{L_2} \ll \sigma_m(\mathcal{F}^{\psi}_{q,r})_{S^2}.$$

Thus, to obtain the required upper estimates, it is sufficient to use Corollary 3.3 for $S^p = S^2$.

If $2 \le p < \infty$, then using the Hausdorff–Young inequality (see, for example, [14] (p. 16)), relation (8) and Corollary 3.3, we get

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll$$

$$\ll G_m(\mathcal{F}^{\psi}_{q,r})_{S^{p'}} \ll \sigma_m(\mathcal{F}^{\psi}_{q,r})_{S^{p'}} \ll \psi(m^{\frac{1}{d}})m^{1-\frac{1}{p}-\frac{1}{q}}.$$

Lower estimate. Let $\mathcal{T}_n$, $n \in \mathbb{N}$, denote the set of all polynomials of the form

$$T_n = \sum_{|k|_\infty \le n} \widehat{T}_n(k)e_k,$$

and let $\mathcal{A}_q(\mathcal{T}_n)$, $0 < q < \infty$, denote the subset of all polynomials $T_m \in \mathcal{T}_m$ such that $||T||_{S^q} \le 1$. From Theorem 5.2 of [1], it follows that for any $0 < q < \infty$, $1 \le p < \infty$, $n = 1, 2, \dots$ and $m = ((2n+1)^d - 1)/2$,

$$\sigma_m(\mathcal{A}_q(\mathcal{T}_n))_{L_p} \ge Km^{1/2-1/q}.$$

For a fixed $n \in \mathbb{N}$, consider the set

$$\psi(dn)\mathcal{A}_q(\mathcal{T}_n) = \{T \in \mathcal{T}_n : ||T||_{S^q} \le \psi(dn)\}.$$

Due to monotonicity ψ, for any polynomial $T \in \psi(dn)\mathcal{A}_q(\mathcal{T}_n)$, we have

$$\sum_{k\in\mathbb{Z}^d} |\widehat{T}(k)/\psi(|k|_1)|^q \le \sum_{|k|_\infty \le n} |\widehat{T}(k)/\psi(d|k|_\infty)|^q \le$$

$$\le \sum_{|k|_\infty \le n} |\widehat{T}(k)/\psi(dn)|^q \le 1$$

Therefore, $\psi(dn)\mathcal{A}_q(\mathcal{T}_n)$ is contained in the set $\mathcal{F}^{\psi}_{q,1}$. In view of definition of the set B, for all $n = 1, 2, \dots$ and $m = ((2n+1)^d - 1)/2$, we obtain

$$\sigma_m(\mathcal{F}^{\psi}_{q,1})_{L_p} \ge \sigma_m(\psi(dn)\mathcal{A}_q(\mathcal{T}_n))_{L_p} \gg \psi(dn)m^{\frac{1}{2}-\frac{1}{q}} \gg \psi(m^{\frac{1}{d}})m^{\frac{1}{2}-\frac{1}{q}}.$$

Taking into account the relations (7) and (47), monotonicity of the quantity σ_m and inclusion $\psi \in B$, we see that for all $1 \le p \le \infty$ and all $1 \le r \le \infty$,

$$G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg$$

$$\gg \sigma_m(\mathcal{F}^{\psi}_{q,1})_{L_p} \gg \psi(m^{\frac{1}{d}})m^{\frac{1}{2}-\frac{1}{q}}.$$

In the case $2 < p < \infty$, for the quantities $\sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p}$ and $G_m(\mathcal{F}^{\psi}_{q,r})_{L_p}$, this estimate can be improved. For this purpose, consider the function

$$f_1 = \sum_{|k|_1 \le n_m} \widehat{f}_1(k)e_k = C_m \sum_{|k|_1 \le n_m} e_k,$$

where $C_m^{-q} := \sum_{|j|_1\le n_m} \psi^{-q}(|j|_1)$, $n_m := [(2m/M_1)^{1/d}]$ and $M_1 = 2^d/d!$. It is obviously that $f_1 \in \mathcal{F}^{\psi}_{q,1}$. Due to (45), for all sufficiently large n, the number $|\widetilde{\Delta}^d_{n,1}|$ of elements of the set $\widetilde{\Delta}^d_{n,1} := \{k \in \mathbb{Z}^d : |k|_1 = n,\ n \in \mathbb{N}\}$ satisfies the condition

$$M_1(n-c_3)^{d-1} < |\widetilde{\Delta}^d_{n,1}| = |\Delta^d_{n,1}| - |\Delta^d_{n-1,1}| \le M_1(n-c_4)^{d-1},$$

where c_3 and c_4 are some positive numbers. Therefore, by virtue of (27),

$$C_m^{-q} \asymp \sum_{n=1}^{n_m} \frac{n^{d-1}}{\psi^q(n)} \asymp \frac{n_m^d}{\psi^q(n_m)} \asymp \frac{m}{\psi^q(m^{\frac{1}{d}})}.$$

For any collection $\gamma_n \subset \mathbb{Z}^d$, using Nikol'skii's inequality [15] and (45), we obtain

$$\Big\|f_1 - \sum_{k\in\gamma_n} \widehat{f}_1(k) e_k\Big\|_{L_p} \gg C_m m^{-\frac{1}{p}} \Big\|\sum_{|k|_1\le n_m:\, k\notin\gamma_n} e_k\Big\|_{L_\infty} \asymp \psi(m^{\frac{1}{d}}) m^{1-\frac{1}{p}-\frac{1}{q}}.$$

Therefore, for all $2 \le p < \infty$, the following estimates are true:

$$G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,1})_{L_p} \gg$$

$$\gg \sigma^{\perp}_m(f_1)_{L_p} \gg \psi(m^{\frac{1}{d}}) m^{1-\frac{1}{p}-\frac{1}{q}}.$$

Theorem 2.6 is proved.

5.2. Proof of Theorem 2.7 is similar to the proof of the upper estimates in Theorem 6.1 [1].

Лема 4.3. *[1] For each $0 < q \le \infty$, each $n = 1, 2, \ldots$, and $1 \le m \le (2n+1)^d$, we have*

$$\sigma_m(\mathcal{A}_q(\mathcal{T}_n))_{L_\infty} \le C m^{\frac{1}{2}-\frac{1}{q}} L(n^d/m), \quad 0 < q \le 1, \tag{48}$$

where $L(x) = (1 + (\ln x)_+)^{1/2}$ and

$$\sigma_m(\mathcal{A}_q(\mathcal{T}_n))_{L_\infty} \le C n^{d-\frac{d}{q}} m^{-\frac{1}{2}} L(n^d/m), \quad 1 < q \le \infty, \tag{49}$$

with C depending only on q and d.

For any $f \in \mathcal{F}^{\psi}_{q,\infty}$, we use the decomposition

$$f = \sum_{j=0}^{\infty} f_j,$$

where $f_j := \sum_{2^{j-1} \le |k|_\infty < 2^j} \widehat{f}(k) e_k$, $j \ge 1$, and $f_0 = \widehat{f}(0)$. We note that

$$f_j / \psi(2^{j-1}) \in \mathcal{A}_q(\mathcal{T}_{2^j}), \quad j = 1, 2, \dots \tag{50}$$

For any $N = 1, 2, \dots$, we approximate f as follows. Let N_0 be the largest integer j such that $m_j := [(j-N)^{-2} 2^{Nd}] \ge 1$ (with $[x]$ denoting the greatest integer in x), i.e. $N_0 = [2^{\frac{Nd}{2}} + N]$. If $j \le N$, we set $P_j := f_j$. If $N < j \le N_0$, then by virtue of (48), (49) and (50), there is polynomial $P_j \in \Sigma_{m_j}$ such that

$$||f_j - P_j||_{L_p} \ll m_j^{\frac{1}{2} - \frac{1}{q}} L(2^{jd}/m_j) \psi(2^{j-1}), \quad 0 < q \le 1. \tag{51}$$

and

$$||f_j - P_j||_{L_p} \ll 2^{j(d - \frac{d}{q})} m_j^{-\frac{1}{2}} L(2^{jd}/m_j) \psi(2^{j-1}), \ 1 < q < \infty. \tag{52}$$

Set $P = \sum\limits_{j=0}^{N_0} P_k$. Since

$$(2 \cdot 2^N + 1)^d + \sum_{j=N+1}^{N_0} (j-N)^{-2} 2^{Nd} \le a 2^{Nd},$$

where a depends only on d, then P is a linear combination of at most $a2^{Nd}$ exponentials e_k. Hence, P is in $\Sigma_{a2^{Nd}}$. We also have

$$||f - P||_{L_p} \le \sum_{j=N+1}^{N_0} ||f_j - P_j||_{L_p} + \sum_{j=N_0+1}^{\infty} ||f_j||_{L_p} =: S_1 + S_2. \tag{53}$$

For all $x \ge 1$, we have $[x] \ge x/2$. Therefore, for sufficiently large N and $N < j \le N_0$, from the definition of $L(x)$, we have

$$L(2^{jd}/m_j) \le (1 + \ln(2^{d(j-N)+1}(j-N)^2))^{\frac{1}{2}} \ll (j-N)^{\frac{1}{2}}. \tag{54}$$

First, consider the case $0 < q \le 1$. Reasoning similar to the proof of (39), it is easy to show that if the function ψ belongs to the set B and for all t, larger than a certain number t_0, ψ is convex and it satisfies condition (13) with a fixed $\beta \ge 0$, then for any $\alpha \in \mathbb{R}$ and sufficiently large $t > N$, the function $h_{\alpha,\beta}(t) := 2^{\beta t}(t-N)^{\alpha}\psi(2^{t-1})$ decreases to zero, as well as

$$\sum_{j=N+1}^{\infty} 2^{\beta j}(j-N)^{\alpha}\psi(2^{j-1}) \ll 2^{\beta N}\psi(2^N). \qquad (55)$$

In this case, $\beta = 0$. By virtue of (51), (54) and (55), we obtain the estimate of the first sum S_1 in (53):

$$S_1 \ll \sum_{j=N+1}^{\infty} (j-N)^{2(\frac{1}{q}-\frac{1}{2})} 2^{-Nd(\frac{1}{q}-\frac{1}{2})} (j-N)^{\frac{1}{2}}\psi(2^{j-1}) \ll$$

$$\ll 2^{-N(\frac{d}{q}-\frac{d}{2})} \sum_{j=N+1}^{\infty} (j-N)^{\frac{2}{q}-\frac{1}{2}}\psi(2^{j-1}) \ll 2^{-N(\frac{d}{q}-\frac{d}{2})}\psi(2^N). \qquad (56)$$

To estimate S_2, we note that from (50)

$$S_2 \le \sum_{j=N_0+1}^{\infty} ||f_j||_{L_\infty} \le \sum_{j=N_0+1}^{\infty} \Big(\sum_{2^{j-1} \le |k|_\infty < 2^j} |\widehat{f}(k)| \Big) \le$$

$$\le \sum_{j=N_0+1}^{\infty} ||f_j||_{S^q} \ll \sum_{j=N_0+1}^{\infty} \psi(2^{j-1}) \ll \psi(2^{N_0}).$$

Further, let us note that if for all t, larger than a certain number t_0, ψ is convex and satisfies the condition (13), then for any $\alpha > 0$, we have $\psi(2^{N(\alpha+1)}) \ll \psi(2^N)2^{-N\alpha}$.

From the definition of N_0, we have $N_0 \ge N + 2^{\frac{Nd}{2}} - 1$. It follows that if N is sufficiently large (depending only on d and q), then $N_0 \ge N(1 + d/q - d/2)$. Hence,

$$S_2 \ll \psi(2^{N(1+\frac{d}{q}-\frac{d}{2})}) \ll 2^{-N(\frac{d}{q}-\frac{d}{2})}\psi(2^N).$$

Using this and (56) in (53), we find that

$$\sigma_{a2^{Nd}}(f)_{L_p} \le ||f-P||_{L_p} \ll 2^{-N(\frac{d}{q}-\frac{d}{2})}\psi(2^N). \qquad (57)$$

In the case $1 < q < \infty$, condition (13) is satisfied with $\beta = d - d/q$. By virtue of (52), (54) and (55), we have

$$S_1 \ll 2^{-\frac{N}{2}} \sum_{j=N+1}^{\infty} 2^{j(d-\frac{d}{q})}(j-N)^{\frac{3}{2}}\psi(2^{j-1}) \ll \psi(2^N)2^{-N(\frac{d}{q}-\frac{d}{2})}. \quad (58)$$

To estimate S_2, we use Hölder's inequality, (50), (55) and the inequalities $N_0 \geq N+2^{\frac{Nd}{2}}-1$ and $h_{\alpha,\beta}(N_0) \leq h_{\alpha,\beta}(N+1)$ with $\alpha = 1$ and $\beta = d-d/q$,

$$S_2 \leq \sum_{j=N_0+1}^{\infty} ||f_j||_{L_\infty} \leq \sum_{j=N_0+1}^{\infty} \psi(2^{j-1})\Big(\sum_{2^{j-1}\leq |k|_\infty < 2^j} \left|\frac{\widehat{f}(k)}{\psi(2^{j-1})}\right|\Big) \leq$$

$$\leq \sum_{j=N_0+1}^{\infty} \psi(2^{j-1})2^{(j-1)(d-\frac{d}{q})} \ll \psi(2^{N_0})2^{N_0(d-\frac{d}{q})} \ll \psi(2^N)2^{N(\frac{d}{2}-\frac{d}{q})}.$$

Using this and (58) in (53), we see that in this case, relation (57) is also true. Therefore, the upper estimate in (13) follows from the monotonicity of σ_m and inclusion $\psi \in B$.

[1] *R. A. DeVore, V. N. Temlyakov.* Nonlinear approximation by trigonometric sums // J. Fourier Anal. Appl. — **2**, № 1. — 1995. — P. 29–48.

[2] *V. N. Temlyakov.* Greedy algorithm and m-term trigonometric approximation // Constr. Approx. — **14**, № 4. — 1998. — P. 569–587.

[3] *R. S. Li, Y. P. Liu.* Best m-term one-sided trigonometric approximation of some function classes defined by a kind of multipliers // Acta Mathematica Sinica, English Series. — **26**, № 5. — 2010. — P. 975–984.

[4] *A. I. Stepanets.* Approximation characteristics of the spaces S^p_φ in different metrics // Ukr. Mat. Zh. — **53**, № 8. — 2001. — P. 1121–1146.

[5] *A. I. Stepanets.* Methods of approximation theory. — VSP, Leiden–Boston, 2005. — 919 p.

[6] *S. B. Stechkin.* On absolute convergence of orthogonal series // Dokl. Akad. Nauk SSSR (N.S.). — **102**. — 1955. — P. 37–40.

[7] *R. A. DeVore.* Nonlinear approximation // Acta Numer. — **7**. — 1998. — P. 51–150.

[8] *V. N. Temlyakov.* Greedy approximation. — Cambridge Monographs on Applied and Computational Mathematics, 20. Cambridge University Press, Cambridge, 2011. — 418 p.

[9] *A. S. Romanyuk.* Approximation characteristics of classes of periodical functions of several variables. (Russian). — Proceed. Inst. Mat. NAS Ukraine, **40**. — Kyiv: Inst. Mat. NAS Ukraine, 2012. — 352 p.

[10] *A. Pinkus.* n-Widths in approximation theory. — Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1985. — 291 p.

[11] *F. Gensun, Q. Lixin.* Approximation characteristics for diagonal operators in different computational settings // J. Approx. Theory. — **140**, № 2. — 2006. — P. 178–190.

[12] *A. I. Stepanets, A. L. Shidlich.* Best approximations of integrals by integrals of finite rank // J. Approx. Theory. — **162**, № 2. — 2010. — P 323–348.

[13] *A. L. Shidlich.* Order equalities for some functionals and their application to the estimation of the best n-term approximations and widths // Ukr. Mat. Zh. — **61**, № 10. — 2009. — P. 1403–1423.

[14] *V. N. Temlyakov.* Approximation of Periodic Functions. — Computational Mathematics and Analysis Series, Commack, New York, Nova Science Publ., 1993. — 271 p.

[15] *S. M. Nikol'skii.* Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables. (Russian) // Trudy Mat. Inst. Steklov. — **38**. — 1951. — P. 244–278.