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COMPLEX MODELING OF RUIN PROCESS
OF INSURANCE COMPANY

The article is dedicated to research and development of an actuarial model for calculating the time of occurrence of bankruptcy probability for an insurance company. The novelty of the article is that we analyze an insurance company with its investment activity, which is its additional income.
Keywords: premium; claim; stochastic capital; ruin time.

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КОМПЛЕКСНЕ МОДЕЛЮВАННЯ ПРОЦЕСУ
БАНКРУТСТВА СТРАХОВОЇ КОМПАНІЇ

У статті досліджено і розроблено актуарну модель розрахунку часу настання банкротства для страхової компанії. Новизна статті полягає у тому, що аналізується страхова компанія, яка здійснює інвестиційну активність, що є додатковою статтею її доходу.
Ключові слова: премії, виплати, стохастичний капітал, час банкрутства.
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КОМПЛЕКСНОЕ МОДЕЛИРОВАНИЕ ПРОЦЕССА
БАНКРОТСТВА СТРАХОВОЙ КОМПАНИИ

В статье исследована и разработана актуарная модель расчета времени наступления банкротства страховой компании. Новизна статьи заключается в том, что анализируется страховая компания, которая осуществляет инвестиционную активность, что является дополнительной статьей ее дохода.
Ключевые слова: премии, выплаты, стохастический капитал, время банкротства.

The problem statement. Today it is impossible to imagine a market economy without risks. They are involved almost in every economic activity. There is a great need in measuring, predicting and minimizing risks. Insurance services are one of the industries, which permanently experience the risks of bankruptcy. That is why calculating the ruin probability for insurance companies are one of the problems that need well-developed mathematical models [1]. Nowadays Ukrainian insurance companies are searching for new ways of profitability and competitiveness. Western European insurance companies has an option of investing their funds for additional profit. This is why there is a great necessity for creation and development of actuarial models for Ukrainian insurance to provide them the possibility to invest their funds for additional profit.

The analysis of the main research and publications. One of the first studies in this area was conducted at the beginning of the XXth century. Since then, mathematical methods of ruin probability calculation developed and accumulated a great variety of models and approaches. With the permanent growth of economic needs, insurance services increase steadily in the economies of all developed countries. Insurance is one of the youngest industries in any economy, it is experiencing a stage of active development. In global practice of developed countries, well-organized insurance

services are involved in many economic sectors like investment activity of insurance companies. This article studies how actuarial mathematical tools can positively affect theoretical and practical development of insurance. Development of theoretical, methodological, organizational and legal bases for insurance market have been contributed by many economists, such as: M. Alexandrova [2], O. Baranovsky [4], N. Vnukova [5], K. Voblyi [6], O. Hamankova [7], A. Hvozdenko [8], O. Zaletov [10], O. Zaruba [11], N. Reitman [12], V. Bazylevych [13], S. Osadets [14], V. Shafov [16], H. Albrecher [17], S. Asmussen [18], J. Beekman [19], R. Cardoso [20], H. Cramer [21], D. Dickson [23], F. De Vilder [22], P. Embrechts [24], A. Frolova [25], V. Kalashnikov [26] and others.

Unsolved issues: one of the main problems at present time for actuarial analysis of the Ukrainian insurance market is the lack of statistical data, which is necessary for any econometric modeling. This is why there is a great necessity for actuarial models that involve fewer statistical information. We analyze methods of calculation for ruin probabilities for an insurance company considering its investing activity. We consider an insurance company in the case when a premium rate is bounded by some nonnegative random function and the capital of an insurance company is invested in a risky asset the price of which follows a geometric Brownian.

The goals and tasks of the article. The goal of the article is a creation of new types of actuarial models for the analysis of ruin probabilities that can be helpful for Ukrainian insurance companies in cases of their investment activities. There are different methods for approximating the distribution of aggregate claims and their corresponding stop-loss premium by means of a discrete compound Poisson distribution and its corresponding stop-loss premium. This discretization is an important step in the numerical evaluation of the distribution of aggregate claims, because recent results on recurrence relations for probabilities only apply to discrete distributions. The discretization technique is efficient in a certain sense, because a properly chosen discretization gives raise to numerical upper and lower bounds on the stop-loss premium, giving the possibility to calculate the numerical estimates for the error on the final numerical results.

The results. We consider an insurance company in the case when the premium rate is a bounded nonnegative random function c_t and the capital of the insurance company is invested in a risky asset the price of which follows a geometric Brownian motion with mean return α and volatility $\sigma > 0$. If $\beta := 2a / \sigma^2 > 0$ we find exact the asymptotic upper and lower bounds for the ruin probability as the initial endowment u tends to infinity, i.e. we show that $C_* u^{-\beta} \leq \Psi(u) \leq C^* u^{-\beta}$ for sufficiently large u. Moreover, if $c_* = c^* r^{\gamma t}$ $\gamma < 0$ we find the exact asymptotics of the ruin probability, namely $\Psi(u) \sim u^{-\beta}$. $\beta \leq 0$, $\psi(u) = 1$ $u \geq 0$. We investigate the problem of consistency of risk measures with respect to usual stochastic order and convex order. It is shown that under weak regularity conditions risk measures preserve these stochastic orders. This result is used to derive bounds for risk measures of portfolios. As a by-product, we extend the characterization of coherent, law-invariant risk measures with the property to unbounded random variables. A surprising result is that the trading strategy yielding the optimal asymptotic decay of the ruin probability simply consists in holding a fixed quantity (which can be explicitly calculated) in a risky asset, independent of the current reserve. This result is in apparent contradiction to

the common believe that "rich" companies should invest more in risky assets than "poor" ones. The reason for this seemingly paradoxical result is that the minimization of the ruin probability is an extremely conservative optimization criterion, especially for "rich" companies [3, *35*].

It is well-known that the analysis of activity of an insurance company under conditions of uncertainty is of great importance [9, *162*]. Starting from the classical papers of Cramer and Lundberg who were the first to consider the ruin problem in stochastic environment, this subject has attracted much attention. Recall that, in the classical Cramer-Lundberg model satisfying the Cramer condition and the positive safety loading assumption, the ruin probability as a function of the initial endowment decreases exponentially [15, *117*]. The problem is subsequently extended to the case when the insurance risk process is a general Levy process.

It is obvious that risky investment can be dangerous: disaster may come in the period when the market value of assets is low and the company will not be able to cover losses by selling these assets because of price fluctuations. Regulators are rather attentive to this issue and impose stringent constraints on company portfolios. Typically, junk bonds are prohibited and a prescribed (large) part of the portfolio should contain non-risky assets (e.g., treasury bonds) while in the remaining part only risky assets with good ratings are allowed. The common notion that investments in an asset with stochastic interest rate may be too risky for an insurance company can be justified mathematically.

We deal with the ruin problem for an insurance company investing its capital in a risky asset specified by a geometric Brownian motion:

$$dV_t = V_t\big(adt + \sigma d\omega_t\big), \tag{1}$$

where $\big(\omega_t, t \geq 0\big)$ is a standard Brownian motion and $\alpha > 0$, $\sigma > 0$.

It turns out that in this case of small volatility, i.e. $0 < \sigma^2 < 2a$, the ruin probability is not exponential but a power function of the initial capital with the exponent $\beta := 2a / \sigma^2 - 1$. It will be noted that this result holds without the requirement of positive safety loading. Also, for large volatility, i.e. $\sigma^2 > 2a$, the ruin probability equals 1 for any initial endowment.

In all these papers the premium rate is assumed to be constant. In practice this means that a company should obtain a premium with the same rate continuously. We think that this condition is too restrictive and it significantly bounds the applicability of the above mentioned results in practical insurance settings.

The numerical calculation of finite time ruin probabilities for two particular insurance risk models are being analyzed. The first model allows for the investment at a fixed rate of interest of the surplus whenever this is above a given level. Our second model is the classical risk model but with the insurer's premium rate depending on the level of surplus.

Our methodology of calculating finite time ruin probabilities is to bound the surplus process by discrete-time Markov chains; the average of the bounds gives an approximation to the ruin probability.

Our primary purpose in this paper is to discuss the numerical calculation of finite time ruin probabilities for two particular insurance risk models. Both models are extensions of the classical risk model. For each model, *U(t)* is a random variable

which denotes the surplus at time $t(\geq 0)$, $\{U(t)\}_{t \geq 0}$ is a continuous time stochastic process; the aggregate claims in $[0,t]$ are denoted $S(t)$, where $S(t)$ has a compound Poisson distribution with Poisson parameter λ; individual claim amounts have *cdf G(t)*, *pdf g(x)* and mean m_1. We assume that *cdf G(0) = 0*, so that all claims are positive. We assume without loss of generality that $\lambda = 1 = m_1$.

It would be possible to have more than two bands for the surplus with a different rate of premium income at time t depending on the band in which $U(t)$ lies. However, all our numerical examples assume just two bands, so we have presented the model in this way.

An essential feature of the two models studied in this paper is that they are time-homogeneous Markov processes; the level of the surplus at any given time is sufficient to determine probabilistically its level at any time h later. This is the feature that we will exploit in this paper to obtain bounds for the finite time ruin probabilities for our two models. We do not need to assume any form of "net profit condition" for our two models, but we do need to assume that c, c_1 and $c_2 > 0$.

Our aim is to produce bounds for this probability; approximate values of the probability can be calculated by averaging the upper and lower bounds. However, it is not always possible to produce absolute bounds.

The surplus process of an insurance portfolio is defined as the wealth obtained by the premium payments minus the reimbursements made at the times of claims. When this process becomes negative (if ever), we say that ruin has occurred. The general setting is the Gambler's Ruin Problem. We address the problem of estimating derivatives (sensitivities) of ruin probabilities with respect to the rate of accidents. Estimating probabilities of rare events is a challenging problem, since naive estimation is not applicable.

Solution approaches are very recent, mostly through the use of Importance Sampling techniques. Sensitivity estimation is an even harder problem for these situations. We study different methods for estimating ruin probabilities: one via importance sampling (IS), and two others via indirect simulation: the storage process (SP), which restates the problems in terms of a queuing system, and the convolution formula (CF). To estimate the sensitivities, we apply the RPA method to importance sampling, the IPA method to storage process and the Score Function method to convolution formula. Simulation methods are compared in terms of their efficiency, a criterion that appropriately weighs precision and CPU time. As well, we indicate how other criteria such as set-up time and prior formulas development may actually be problem-dependent. The canonical model in Risk Theory assumes that claims due to accidents arrive according to a Poisson process $N(t)$ of rate λ. The successive claim amounts, denoted $\{Y_i\}$, are i.i.d. random variables with general distribution G and premiums are received at a constant rate c.

The event epochs of the process $N(t)$ are denoted by $\{T_n n \geq 0\}$, and $W_n = T_n - T_{n-1}$ are the interarrival times. The cumulative claims process:

$$S(t) = \sum_{i=1}^{N(t)} Y_i, \tag{2}$$

is a compound Poisson process. We shall often write $U_n = U(T_n)$ to denote the embedded discrete event process and $S_n = \sum_{i=1}^{n} Y_i$ with an obvious abuse of notation. If

we set $\tau = \min\{n : U(T_n < 0)\} = \min\{n : u + cT_n \le S_n\}$ then the ruin probability is [9, 217]. Call $\beta = E[Y_1]$. $c \le \lambda\beta$, $\psi(u, \lambda)$ for all initial endowment u. As a consequence of this result, it is common to assume that premiums satisfy $c \le \lambda\beta$.

Conclusions. We analyzed methods of calculation of ruin probabilities for insurance company in presents of its investing activity. We considered an insurance company in the case when the premium rate is a bounded by some nonnegative random function and the capital of the insurance company is invested in a risky asset whose price follows a geometric Brownian.

The weak development of insurance market in Ukraine is explained by the low incomes of Ukrainians and their disinterest in spending money on insurance, but for some exceptional cases.

The analyzed economic and mathematical models are recommended to be used by Ukrainian insurance companies for increasing their profitability and diversification of ruin risks.

The prospects for further development of the problem. Although there are several methods for calculation the ruin probabilities for insurance companies, this study may enrich existing methods, for cases of investment activities for Ukrainian insurance companies.

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