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Geometry of highly inclined protoplanetary disks

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We present a geometric model for the modelling of spectral energy distribution of inclined protoplanetary disks. We investigate peculiarities in the geometry of nearly edge-on disks with an inner hole and a central object. In the investigation we consider two cases: that of geometrically thin disks (where the star is larger than the rim of the inner edge of the disk) and that of geometrically thick disks (when the star is smaller than the inner rim of the disk). Our model is appropriate for modelling substellar objects with primordial gas-rich disks, as activity (such as accretion or outflows) in such disks has low amplitude and can be ignored even when modelling early evolution stages. Furthermore, it can also be used to model any symmetric system with a disk and a spherical central body (star, brown dwarf or giant planet).

Key words: protoplanetary disks, stars: planetary systems

INTRODUCTION

A circumstellar disk is a torus, annular or ring-shaped structure, which consists of gas, dust, planetesimals, rocks and parent bodies such as asteroids and comets, surrounding a host star. Disk structure modelling has developed significantly in the past several years, from simple models (see e. g. [2, 3, 6, 9]) to detailed dust continuum radiative transfer models (see e. g. [4, 5, 12]) and models which, in addition to dust continuum radiative transfer, take into account gas radiative transfer in the disks [8, 13] with dust and gas temperature deviations in the upper layers. Radiative transfer models enable us to simulate disk structure with high precision, taking into account disk physical and dynamical properties, as well as disk inclination. In most cases, simple models also provide a plausible interpretation of the observational data.

In the simple disk models for spectral energy distribution simulations, the system's geometry of the inclined disk is taken into account by means of projecting the emitting area (i. e., multiplying by $\cos i$: see e. g. [3, 6]), while limiting the viewing angles for which the disk does not occult the central star. Previous modelling results for such configurations and for the systems with typical primordial disks around brown dwarfs [15] indicate, that the intensity of the spectral energy distribution profile decreases at wavelength $> 1\,\mu$m by means of increasing the inclination angle toward the observer. In this paper we investigate the geometrical peculiarity of a disk with an inner hole and a central object (potentially a star, brown dwarf, or giant planet), when the system is highly inclined toward the observer. Specifically we focus on the configurations when the disk partly/completely occults the central star. The basic ideas of the model for geometrically thin disks (i. e., the star is larger than the rim of the inner edge of the disk) have already been presented in [14] and [16]. In this paper, we generalize the existing model for geometrically thin disks and present the calculation method for geometrically thick disks (i. e., when the star is smaller than the inner rim of the disk) to cover all possible geometric combinations for protoplanetary disks with an inner hole.

GENERAL REMARKS AND CRITICAL ANGLES DETERMINATION

Non-zero inclined protoplanetary disks consist of four elements: the central object, the disk, and the inner and outer rims of the disk. The inner and outer rims are the inner and outer edges of the disk which, as we assume, have a flat shape and are perpendicular to the disk midplane. In this section we describe the procedure for calculating the projected emitting areas of these parts as a function of the disk inclination angle. The calculations are done in two steps. Firstly, the critical angles are calculated. The critical angles are the limiting values that determine the area configurations as presented in Fig. 1–2. Secondly, the projected emitting areas are calculated for the disk parts.

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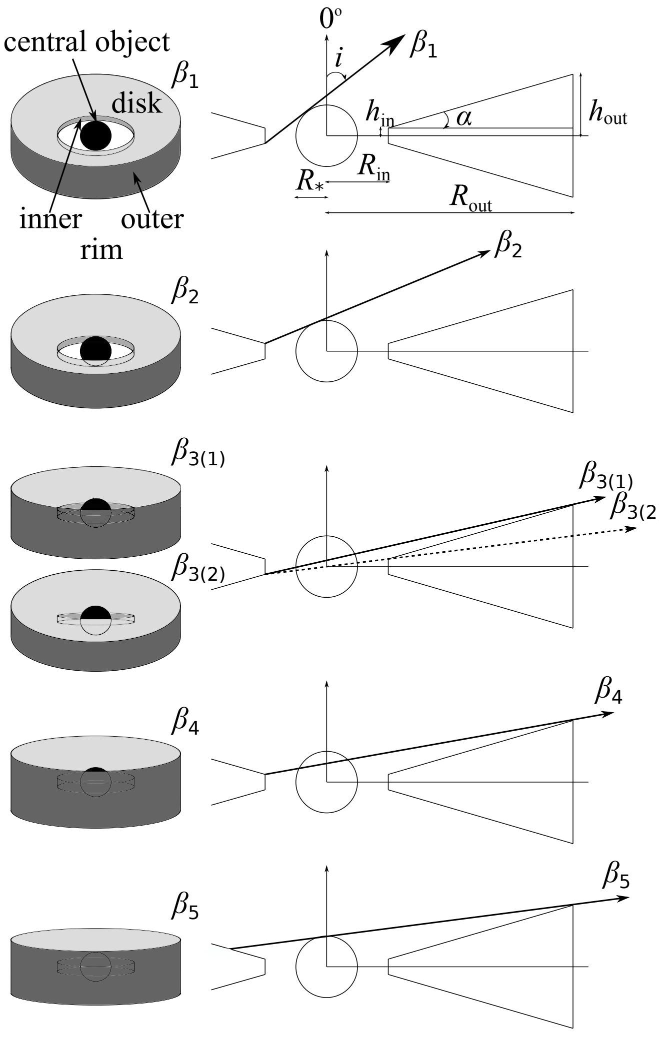

Fig. 1: Sketches of the key configurations for the critical angles and the corresponding projected areas for the circular central object with flaring disk ($h_{\rm in} < R_*$). The arrows are pointing to the observer.

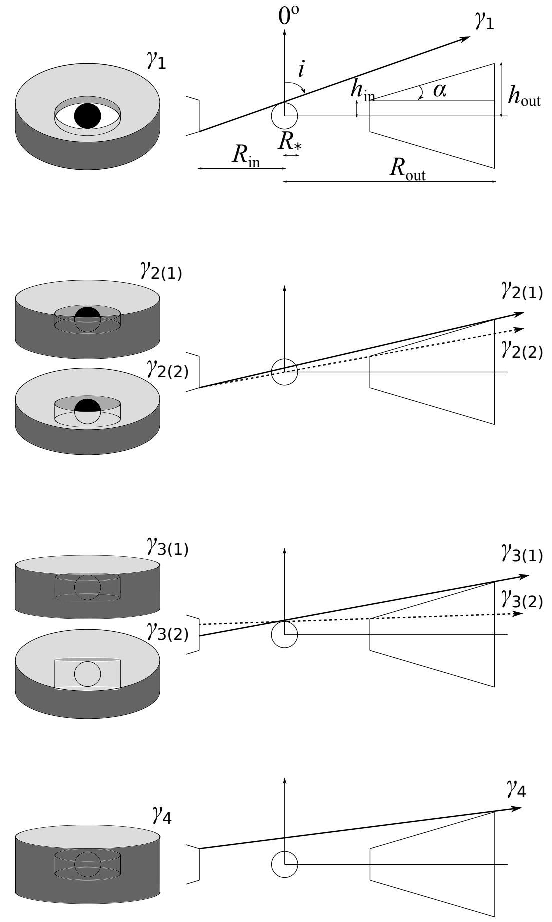

Fig. 2: Sketches of the key configurations for the critical angles and the corresponding projected areas for the circular the central object with flaring disk ($h_{\rm in} > R_*$).

CRITICAL ANGLES FOR THE SYSTEM WITH A GEOMETRICALLY THIN DISK

Fig. 1 depicts the sketches of the key configurations for the critical angles (right side of figure) and the corresponding projected areas of the spherical central object with a geometrically thin flaring disk as it is seen (although not resolved) by the observer (left panels of figure). For all disk configurations, the surface of the central object that is not occulted by the disk material is shown in black, the disk inner rim is in medium grey, the disk is in light grey and the disk outer rim in dark grey. The central object radius (R_*), disk inner and outer radii ($R_{\rm in}$ and $R_{\rm out}$, correspondingly), disk inner and outer half height ($h_{\rm in}$ and $h_{\rm out}$, correspondingly) are indicated in the figure. The arrows are pointing to the observer, indicating the inclination angle. Angles are measured from the 0° (orientation face-on disk).

These configurations correspond to the case when the disk inner rim half-thickness is lower than the radius of the central object (viz. $h_{\rm in} < R_*$). Such a system is characterized by 5 critical angles as is shown in Fig. 1. At inclinations i lower than the first critical angle β_1 the disk and the central object do not occult each other, being β_1:

$$\beta_1 = \\ = \arctan\left(\frac{h_{\rm in}R_{\rm in} - \sqrt{h_{\rm in}^2 R_{\rm in}^2 - (h_{\rm in}^2 - R_*^2)\,(R_{\rm in}^2 - R_*^2)}}{h_{\rm in}^2 - R_*^2}\right).$$

For the angle interval $\beta_1 < i \leq \beta_2$, the central object blocks the fraction of the disk inner rim, and the opposite side of the disk inner rim blocks the equal part of the central object. The second critical angle is determined as following:

$$\beta_2 = 2\arctan\left(\frac{h_{\rm in}}{(R_{\rm in}+R_*)} + \right. \left. + \sqrt{\frac{h_{\rm in}^2}{(R_{\rm in}+R_*)^2} + \frac{R_{\rm in}-R_*}{R_{\rm in}+R_*}}\right).$$

Once $i > \beta_2$ the central object shields the fraction of the disk as well. When the disk inclination exceeds the angle β_3 the inner hole is not visible anymore. The value of this angle depends on the disk flaring angle α. If $\alpha > \arctan(h_{\rm in}/R_{\rm in})$, then

$$\beta_{3(1)} = \frac{\pi}{2} - \arctan\left(\frac{h_{\rm out}+h_{\rm in}}{R_{\rm in}+R_{\rm out}}\right). \qquad (1)$$

If $\alpha \leq \arctan(h_{\rm in}/R_{\rm in})$, then

$$\beta_{3(2)} = \frac{\pi}{2} - \arctan\left(\frac{h_{\rm in}}{\sqrt{R_{\rm in}^2 - R_*^2}}\right). \qquad (2)$$

The angle β_4 determines the disk inclination when the inner rim is not visible anymore:

$$\beta_4 = \frac{\pi}{2} - \arctan\left(\frac{h_{\rm out}-h_{\rm in}}{R_{\rm out}+R_{\rm in}}\right).$$

Finally, the last critical angle is β_5, when the central object is completely hidden by the disk:

$$\beta_5 = \frac{\pi}{2} - \arctan\left(\frac{h_{\rm out}-R_*}{R_{\rm out}}\right).$$

CRITICAL ANGLES FOR THE SYSTEM WITH A GEOMETRICALLY THICK DISK

Systems with geometrically thick disks ($h_{\rm in} > R_*$) would have different critical angles, because in this configuration the central object can only shield the disk inner rim. Sketches of the key configurations for the geometrically thick disk are presented in Fig. 2.

Comparing Fig. 2 with Fig. 1 the correspondence of the calculation procedure for the critical angles of these two cases is evident:

$\gamma_1 = \beta_1$; $\gamma_{2(1)} = \beta_{3(1)}$; $\gamma_{2(2)} = \beta_{3(2)}$; $\gamma_4 = \beta_4$.

The angle γ_3 determines the moment when the central object is not seen anymore (i. e. it is an analogue of the angle β_5 for the geometrically thin disk). For the geometrically thick disk the value of this angle depends on α: if $\alpha > \arctan((h_{\rm in} - R_*)/R_{\rm in})$, then $\gamma_{3(1)} = \beta_5$; if $\alpha \leq \arctan((h_{\rm in} - R_*)/R_{\rm in})$, then

$$\gamma_{3(2)} = \frac{\pi}{2} - \arctan\left(\frac{h_{\rm in}-R_*}{R_{\rm in}}\right).$$

PROJECTED AREAS FOR SYSTEM WITH GEOMETRICALLY THIN DISK

After determining the values for the critical angles we can calculate the areas of the emitting surfaces projections for every system component. The analytical expressions for all projected areas have been derived using the classical method of mathematical analysis. The detailed description of this procedure for the geometrically thin disk has already been presented in [14, 16]. Here, we skip this part and present only the resulting generalized equations.

The area of the *central object* projection equals πR_*^2 if $i \leq \beta_1$. When $i > \beta_1$, the inner rim of the disk overlaps part of the central object, so its area should now be calculated with the expression below.

If $\beta_1 < i < \beta_3$ (Hereinafter β_3 stands for either $\beta_{3(1)}$ or $\beta_{3(2)}$, using equation (1) or (2), as appropriate.), then

$$S_{*(\beta_1<i<\beta_{3(1)})} = \pi R_*^2 + k(R_{\rm in}, x_1)\cos i - \\ - k(R_*, x_1) - 2h_{\rm in}x_1\sin i, \qquad (3)$$

where x_1 is a variable that is determined by R_*, $R_{\rm in}$, $h_{\rm in}$ and i (in Appendix A) we show how to compute x_1), k is a function introduced to shorten the equations, in terms of $R_{\rm in}$ and x_1 is given by:

$$k(R_{\rm in}, x_1) = R_{\rm in}^2 \arcsin\left(\frac{x_1}{R_{\rm in}}\right) + x_1\sqrt{R_{\rm in}^2 - x_1^2}.$$

The expression 3 must also be equal for $\beta_{3(2)}$, while using the outer rim parameters (radius and half thickness) of the disk, including the calculations for x_1.

If $\beta_{3(2)} \leq i < \beta_4$, then

$$S_{*(\beta_{3(2)}\leq i<\beta_4)} = k(R_{\rm in}, x_1)\cos i + \\ + k(R_*, x_1) - 2h_{\rm in}x_1\sin i. \qquad (4)$$

For the inclination angle $\beta_{3(1)} \leq i < \beta_5$ (if $\alpha > \arctan(h_{\rm in}/R_{\rm in})$) and for $\beta_4 \leq i < \beta_5$ (if $\alpha \leq \arctan(h_{\rm in}/R_{\rm in})$), the central object projected area is calculated using equation 4 also, while using the outer rim parameters, including the calculations for x_1.

For the disk inclination $i < \beta_3$ the projected area of the *inner rim* approximately equals

$$S_{in(i<\beta_3)} = 4h_{\rm in}\sin i\sqrt{R_{\rm in}^2 - (h_{\rm in}\tan i)^2} - \\ - \Delta S_{in(i<\beta_3)}, \qquad (5)$$

where $\Delta S_{\mathrm{in}(i<\beta_3)}$ is an area that is shielded by the central object, $\Delta S_{\mathrm{in}(i<\beta_3)} = 0$ for $i \le \beta_1$. For $\beta_1 < i \le \beta_2$,

$$\Delta S_{\mathrm{in}(i<\beta_3)} = k(R_*, x_1) - k(R_{\mathrm{in}}, x_1)\cos i + + 2h_{\mathrm{in}} x_1 \sin i. \quad (6)$$

For $\beta_2 < i < \beta_3$:

$$\Delta S_{\mathrm{in}(i<\beta_3)} = \cos i\,[k(R_{\mathrm{in}}, x_2) - k(R_{\mathrm{in}}, x_1)] + + k(R_*, x_1) - k(R_*, x_2) + 2h_{\mathrm{in}}(x_1 + x_2)\sin i,$$

where x_2 is a variable that is determined together with x_1 in Appendix A.

If $\beta_{3(1)} \le i < \beta_4$, then the inner rim area is

$$S_{in(\beta_{3(1)} \le i<\beta_4)} = \cos i\,[k(R_{\mathrm{in}}, x_3) + k(R_{\mathrm{out}}, x_3)] + + 2(h_{\mathrm{in}} - h_{\mathrm{out}})x_3 \sin i - \Delta S_{in(\beta_{3(1)}<i\le\beta_4)},$$

where x_3 is a variable that is determined by R_{in}, h_{in}, R_{out}, h_{out} and i. In Appendix A we show how to compute x_3.

If $\beta_{3(2)} \le i < \beta_4$ then

$$S_{in(\beta_{3(2)} \le i<\beta_4)} = \cos i \pi R_{\mathrm{in}}^2 - \Delta S_{in(\beta_{3(2)}\le i<\beta_4)},$$

where $\Delta S_{in(\beta_{3(1)}\le i<\beta_4)}$ and $\Delta S_{in(\beta_{3(2)}\le i<\beta_4)}$ are the areas that are shielded by the central object, and are calculated using the same equation, while using the outer or inner rims geometrical parameters, as appropriate. In case of inner rim parameters ($\beta_{3(2)} \le i < \beta_4$) it is:

$$\Delta S_{in(\beta_{3(2)}\le i<\beta_4)} = \cos i\,[k(R_{\mathrm{in}}, x_2) + k(R_{\mathrm{in}}, x_1)] + + k(R_*, x_1) - k(R_*, x_2) - 2h_{\mathrm{in}}(x_1 - x_2)\sin i.$$

The *outer rim* area equals $S_{\mathrm{out}} = 4R_{\mathrm{out}} h_{\mathrm{out}} \sin i$ for $i \ne 0$.

The area of the *disk* inclined at $i < \beta_3$ is described with the equation:

$$S_{d(i<\beta_3)} = (\pi R_{\mathrm{out}}^2 - \pi R_{\mathrm{in}}^2)\cos i - \Delta S_{d(i<\beta_3)},$$

where $\Delta S_{d(i<\beta_3)}$ is an area that is shielded by the central object, it $= 0$ for $i \le \beta_2$. And for $\beta_2 < i < \beta_3$:

$$\Delta S_{d(\beta_2<i<\beta_3)} = k(R_*, x_2) - - k(R_{\mathrm{in}}, x_2)\cos i - 2h_{\mathrm{in}} x_2 \sin i.$$

For the inclination angle $i \ge \beta_3$ the disk projected area is:

$$S_{d(i\ge\beta_3)} = \pi R_{\mathrm{out}}^2 \cos i - \Delta S_{d(i\ge\beta_3)},$$

where $\Delta S_{d(i\ge\beta_3)}$ is a part of the disk that is shielded by the central object and the disk inner rim, it depends on the i:
$\Delta S_{d(i\ge\beta_3)} = S_{*(\beta_3\le i<\beta_4)} + S_{in(\beta_3\le i<\beta_4)}$ if $\beta_3 \le i < \beta_4$;
$\Delta S_{d(i\ge\beta_3)} = S_{*(\beta_4\le i<\beta_5)}$ if $\beta_4 \le i < \beta_5$;
$\Delta S_{d(i\ge\beta_3)} = 0$ if $i \ge \beta_5$.

PROJECTED AREAS FOR THE SYSTEM WITH GEOMETRICALLY THICK DISK

As for the geometrically thin disk, the *central object* projected area equals πR_*^2 for $i \le \gamma_1$. For angles ranging $\gamma_1 < i < \gamma_2$ it is calculated from equation 3 (using the inner or outer rim parameters for $\gamma_{2(1)}$ and $\gamma_{2(2)}$, correspondingly). For $\gamma_{2(2)} \le i < \gamma_{3(2)}$ it is calculated using the equation (4) and for $\gamma_{2(1)} \le i < \gamma_{3(1)}$ also from equation (4), but for the outer rim geometrical parameters of the disk (including the calculations of the value for x_1).

The *inner rim* projected area for inclination $i < \gamma_2$ is calculated using equation (5). Where $\Delta S_{\mathrm{in}} = 0$ for $i \le \gamma_1$ and for $\gamma_1 < i < \gamma_2$ (Hereinafter γ_2 and γ_3 stand for either $\gamma_{2(1)}$ or $\gamma_{2(2)}$ and $\gamma_{3(1)}$ or $\gamma_{3(2)}$, correspondingly), ΔS_{in} is calculated with equation (6). For $\gamma_2 \le i < \gamma_4$, the inner rim projected area, depending on α, is

$$S_{in(\gamma_{2(1)}\le i<\gamma_4)} = \cos i\,[k(R_{\mathrm{in}}, x_3) - k(R_{\mathrm{out}}, x_3)] + + 2(h_{\mathrm{in}} - h_{\mathrm{out}})x_3 \sin i - S_{*(\beta_{3(1)}<i\le\beta_4)},$$

$$S_{in(\gamma_{2(2)}\le i<\gamma_4)} = \pi R_{\mathrm{in}}^2 \cos i - S_{*(\beta_{3(2)}<i\le\beta_4)},$$

$$S_{*(\gamma_3\le i<\gamma_4)} = 0.$$

The projected area of the disk *outer rim* is calculated the same way as that of the geometrically thin disk.

In case of the geometrically thick disk, the disk projected area is not shielded by the central object, and for all inclinations it equals $(\pi R_{\mathrm{out}}^2 - \pi R_{\mathrm{in}}^2)\cos i$. Exceptions are the cases when $\alpha > \arctan(h_{\mathrm{in}}/R_{\mathrm{in}})$:

$$S_{d(\gamma_{2(1)}\le i<\gamma_4)} = \pi R_{\mathrm{out}}^2 \cos i - - S_{in(\gamma_{2(1)}\le i<\gamma_4)} - S_{*(\beta_3\le i<\beta_4)};$$

and when $\alpha > \arctan((h_{\mathrm{in}} - R_*)/R_{\mathrm{in}})$:

$$S_{d(\gamma_{3(1)}\le i<\gamma_4)} = \pi R_{\mathrm{out}}^2 \cos i - S_{in(\gamma_{3(1)}\le i<\gamma_4)}.$$

CONCLUSIONS

We have presented a geometrical model for young stellar, substellar, and planetary objects, valid for highly inclined protoplanetary disks. We considered two cases: geometrically thin ($h_{\mathrm{in}} < R_*$) and geometrically thick ($h_{\mathrm{in}} > R_*$) disks. A typical example of a geometrically thin disk is a gas-rich primordial disk with a relatively small inner hole. We have also developed a geometrically thick disk model for transitional disks with large inner radii and, therefore, $h_{\mathrm{in}} \gg R_*$. The proposed model can be applied directly to the optically thick primordial disks coupled with a simple radiative transfer model. In the following paper we will present our model application on the example of circumsubstellar primordial disks, whose activity (like accretion, outflows) have smaller rates and for SEDs modelling can be ignored even at early evolution stages [1, 7, 10, 11].

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APPENDIX A:

EMITTING AREAS PROJECTIONS

To derive the equations for x_1 and x_2, that are required for the calculations of the emitting areas projections, let us consider the configuration that corresponds to the disk inclination $\beta_2 < i < \beta_3$. Fig. 3 shows the inner region of that disk.

Shaded dark grey area shows the projection of the part of the central object and shaded light grey area shows the projection of the part of the inner rim as seen by the observer. The central object projection area is limited by the circle (that describes the central object projection) and ellipse (that describes the upper edge of the disk inner rim); they are described by the equations

$$x^2 + y^2 = R_*^2,$$

$$\frac{x^2}{R_{\rm in}^2} + \frac{(y - h_{\rm in}\sin i)^2}{R_{\rm in}^2\cos^2 i} = 1. \qquad \text{(A1)}$$

Presenting these equations explicitly we obtain the system of equations

$$\begin{cases} y_1(x) = \pm\sqrt{R_*^2 - x^2}, \\ y_2(x) = h_{\rm in}\sin i \pm \cos i\sqrt{R_{\rm in}^2 - x^2}. \end{cases}$$

These two curves intersect in 4 points. The positive solutions of this system are

$$x_1 = \sqrt{\frac{-\xi_2 + \sqrt{\xi_2^2 - 4\xi_1\xi_3}}{2\xi_1}},$$

$$x_2 = \sqrt{\frac{-\xi_2 - \sqrt{\xi_2^2 - 4\xi_1\xi_3}}{2\xi_1}},$$

where:

$$\xi_1 = \sin^4 i,$$

$$\xi_2 = 2\sin^2 i\left[R_{\rm in}^2\cos^2 i - R_*^2 + h_{\rm in}^2(\cos^2 i + 1)\right],$$

$$\xi_3 = \left(R_*^2 - h_{\rm in}^2\sin^2 i - R_{\rm in}^2\cos^2 i\right)^2 - \left(2h_{\rm in}R_{\rm in}\cos i\sin i\right)^2.$$

As a result of the symmetry of the upper and lower edge of the disk inner rim relative to the OX axis, and as it is seen from the geometrical constructions in Fig. 3, x_1 is also the solution for the equations describing the central object circle and the ellipse of the lower edge of the disk inner rim located in the first quarter.

To derive the equation for x_3, let us consider the configuration that corresponds to the disk inclination $\beta_{3(1)} \le i < \beta_4$. In Fig. 4 grey colour shows the part of the disk inner rim as seen by the observer. The central object is not shown in Fig. 4, to avoid figure obstruction.

Hence the variable is an intersection point of two ellipses describing the upper edges of the inner $y_2(x)$ and outer $y_3(x)$ rims. According to the geometrical constructions we use the positive half of the $y_2(x)$ (Eq. (A1)) and the negative half for the $y_3(x)$. The system of equations is

$$\begin{cases} y_2(x) = h_{\rm in}\sin i + \cos i\sqrt{R_{\rm in}^2 - x^2}, \\ y_3(x) = h_{\rm out}\sin i - \cos i\sqrt{R_{\rm out}^2 - x^2}. \end{cases}$$

The positive solution of this system is

$$x_3 = \frac{\sqrt{4R_{\rm in}^2R_{\rm out}^2 - \left[\tan^2 i(h_{\rm out} - h_{\rm in})^2 - R_{\rm in}^2 - R_{\rm out}^2\right]^2}}{2\tan i\,(h_{\rm out} - h_{\rm in})}.$$

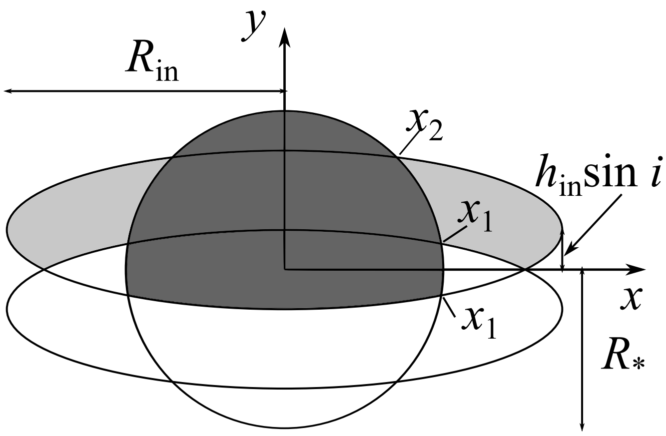

Fig. 3: Schematic of the projected area of the central object shielded by the inner rim, for the $\beta_2 < i < \beta_3$.

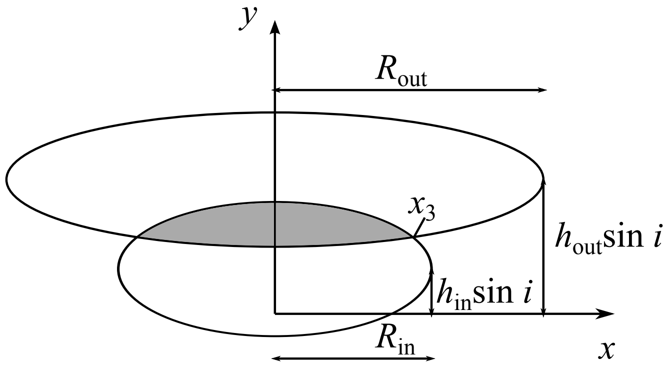

Fig. 4: Schematic of the projected area of the inner rim shielded by the outer rim, for the $\beta_{3(1)} < i \leq \beta_4$.