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OPTIMAL ADVERTISING POLICY FOR THE DIFFUSION MODEL OF INNOVATIONS

This paper offers an optimal advertising strategy for new product diffusion model. We develop a diffusion model of innovations based on the Bass model by including the advertising factor. This factor is incorporated as an integral of advertising intensity weighed on exponential term, so all advertising strategies during the time of existence of a new product at a market have been considered. Using the standard methods of optimization the optimal strategy is found for a linear advertising intensity function. A numerical example is given to illustrate the model.
Keywords: *diffusion of innovations; the Bass model; optimal advertising policy.*

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ОПТИМАЛЬНІ РЕКЛАМНІ СТРАТЕГІЇ ПРИ МОДЕЛЮВАННІ ПОПИТУ НА ІННОВАЦІЙНИЙ ПРОДУКТ

У статті побудовано модель дифузії інновацій з урахуванням впливу всіх рекламних заходів, що було здійснено протягом часу існування на ринку інноваційного продукту. Проаналізовано різні рекламні стратегії з точки зору інтенсивності витрат на рекламу в часі, знайдено оптимальну з них за однакових витрат та лінійної функції інтенсивності. Теоретичний матеріал проілюстровано числовим прикладом.
Ключові слова: *дифузія інновацій; модель Басса; оптимальна рекламна стратегія.*
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ОПТИМАЛЬНЫЕ РЕКЛАМНЫЕ СТРАТЕГИИ ПРИ МОДЕЛИРОВАНИИ СПРОСА НА ИННОВАЦИОННЫЙ ПРОДУКТ

В статье построена модель диффузии инноваций с учетом всех рекламных мероприятий, которые имели место на протяжении всего времени существования инновационного продукта на рынке. Проведен анализ разных рекламных стратегий с точки зрения интенсивности затрат на рекламу во времени, найдена оптимальная из них при постоянных затратах и линейной функции интенсивности. Теоретический материал проиллюстрирован числовым примером.
Ключевые слова: *диффузия инноваций; модель Басса; оптимальная рекламная стратегия.*

Introduction and brief historical overview. There are many problems in economics associated with forecasting the demand for a new product, a service or a technology. In this case statistical methods cannot be used due to the lack of historical data on demand.

The diffusion model in marketing was initially developed by F.M. Bass (1969). The was model designed to answer the question of how many customers will adopt a new product in time t. It can be used in making pre-launch, launch and post-launch strategic choices for capturing life-cycle dynamics of new products.

The Bass model is built on two basic assumptions, that potential adopters of an innovation are influenced by two types of communication channels: media and interpersonal channels. The main idea of the model is that the probability of a current purchase is a linear function of the number of prior purchases. More precisely,

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$$h(t) = \frac{f(t)}{1 - F(t)} = p + qF(t), \tag{1}$$

where $h(t)$ is the hazard function; p and q are the constant parameters representing the coefficient of innovation (or the coefficient of external influence) and the coefficient of imitation (or the coefficient of internal influence), respectively. The hazard function $h(t) = f(t) / (1 - F(t))$ is the rate of the first-purchase times density, where $F(t)$ are the cumulative density function of adopters at time t, $f(t) = dF(t) / dt$. The interpretation of the hazard function is that if it is multiplied by small time increment dt it represents the conditional probability that a random customer who has not made the purchase will do it in the next small period dt.

After some rearrangement the equation (1) can be formulated as follows:

$$\frac{dN}{dt} = (pm + (q - p)N(t) - \frac{q}{m}[N(t)]^2), \tag{2}$$

where m describes the market potential size of the new product and $N(t)$ is the cumulative number of adopters by time t.

Estimating the Bass model parameters can be provided in two ways. If we have some set of historical sales data, linear or non-linear regression can be used (Kenneth, Dinesh and Sheila, 2009). There are more sophisticated approaches for the estimation of these parameters, including the maximum likelihood estimation (Srinivasan and Mason, 1986) and the hierarchical Bayes estimation (Lenk and Rao, 1990). Otherwise, parameters can be estimated by judgmental methods such as the analogy method or the Delphi method. Furthermore, using fuzzy logic for modelling a diffusion process can be a future research on this direction.

By integrating the equation (2) considering the initial condition $N(0) = 0$ we have

$$N(t) = m\frac{1 - e^{-(p+q)t}}{1 + (q/p)e^{-(p+q)t}}. \tag{3}$$

Since the appearance of the Bass model in 1969 diffusion models have been increasingly complex. Several researchers generalized the original model by introducing marketing variables such as price and advertising.

B. Robinson and C. Lakhani (1975) were the first to introduce into the model a price factor as an exponential term that multiplies the original Bass expression. G. Feichtinger (1982) presented a model with varying market potential as a function of product price. D. Horsky and L.S. Simon (1983) modified the Bass model by introducing the coefficient of innovation p as a logarithmic function of advertising.

S. Kalish (1985) built a two-step theoretical model grounded in the utility maximization principle. In his model advertising and price were included in some way.

F.M. Bass, D. Jain and T. Krishnan (1994) proposed a generalized form that incorporates the effects of marketing mix variables on the likelihood of adoption. They do so by adding to the original model a multiplicative factor

$$x(t) = 1 + \alpha \Pr'(t)/\Pr(t) + \beta A'(t)/A(t),$$

where α, β — coefficients; $PR(t)$, $A(t)$ — price and advertising at time t, respectively. There is the generalized Bass model software based on F.M. Bass, D. Jain and

T. Krishnan (1994) extension. In particular, it is an Excel model to forecast the adoption of a new product.

Note that all mentioned here extensions of the Bass model link diffusion speed only to instantaneous change of advertising and do not take into account advertising "life-time". In this paper we introduce into the diffusion model a multiplicative factor which contains an integral of advertising intensity weighed on exponential term. So all advertising strategies during the time of existence of a new product at a market have been considered.

For an extensive review of the literature in this field the reader is referred to (Radas, 2006).

Diffusion model with advertising.

1. Model assumptions and formulation. The main assumptions of the model are the same as for the original Bass model:

1) diffusion process is binary (consumer either adopts, or waits to adopt);
2) constant maximum potential number of buyers (m);
3) eventually, all m will buy the product;
4) no repeat purchase or replacement purchase.

But the main difference in assumptions is the dependence of mass media and word-of-mouth impact on adoption time through marketing strategies. It is represented in the model as a multiplier, which reflects the intensity of advertising till time t.

Our model is developed to estimate the diffusion in the pre-established time horizon of the product existence at a market $[0;T]$. Let $I(t)$ – intensity of advertising at time t; α, λ – parameters; p_0 – interest rate; $G(T)$ – bound of costs on advertising in time T.

Then the cumulative number of buyers $N(t)$ at time t is the solution of the differential equation:

$$\frac{dN}{dt}=\left(pm+(q-p)N(t)-\frac{q}{m}[N(t)]^2\right)\left(1+\alpha\int_0^t \exp[\lambda(x-t)]I(x)dx\right), \quad (4)$$

$$0\le t\le T,\ N(0)=0;$$

$$\int_0^T \exp[-p_0x]I(x)dx\le G(T). \quad (5)$$

The condition (5) allows estimating cumulative demand in time T under different intensive function of advertising $I(x)$ when the costs on this campaign are constant by accounting time value of money. Thus, investigating the solution function of the problem (4–5) let us choose the optimal advertising strategy. Multiplier $\exp[\lambda(x-1)]$ in the integral on the right part of the equation (4) weighs the influence of advertising represented by $I(x)$. For example, on Figure 1 linear function $I(t)=40-\frac{40}{3}t$ is represented by solid line, while the integral $\int_0^t \exp[1.5(x-t)]I(x)dx$ as a function of t is drawn by dotted line. Thus, the rate of diffusion change reflects the intensity of advertising.

Note, that the model (4) is classical if $\alpha = 0$.

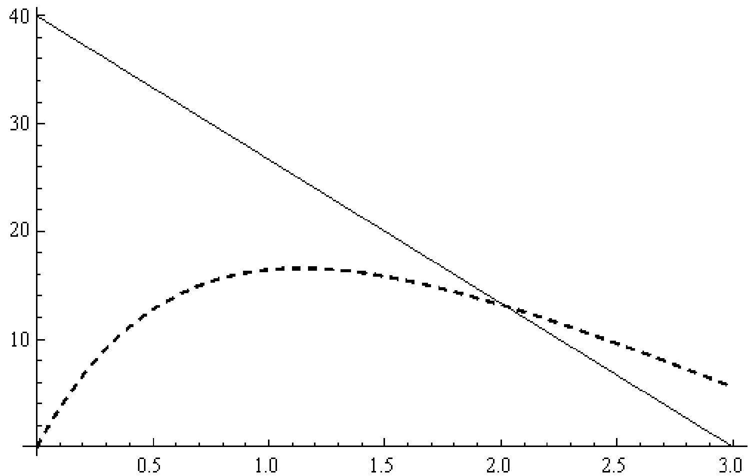

Figure 1. **Reflection of the diffusion rate by advertising intensity,**
developed by the author

2. Solving the model in general case. The solution of (4) can be found by means of integrating both parts of the differential equation:

$$\frac{dN}{pm+(q-p)N-\frac{q}{m}N^2}=\left(1+\alpha\int_0^t \exp[\lambda(x-t)]I(x)dx\right)dt. \qquad (6)$$

Part of (6) by integrating is given as

$$\int\frac{dN}{pm+(q-p)N-\frac{q}{m}N^2}=\frac{1}{q}\ln\left|\frac{N+mp/q}{N-m}\right|+c,\ c=const.$$

And taking into account the initial condition $N(0)=0$ we have

$$c=-\frac{1}{q+p}\ln\frac{p}{q}.$$

Integrating the right part of (6) and taking into consideration the condition $m > N$ after all the general integral of (6) is obtained:

$$\ln\left(\frac{Nq/p+m}{m-N}\right)=(p+q)\left(t+\alpha\int_0^t\left(\int_0^y \exp[\lambda(x-y)]I(x)dx\right)dy\right).$$

After rearranging the general solution of (4) is as follows

$$N=m\frac{\exp\left[(p+q)\left(t+\alpha\int_0^t\left(\int_0^y \exp[\lambda(x-y)]I(x)dx\right)dy\right)\right]-1}{q/p+\exp\left[(p+q)\left(t+\alpha\int_0^t\left(\int_0^y \exp[\lambda(x-y)]I(x)dx\right)dy\right)\right]},\ 0\le t\le T, \qquad (7)$$

where the function $I(x)$ is so that $\int_0^T \exp[-p_0x]I(x)dx\le G(T)$.

The case of a linear intensity of advertising.

1. Solving the model. The most natural function of advertising intensity is linear one because of its simplicity. Hence, let $I(x) = a + bx$ which satisfies the condition

$$\int_0^T \exp[-p_0 x] I(x) dx = G_0. \tag{8}$$

Then, the function (7) transforms into

$$N(t) = m\frac{\exp[(p+q)(t+\alpha S(t))]-1}{q/p+\exp[(p+q)(t+\alpha S(t))]}, \tag{9}$$

where

$$S(t) = \frac{a}{\lambda}\left(t+\frac{1}{\lambda e^{\lambda t}}-\frac{1}{\lambda}\right)+\frac{b}{\lambda}\left(\frac{t^2}{2}-\frac{t}{\lambda}-\frac{1}{\lambda^2 e^{\lambda t}}+\frac{1}{\lambda^2}\right). \tag{10}$$

So we want to research different strategies of advertising expenditures in time T when advertising budget have the same level G_0. Hence, one parameter of the a and b can be excluded from (9).

Thus, from the equation (8) we obtain:

$$a = \frac{G_0 p_0 e^{pTt} + bT}{e^{p_0 T}-1} - \frac{b}{p_0}. \tag{11}$$

Applying (11) to (10) the function $N(t,b)$ is finally obtained.

2. Optimal strategy of advertising. The problem of optimal strategy choice is the choice of parameters a and b, which maximize the cumulative number of buyers in time T. As it was accepted the parameter a can be excluded from the model. Therefore the problem is: $N(t,b)\big|_{t=T} = N(b) \to \max_b$.

Note that the function $N(b)$ can be rewritten as follows:

$$N(b) = m\frac{\gamma e^{\beta b}-1}{q/p+\gamma e^{\beta b}},$$

where

$$\gamma = \exp\left[(p+q)\left(T+\frac{\alpha G_0 p_0 e^{p_0 T}}{\lambda^2(e^{p_0 T}-1)}(e^{-\lambda T}+\lambda T-1)\right)\right],$$

$$\beta = \alpha(p+q)\left[\frac{(\lambda T)^2}{2}-\lambda T-\frac{1}{e^{\lambda T}}+1+\left((\lambda T)^2-\lambda T+\frac{\lambda T}{e^{\lambda T}}\right)\left(\frac{1}{e^{p_0 T}-1}-\frac{1}{p_0 T}\right)\right].$$

The main properties of $N(b)$ are:

a) continuity on b (as a superposition of elementary functions);

b) it is defined for $\frac{G_0 p_0^2}{1-p_0 T-e^{-p_0 T}} \le b \le \frac{G_0 p_0^2 e^{p_0 T}}{e^{p_0 T}-1-p_0 T}$;

c) its derivative is $N'(b) = m\frac{\gamma\beta e^{\beta b}(q/p+1)}{(q/p+\gamma e^{\beta b})^2}$.

Consequently, the sign of $N'(b)$ depends on the sign of β because of $\gamma > 0$. Thus if

$$\frac{(\lambda T)^2}{2}-\lambda T-\frac{1}{e^{\lambda T}}+1+\left((\lambda T)^2-\lambda T+\frac{\lambda T}{e^{\lambda T}}\right)\left(\frac{1}{e^{p_0 T}-1}-\frac{1}{p_0 T}\right)<0,$$

than the function $N(b)$ is monotonically decreasing on its domain owing to $\alpha > 0$, $p + q > 0$. In the opposite, this function is increasing.

Given the properties 1—3, it is possible to conclude that the optimal strategies are as following:

1) if parameters λ, T, p_0 are so that $\beta > 0$, than

$$b_{opt} = \frac{G_0 p_0^2 e^{p_0 T}}{e^{p_0 T} - 1 - p_0 T}, \; a_{opt} = 0; \tag{12}$$

2) if parameters λ, T, p_0 are so that $\beta < 0$, than

$$b_{opt} = \frac{G_0 p_0^2}{1 - p_0 T - e^{-p_0 T}}, \; a_{opt} = -b_{opt} T. \tag{13}$$

Let $x = \lambda T$, $s = p_0 T$. Note, when s is fixed than the function $\beta = \beta(z,s) > 0$ for $z > z^*$, moreover the more is s, the less is some point z^*.

So $\beta > 0$ when there is long-term project, advertising is inefficient, the value of interest rate is quite large. But generally in real situations $\beta < 0$ and (13) is used for finding the optimal solution.

3. Numerical example. Let us illustrate the theoretical research by numerical examples. Differences in $N(T)$ under different values of parameters a, b (including optimal values) are presented in Table 1.

Table 1. **Numerical examples,** *developed by the author*

Data-in	β	***a, b***	***N(T)***
$m = 80000$, $T = 1$, $p = 0.04$, $q = 0.001$, $\alpha = 0.00005$, $\lambda = 2.5$, $G_0 = 20000$, $p_0 = 0.08$	-0.408 < 0	$a = 0$, $b = 0$ (classical model without advertising)	3138.36
		$a = 0$, $b = 42183.8$	3776.72
		$a = 20810.7$, $b = 0$	3944.79
		$a_{opt} = 41073.8$, $b_{opt} = -41073.8$ (due to formula (13))	4108.1
$m = 80000$, $T = 5$, $p = 0.04$, $q = 0.001$, $\alpha = 0.00005$, $\lambda = 2.5$, $G_0 = 30000$, $p_0 = 0.1$	0.715 > 0	$a = 0$, $b = 0$ (classical model without advertising)	14532.2
		$a = 7624.48$, $b = 0$	16352.2
		$a_{opt} = 0$, $b_{opt} = 3325.8$ (due to formula (12))	16371.7

Conclusions. In this study optimal strategies of advertising expenditures in time T for the prediction of a demand for innovation product has been developed. In particular, the algorithm for finding optimal parameters a and b for intensity advertising function $I(x) = a + bx$ depending on the in-data λ, p_0, T.

Future research in this area can be devoted to the creation of the diffusion innovation model taking into consideration not only an advertising factor but also other factors affecting the model, such as price etc. It would be interesting to consider the case of fuzzy parameters λ, α because of their imprecise nature.

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