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PRACTICAL ASPECTS OF MARKOWITZ THEORY APPLICATION FOR PORTFOLIO ANALYSIS

The paper deals with H. Markowitz's model for optimal portfolio selection of European and American instruments. The procedure for construction and analysis of a multi-asset portfolio is presented. Combinations of assets of 20 companies selected from DAX30, DJIA, EURONEXT100, NASDAQ100 and WIG30 are analyzed. It is shown that the theoretical values of the return on a portfolio and the risk of a portfolio, determined on the basis of historical data, differ from their real values. It is noted that average value should not be used for portfolio selection.

Keywords: *Markowitz's model; portfolio selection; rate of return; risks.*

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ПРАКТИЧНІ АСПЕКТИ ЗАСТОСУВАННЯ ТЕОРІЇ МАРКОВІЦА В ПОРТФЕЛЬНОМУ АНАЛІЗІ

У статті розглянуто застосування портфельного аналізу в моделі Марковіца. Представлено процедуру формування та аналізу багатокомпонентного портфеля. Портфелі для прикладу сформульовано для європейського та американського фінансових ринків. Для аналізу обрано по 20 компаній з біржовими індексами на: DAX30, DJIA, EURONEXT100, NASDAQ100, а також WIG30. Показано, що норма прибутковості, а також ризик, визначений на основі історичних даних, відхиляються від реальних значень норм прибутковості та ризику. Кореляція між реальними значеннями норм прибутковості для портфелів, визначених за Марковіцем, та їх теоретичними значеннями, розрахованими при проектування складу портфеля, є досить слабкою. Додатково відмічено, що середнє значення не може слугувати основою для побудови портфеля.

Ключові слова: *модель Марковіца; вибір портфеля; норма прибутковості; ризик.*
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ПРАКТИЧЕСКИЕ АСПЕКТЫ ПРИМЕНЕНИЯ ТЕОРИИ МАРКОВИЦА В ПОРТФЕЛЬНОМ АНАЛИЗЕ

В статье рассмотрено применение в портфельном анализе модели Марковица. Представлено процедуру формирования и анализа многокомпонентного портфеля. Примерные портфели были сформулированы для европейского и американского финансовых рынков. Для анализа выбраны по 20 компаний с биржевыми индексами на: DAX30, DJIA, EURON\EXT100, NASDAQ100, а также WIG30. Показано, что норма прибыльности, а также риск, определённые на основе исторических данных, отклоняются от реальных значений норм прибыли и риска. Корреляция между реальными значениями норм прибыльности для портфелей, определённых по Марковицу, и их теоретическими значениями, рассчитанными при проектировании состава портфеля, является очень слабой. Дополнительно отмечено, что среднее значение не может служить основой при построении портфеля.

Ключевые слова: *модель Марковица; выбор портфеля; норма прибыли; риск.*

1. Introduction. Modern portfolio theory (MPT), developed in the 1950s by H. Markowitz (1952; 1991) and simply called "portfolio theory" by him, has revolutionized investment practice (Bernstein, 1992). MPT explains how to find the best possible diversification strategy for a portfolio. On the basis of Markowitz's theory,

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rational investors can construct optimal portfolios under uncertainty. From a large number of portfolios, a rational investor can choose the one that offers the highest level of expected return for a given level of risk, and the one that offers the lowest level of risk for a given level of return. Portfolio theory assumes that a rational investor will not invest in a portfolio if a second portfolio exists with the same expected return and lower risk or with the same risk and higher expected return. Expected return and the risk associated with investment are represented by the mean and the variance or standard deviation, respectively.

But since 1970s (Jensen et al., 1972) some criticism has been levelled against MPT. The shortcomings of the Markowitz model proceed from that investors in real life are not rational and markets may not be efficient (Shleifer, 2000; Koponen, 2003). In reply to criticism of Markowitz Model some modifications have been applied (Brodie et al., 2009).

In this work we deal with portfolios that consist of a large number of assets. This case was also studied by Bai et al. (2009). These authors proved that "the traditional return estimate for the optimal self-financing portfolio obtained by plugging the sample mean and covariance matrix into its theoretic value is always overestimated and, in return, makes the self-financing MV optimization procedure impractical". In order to reduce this overprediction they used the bootstrap technique (Kosowski et al., 2006; Scherer, 2002).

In this work we would like to present a procedure for the construction of a multi-asset portfolio for which risk has the minimal value. This is a generalization of the procedure for two assets previously described by (Rzymowski et al., 2013). We show that:

- there is no satisfactory correlation between the real and the theoretical average rates of return, and that there is no satisfactory correlation between the real and the theoretical risk measures;
- average value should not be used for portfolio selection.

2. Methodology description. In this paper the following notation is used:

$$A_1 = \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,n} \end{bmatrix}, \; A_2 = \begin{bmatrix} a_{2,1} \\ a_{2,2} \\ \vdots \\ a_{2,n} \end{bmatrix}, \; \ldots, \; A_m = \begin{bmatrix} a_{m,1} \\ a_{m,2} \\ \vdots \\ a_{m,n} \end{bmatrix} \in R^n \qquad (1)$$

are the vectors of the successive asset prices of m selected companies in a fixed period. Therefore,

$$X_1 = \begin{bmatrix} \frac{a_{1,2} - a_{1,1}}{a_{1,1}} \\ \frac{a_{1,3} - a_{1,2}}{a_{1,2}} \\ \vdots \\ \frac{a_{1,n} - a_{1,n-1}}{a_{1,n-1}} \end{bmatrix}, \; \ldots, \; X_m = \begin{bmatrix} \frac{a_{m,2} - a_{m,1}}{a_{m,1}} \\ \frac{a_{m,3} - a_{m,2}}{a_{m,2}} \\ \vdots \\ \frac{a_{m,n} - a_{m,n-1}}{a_{m,n-1}} \end{bmatrix} \in R^{n-1} \qquad (2)$$

are the vectors of return rates. Additionally, we create the following vectors:

$$M = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix} \in R^m, \tag{3}$$

where $\mu_j = \frac{1}{n-1}\sum_{i=2}^{n}\frac{a_{j,i} - a_{j,i-1}}{a_{j,i-1}}$ is the mean value of the coordinates of the vector X_j (the average rate of return for the company shares A_j), and vectors

$$J_m = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \in R^m,\ J_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \in R^{n-1}, \tag{4}$$

and the matrix

$$\mathbf{X_M} = \begin{bmatrix} \frac{a_{1,2} - a_{1,1}}{a_{1,1}} - \mu_1 & \cdots & \frac{a_{m,2} - a_{m,1}}{a_{1,1}} - \mu_m \\ \frac{a_{1,3} - a_{1,2}}{a_{1,2}} - \mu_1 & \cdots & \frac{a_{m,3} - a_{m,2}}{a_{1,2}} - \mu_m \\ \vdots & \ddots & \vdots \\ \frac{a_{1,n} - a_{1,n-1}}{a_{1,n-1}} - \mu_1 & \cdots & \frac{a_{m,n} - a_{m,n-1}}{a_{1,n-1}} - \mu_m \end{bmatrix}. \tag{5}$$

The dimension of the matrix X_M is $(n - 1)$ x m ($n - 1$ rows, m columns). The columns of matrix X_M are created by the coordinates of the vectors $X_1 - \mu_1 J_n$, …, $X_m - \mu_m J_n$.

Each vector

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} \in R^m \tag{6}$$

that satisfies the condition

$$\sum_{j=1}^{m} w_j = 1 \tag{7}$$

we call a portfolio. The set of all portfolios $w \in R^m$ is denoted by P_m. $w \in P^m$ is an efficient portfolio if

$$w_j > 0,\quad j = 1, 2, \ldots, m. \tag{8}$$

According to Markowitz, the mean value of the portfolio

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} \in P^m \tag{9}$$

is the sum

$$\mu_M(w) \overset{\text{def}}{=} \langle w, M \rangle = \sum_{j=1}^{m} w_j \mu_j, \tag{10}$$

and the risk of the portfolio is the number

$$\sigma(w) \overset{\text{def}}{=} \frac{1}{\sqrt{n-2}} \|X_M w\|. \tag{11}$$

2.1.Auxiliary lemma. *Lemma:* If the vectors J_n, X_1, ..., X_m are linearly independent, then the minimal risk portfolio $w_{\min}$ is presented by the formula:

$$w_{\min} = \frac{1}{\left\langle \left(X_M^T X_M\right)^{-1} J_m, J_m \right\rangle} \left(X_M^T X_M\right)^{-1} J_m. \tag{12}$$

Warning: $w_{\min}$ does not have to be an efficient portfolio.

Proof: Let us define:

$$f(w) = \|X_M w\|^2, w \in R^m. \tag{13}$$

Of course

$$\sigma(w_{\min}) = \min_{w \in P^m} \sigma(w) \Leftrightarrow f(w_{\min}) = \min_{w \in P^m} f(w). \tag{14}$$

Note that

$$\sum_{j=1}^{m} w_j X_j - \sum_{j=1}^{m} w_j \mu_j J_m = \sum_{j=1}^{m} w_j \left(X_j - \mu_j J_m\right) = X_M w, \tag{15}$$

where $X_j - \mu_j J_m, \mathrm{j} = 1, 2, \ldots, m$ are linearly independent. Therefore, for each $w \in R^m$

$$f''(w) = 2 X_M^T X_M \tag{16}$$

is a positively determined matrix. It follows from this that $f: R^m \to R$ is a differentiable, strictly convex function satisfying the condition

$$\lim_{\rho \to \infty} \min_{\|w\| \geq \rho} f(w) = \infty. \tag{17}$$

Thus, there exists exactly one $w_{\min} \in P^m$ such that

$$f(w_{\min}) = \min_{w \in P^m} f(w). \tag{18}$$

Moreover, since J_m is perpendicular to the affine space $P^m \subset R^m$, there exists a $\lambda \in R$ such that

$$f'(w_{\min}) = 2 X_M^T X_M w_{\min} = 2 \lambda J_m. \tag{19}$$

Consequently, $w_{\min} = \lambda \left(X_M^T X_M\right)^{-1} J_m$ and by the relation $w_{\min} \in P^m$

$$1 = \langle w_{\min}, J_m \rangle = \lambda \left\langle \left(X_M^T X_M\right)^{-1} J_m, J_m \right\rangle, \tag{20}$$

which completes the proof.

2.2. Portfolio construction procedure. In order to compare the theoretical and the real average rates of return, and the theoretical and the real risk measures, we propose the following procedure of portfolio construction.

Step 0. The full period of observation $t = 1, 2, ..., N$ is divided into k separate subperiods.

$$O_1 = (1, 2, \ldots, n),\ O_2 = (n+1, n+2, \ldots, 2n),\ \ldots,\ O_k = ((k-1)n+1, \ldots, kn). \tag{21}$$

Each subperiod has n observations.

Step 1. The asset prices of m selected companies $A_1, A_2, \ldots, A_m$ in a fixed first period are observed. Therefore, m vectors $A_1, A_2, \ldots, A_m$ of asset prices, according to equation (1), are obtained. Then for those vectors the following quantities are calculated in the first period:

- vectors of rates of return according to equation (2);
- average rate of return according to equation (3);
- efficient portfolio according to equation (12), introducing the following notation

$$w_{\min} = w_1^* = \begin{pmatrix} w_{1,1}^* \\ w_{1,2}^* \\ \vdots \\ w_{1,m}^* \end{pmatrix}; \tag{22}$$

- average rate of return of efficient portfolio according to equation (10), where $w = w_1^*$;
- risk according to equation (11), where $w = w_1^*$.

Step 2. Let K be the capital that is to be invested. At the beginning of the next subperiod $O_2 = (n + 1, n + 2, ..., 2n)$, we assume that we are buying shares in company A_1 for $w_{1,1}^* K$, shares in company A_2 for $w_{1,2}^* K$, .., and shares in company A_m for $w_{1,m}^* K$. In view of the indivisibility of shares we buy approximately $q_{A1} \approx \frac{w_{1,1}^* K}{a_{1,n+1}}$ shares in company A_1, $q_{A2} \approx \frac{w_{1,2}^* K}{a_{2,n+1}}$ shares in company A_2, ..., and $q_{Am} \approx \frac{w_{1,m}^* K}{a_{m,n+1}}$ shares in company A_m, where $q_{A1}, q_{A2}, \ldots, q_{Am}$ are integers. Therefore, we obtain the vector

$$q_A = \begin{pmatrix} q_{A1} \\ q_{A2} \\ \vdots \\ q_{Am} \end{pmatrix}. \tag{23}$$

Then, in the period $O_2 = (n + 1, n + 2, ..., 2n)$, we calculate:

- the subsequent values of the portfolio w_1^*

$$P_t = \langle q_A, A_t \rangle,\ t = n+2, n+3, \ldots, 2n, \text{ where } A_t = (a_{1,t}, a_{2,t}, \ldots, a_{m,t}); \tag{24}$$

- the portfolio rates of return w_1^*

$$Z_t = \frac{P_t - P_{t-1}}{P_{t-1}},\ t = n+2, n+3, \ldots, 2n; \tag{25}$$

- the portfolio average rate of return w_1^*

$$\bar{z}_1 = \frac{1}{n-1}\sum_{t=n+2}^{2n} Z_t; \tag{26}$$

- the portfolio risk measure w_1^*

$$\sigma_1 = \sqrt{\frac{1}{n-2}\sum_{t=n+2}^{2n}\left(Z_t - \bar{z}_1\right)^2}. \tag{27}$$

Step 3. We repeat the procedure described above for subsequent subperiods $O_2 = (n + 1, n + 2, ..., 2n)$, $O_3 = (2n + 1, ..., 3n)$, $O_k = ((k - 1)\, n + 1, ..., kn)$. The portfolio w_k^* in the last subperiod O_k is not calculated. As a result, we obtain the theoretical sequences of average rates of return and the sequences of risk measures of the portfolios w_j^* in the periods $O_1, ..., O_{k-1}$:

$$\left(z_1^*, \ldots, z_{k-1}^*\right), \left(\sigma_1^*, \ldots, \sigma_{k-1}^*\right) \tag{28}$$

and, corresponding to them, in the next period their real values:

$$\left(\bar{z}_2, \ldots, \bar{z}_k\right),\ \left(\sigma_2, \ldots, \sigma_k\right). \tag{29}$$

If Markowitz portfolio was a good tool that could help make better investment decisions, there should be a strong positive correlation between the sequences of average rates of return $\left(z_1^*, \ldots, z_{k-1}^*\right)$ and $\left(\bar{z}_2, \ldots, \bar{z}_k\right)$. Analogously, a strong positive correlation should be observed between the sequences of standard deviations $\left(\sigma_1^*, \ldots, \sigma_{k-1}^*\right)$ and $\left(\sigma_2, \ldots, \sigma_k\right)$. To verify this, we calculate the correlation coefficients $r_z = r\left(\left(z_1^*, \ldots, z_{k-1}^*\right), \left(\bar{z}_2, \ldots, \bar{z}_k\right)\right)$ and $r_\sigma = r\left(\left(\sigma_1^*, \ldots, \sigma_{k-1}^*\right), \left(\sigma_1, \ldots, \sigma_{k-1}\right)\right)$, which measure the correlation between the theoretical and the real sequences of average return rates and between the theoretical and the real sequences of risk measures, respectively.

3. Research objective and data sources. The portfolio construction procedure is applied to the companies listed on various European and US stock exchanges. For European stock exchanges, the companies used to calculate the following stock market indices are chosen at:

- DAX30 (Frankfurt Stock Exchange);
- EURONEXT100 (Paris Stock Exchange);
- WIG30 (Warsaw Stock Exchange).

For the US stock exchange, the companies used to calculate the stock market indices are chosen:

- DJIA (New York Stock Exchange – NYSE);
- NASDAQ100 (NASDAQ Stock Market).

20 companies are chosen for each stock market in Europe and the US to be considered (Table 1).

Because the portfolio construction procedure is associated with the division of the full observation period $t = 1, 2, ..., N$ into k disjoint subperiods with the length n, and because it is known that the length of the observation period has significant influence on the results, the portfolios for each selected stock market are analysed in two ways. For variant I, $N = 936$, $n = 26$ and $k = 36$. $n = 26$ corresponds to the period of approximately one month. For variant II the number of observations is the same, but the length and the number of periods are changed: we choose $n = 36$, $k = 26$.

Table 1. **The selected companies listed at stock markets under consideration: DAX30, EURONEXT100, WIG30, DJIA and NASDAQ100,** *authors' own research on the basis* (www.gielda.onet.pl; finance.yahoo.com)

No.	DAX30	EURONEXT100	WIG30	DJIA	NASDAQ100
1	ADS.DE	ACA.PA	ACP	AXP	AAPL
2	ALV.DE	ALU.PA	ATT	BA	ADBE
3	BMW.DE	BN.PA	BHW	CVX	ADSK
4	CBK.DE	BNP.PA	BRE	DIS	AMZN
5	CON.DE	CA.PA	BRS	GE	AMGN
6	DAI.DE	CO.PA	CPS	JPM	CERN
7	DB1.DE	CS.PA	GTC	KO	DLTR
8	DBK.DE	DEC.PA	ING	MCD	GMCR
9	DPW.DE	DG.PA	KER	MRK	GOOG
10	DTE.DE	LG.PA	KGH	NKE	INTC
11	EOAN.DE	MC.PA	LTS	PFE	MAT
12	HEN3.DE	ML.PA	NET	PG	MSFT
13	FRE.DE	OR.PA	PEO	T	MU
14	IFX.DE	ORA.PA	PKN	TRV	NVDA
15	LHA.DE	RCO.PA	PKO	UNH	SHLD
16	LXS.DE	RI.PA	PGN	UTX	SPLS
17	MRK.DE	RNO.PA	SNS	V	VIAB
18	RWE.DE	SU.PA	TPS	VZ	VOD
19	SDF.DE	VIV.PA	TVN	WMT	WYNN
20	SIE.DE	ZC.PA	LBW	XOM	YHOO

Preservation of the same number of observations for all the markets analysed leads to different periods for different markets. The first day of observation (31st December 2009) is the same for all markets. In Table 2 the stock markets (represented by the given stock indices) and the corresponding time periods are presented.

Table 2. **Considered periods for the selected stock markets represented by stock market indices,** *authors' construction on the basis of* (www.gielda.onet.pl; finance.yahoo.com)

No.	Index	Period
1	DAX30	31.12.2009–12.08.2013
2	EURONEXT100	31.12.2009–6.08.2013
3	WIG30	31.12.2009–23.09.2013
4	DJIA	31.12.2009–19.09.2013
5	NASDAQ100	31.12.2009–19.09.2013

The price of shares at Warsaw Stock Exchange is a closing price (www.gielda.onet.pl). The prices of shares on the other stock exchanges are closing prices adjusted for dividends and splits (finance.yahoo.com). The value of the capital for investments, which is described in step 2 of the procedure, is K = 10,000 units, regardless the market.

In this work all portfolios for which condition (8) is not met are rejected. Moreover, we only consider those portfolios for which condition (8) is satisfied in at least 11 subperiods. Therefore, in variant I we consider only those portfolios that have at least 11 x 26 = 286 observations, and in variant II those that have at least 11 x 36 = 396 observations.

4. Results.

4.1. Results concerning numbers of portfolios. Applying the portfolio construction procedure described in Section 2.1 to companies listed in Table 1 we did not obtain any portfolios with at least 6 and more assets that exist for at least 11 subperiods starting from the first subperiod.

The numbers of all possible portfolios with many assets (2, 3, 4 and 5 assets) for all the stock markets in variant I that are analysed are presented in Table 3), and the same results but for variant II are presented in Table 4). There are no portfolios with more than 5 assets for any of the stock exchanges that we analysed.

In Tables 3 and 4: 1) denotes the portfolios that exist for at least 11 subperiods starting from the first subperiod; 2) denotes the portfolios that exist for exactly 11 subperiods; 3) denotes the portfolios that exist for the maximum number of subperiods in the analysis (k = 36 in variant I and k = 26 in variant II).

Comparing the results presented in Tables 3 and 4, one can see that for each stock market analysed the number of all portfolios that exist in the case of variant I (n = 26, k = 36) is less than the number of all portfolios that exist in the case of variant II (n = 36 and k = 26), for all cases 1), 2) and 3).

The unique exception to this rule in our results concerns the three-asset portfolio at the Frankfurt Stock Exchange – there are no portfolios in variant II with 3 or more assets that exist for the full period of observation.

Table 3. **Number of portfolios with many assets for different stock markets for variant I (*n* = 26, *k* = 36),** *authors' own research*

Number of subperiods	1)	2)	3)	1)	2)	3)	1)	2)	3)	1)	2)	3)	1)	2)	3)
Portfolios/ Markets	DAX30			EURONEXT100			WIG30			DJIA			NASDAQ100		
2-asset	**64**	0	12	**41**	1	6	**107**	0	46	**59**	3	9	**85**	10	13
3-asset	**22**	1	1	**1**	0	0	**96**	4	2	**6**	2	0	**18**	8	0
4-asset	**0**	0	0	**0**	0	0	**14**	2	0	**0**	0	0	**1**	1	0
5-asset	**0**	0	0	**0**	0	0	**0**	0	0	**0**	0	0	**0**	0	0
Sum	**86**	1	13	**42**	1	6	**217**	6	48	**65**	5	9	**104**	19	13

1) denotes the portfolios that exist for at least 11 subperiods starting from the first subperiod; 2) denotes the portfolios that exist for exactly 11 subperiods; 3) denotes the portfolios that exist for the maximum number of subperiods under analysis.

Regarding the results presented in Tables 3 and 4, the portfolios that exist for the maximum number of subperiods in the analysis consist of 3 assets at most.

A unique five-asset portfolio exists for exactly 11 subperiods (Table 4). This portfolio can be constructed for the Warsaw Stock Exchange and contains shares of the following companies: BHW, GTC, KER, PEO and PGN.

Table 4. **Number of multi-asset portfolios for different stock markets for variant II I (*n* = 36, *k* = 26),** *authors' own research*

Number of subperiods	1)	2)	3)	1)	2)	3)	1)	2)	3)	1)	2)	3)	1)	2)	3)
Portfolios/ Markets	DAX30			EURONEXT100			WIG30			DJIA			NASDAQ100		
2-asset	**74**	16	19	**57**	15	14	**133**	4	98	**74**	34	25	**98**	44	33
3-asset	**30**	16	0	**8**	4	0	**190**	33	39	**27**	20	1	**51**	35	2
4-asset	**0**	0	0	**1**	1	0	**35**	23	0	**0**	0	0	**2**	1	0
5-asset	**0**	0	0	**0**	0	0	**1**	1	0	**0**	0	0	**0**	0	0
Sum	**104**	32	19	**66**	20	14	**359**	61	137	**101**	54	26	**151**	80	35

1) denotes the portfolios that exist for at least 11 subperiods starting from the first subperiod; 2) denotes the portfolios that exist for exactly 11 subperiods; 3) denotes the portfolios that exist for the maximum number of subperiods under analysis.

4.2. Results concerning correlation. The next results for portfolios of many assets concern correlation between the theoretical and real average rates of return and between theoretical and real risk measures.

The values of the correlation coefficient between theoretical and real average rates of return are presented in Table 5 for variant I and in Table 6 for variant II.

For all stock markets analysed, Tables 5 and 6 present the minimal and maximal values of the correlation coefficient between theoretical and real average rates of return for two-, three-, four- and five-asset portfolios.

The biggest values for the correlation coefficient between theoretical and real average rates of return are observed for the portfolios at the Warsaw Stock Exchange. In variant I this value is equal 0.65 (Table 5) for the three-asset portfolio (ATT, BHW, LBW), and in variant II it is equal to 0.57 (Table 6) for the four-asset portfolio (BRE, KER, LTS, TVN). The correlation between the theoretical and real average rates of return is therefore poor.

Table 5. **Minimal and maximal values of the correlation coefficient between theoretical and real average rates of return for multi-asset portfolios at European and US stock markets for variant I (*n* = 26, *k* = 36),** *authors' own research*

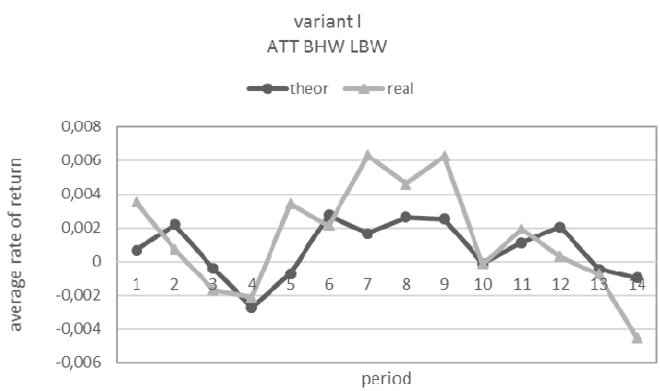
Portfolios/ Markets	DAX30		EURONEXT100		WIG30		DJIA		NASDAQ100	
Correlation coefficients for average rates of return	min	max	min	max	min	max	min	max	min	max
2-asset	-0.59	0.32	-0.77	0.17	-0.49	0.45	-0.59	0.51	-0.44	0.39
3-asset	-0.48	0.02	-0.59	-0.59	-0.72	0.65	-0.52	0.23	-0.39	0.36
4-asset	-	-	-	-	-0.72	0.50	-	-	0.04	0.04

The comparison between theoretical and real average rates of return for the portfolio with the maximal value of the correlation coefficient for average rates of return is presented in Figure 1a. This portfolio exists for 14 subperiods.

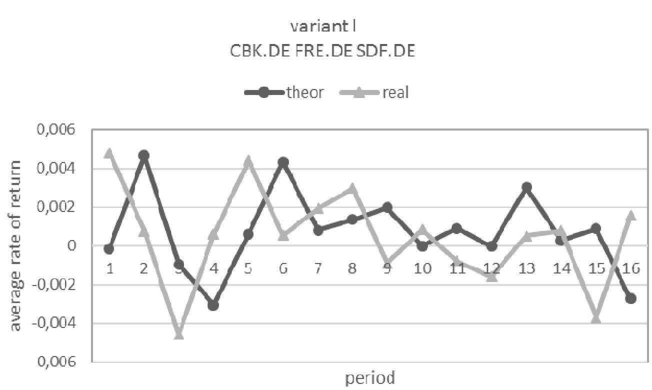
The smallest positive value of the correlation coefficient between theoretical and real average rates of return corresponds to the three-asset portfolio that has been constructed in variant I for the Frankfurt Stock Exchange. This value is equal to 0.02 (Table 5). The portfolio consists of the following assets: CBK.DE, FRE.DE and SDF.DE. Figure 1b presents the comparison between theoretical and real average rates of return for this portfolio. This portfolio exists for 16 subperiods.

Table 6. **Minimal and maximal values of the correlation coefficient between theoretical and real average rates of return for the multi-asset portfolios at European and US stock markets for variant II (*n* = 36, *k* = 26),** *authors' own research*

Portfolios/ Markets	DAX30		EURONEXT100		WIG30		DJIA		NASDAQ100	
Correlation coefficients for average rates of return	min	max	min	max	min	max	min	max	min	max
2-asset	-0.51	0.24	-0.54	0.23	-0.58	0.27	-0.57	0.41	-0.78	0.29
3-asset	-0.30	0.23	-0.32	0.17	-0.57	0.48	-0.50	0.16	-0.60	-0.15
4-asset	-	-	-0.07	-0.07	-0.37	0.57	-	-	-0.28	-0.18
5-asset	-	-	-	-	0.14	0.14	-	-	-	-

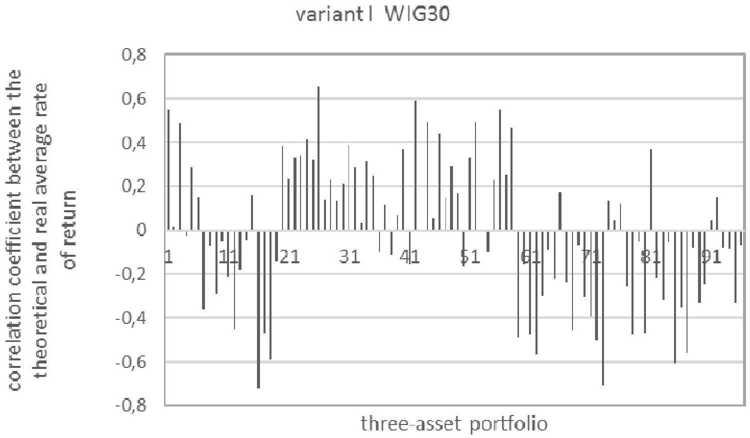


a) for the three-asset portfolio (ATT, BHW, LBW) in variant I (n = 26, *k* = 36) for the Warsaw Stock Exchange

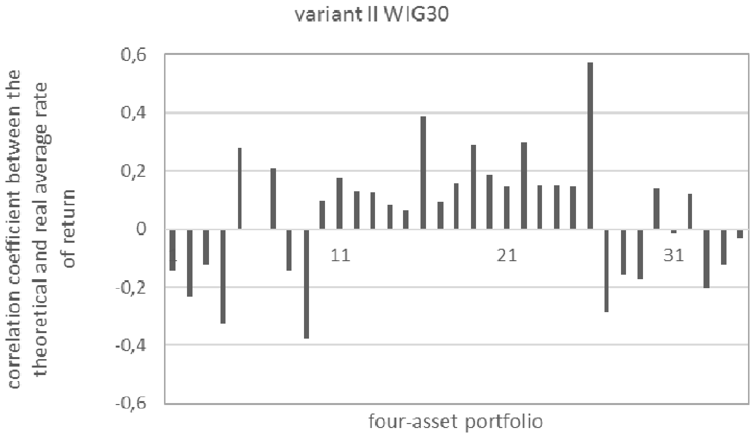


b) for the three-asset portfolio (CBK.DE, FRE.DE, SDF.DE) in variant I (*n* = 26, *k* = 36) for the Frankfurt Stock Exchange

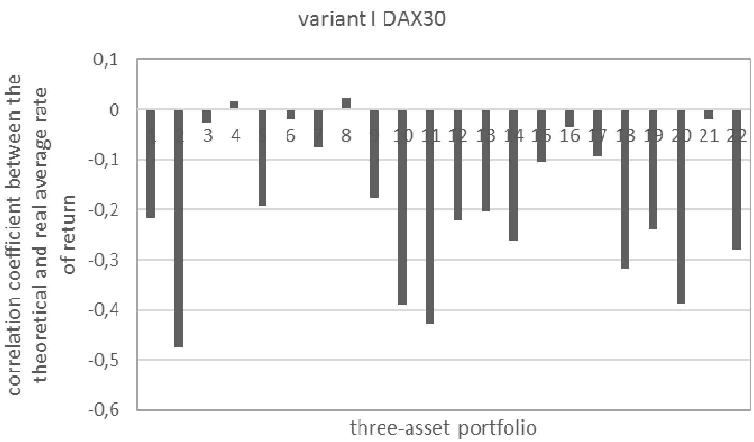
Figure 1. **Comparison between theoretical (teor) and real (real) average rates of return,** *authors' own research*



a) for all 96 three-asset portfolios in variant I (n = 26, k = 36) for the Warsaw Stock Exchange



b) for all 35 four-asset portfolios in variant II (n = 36, k = 26) for the Warsaw Stock Exchange



c) for all 22 three-asset portfolios in variant I (n = 26, k = 36) for the Frankfurt Stock Exchange

Figure 2. **Correlation coefficients between theoretical and real average rates of return,** *authors' own research*

Figure 2 presents the values of the correlation coefficient between theoretical and real average rates of return. Figure 2a presents the results for all three-asset portfolios in our analysis that were constructed from the companies listed at the Warsaw Stock Exchange in variant I; Figure 2b presents the results for four-asset portfolios in variant II; Figure 2c presents the results for all three-asset portfolios in our analysis that were constructed for the companies listed at the Frankfurt Stock Exchange for variant I. In the case of the three-asset portfolios at this stock exchange there is a large asymmetry. Among the 22 portfolios presented in Figure 2c, there are only two with a positive correlation coefficient. This result attests to the disadvantage of the Markowitz theory.

In the same way as for the average rates of return, the analysis of investment risk for multi-asset portfolios was performed. The results concerning the minimal and maximal values of the correlation coefficient between theoretical and real risk measures are presented in Table 7 for variant I and in Table 8 for variant II.

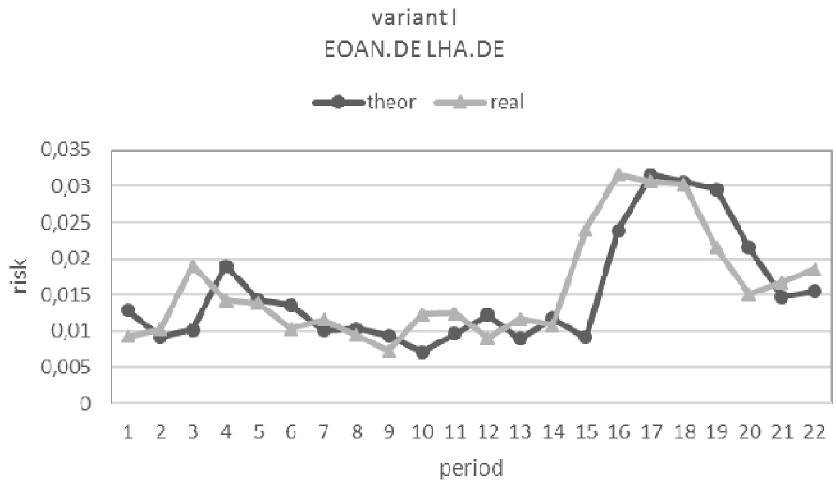
Table 7. **Minimal and maximal values of the correlation coefficient between theoretical and real standard deviations for multi-asset portfolios at European and US stock markets for variant I (n = 26, k = 36),** *authors' own research*

Portfolios/ Markets	DAX30		EURONEXT100		WIG30		DJIA		NASDAQ100	
Correlation coefficients for standard deviation	min	max	min	max	min	max	min	max	min	max
2-asset	-0.25	0.75	-0.31	0.67	-0.42	0.64	-0.22	0.46	-0.41	0.58
3-asset	-0.20	0.71	-0.07	-0.07	-0.38	0.63	-0.02	0.26	-0.19	0.66
4-asset	-	-	-	-	-0.26	0.22	-	-	0.37	0.37

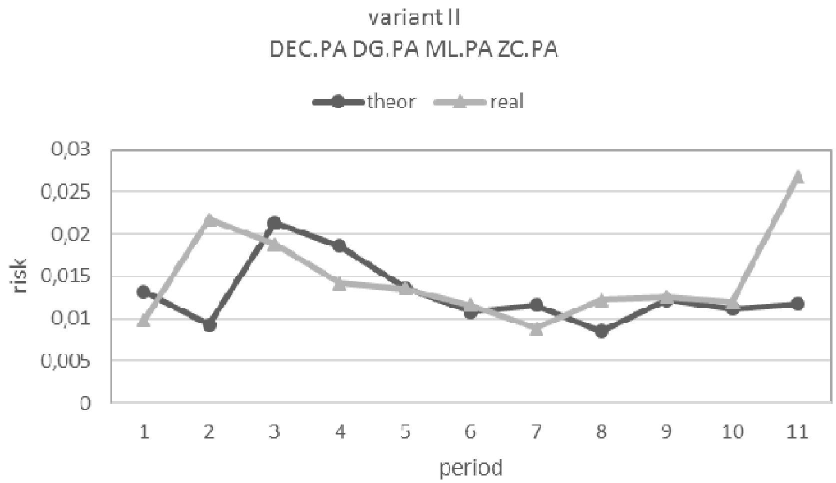
Table 8. **Minimal and maximal values of the correlation coefficient between theoretical and real standard deviations for multi-asset portfolios at European and US stock markets for variant II (n = 36, k = 26),** *authors' own research*

Portfolios/ Markets	DAX30		EURONEXT100		WIG30		DJIA		NASDAQ100	
Correlation coefficients for standard deviation	min	max	min	max	min	max	min	max	min	max
2-asset	-0.34	0.67	-0.29	0.61	-0.30	0.60	-0.41	0.56	-0.51	0.58
3-asset	-0.21	0.62	-0.09	0.51	-0.44	0.57	-0.43	0.48	-0.32	-0.37
4-asset	-	-	0.1	0.1	-0.45	0.47	-	-	-0.31	0.22
5-asset	-	-	-	-	0.25	0.25	-	-	-	-

The biggest value for the correlation coefficient between theoretical and real risk measure is observed for the two-asset portfolios constructed for the Frankfurt Stock Exchange. The biggest value for the portfolio constructed in variant I is equal to 0.75, and in variant II – 0.67. Even these values indicate that the correlation between theoretical and real risk measures is not strong. Figure 3a presents the comparison between theoretical and real risk measures for the portfolio with the maximal value of the standard deviation in variant I. This result is obtained for a two-asset portfolio that consists of EOAN.DE and LHA.DE. This portfolio exists for 22 subperiods. Figure 3b presents a comparison between theoretical and real risk measures for the portfolio with a minimal value of the standard deviation. This minimal value is equal to 0.1 (Table 8). This result is obtained for the four-asset portfolio (DEC.PA, DG.PA, ML.PA and ZC.PA) in variant II for the Paris Stock Exchange. This portfolio exists only for 11 subperiods.



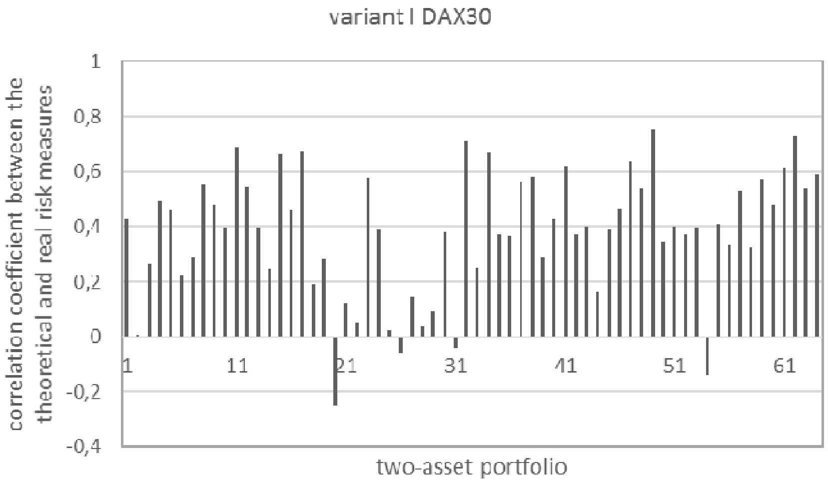
a) for the two-asset portfolio (EOAN.DE, LHA.DE) in variant I (n = 26, k = 36) for the Frankfurt Stock Exchange



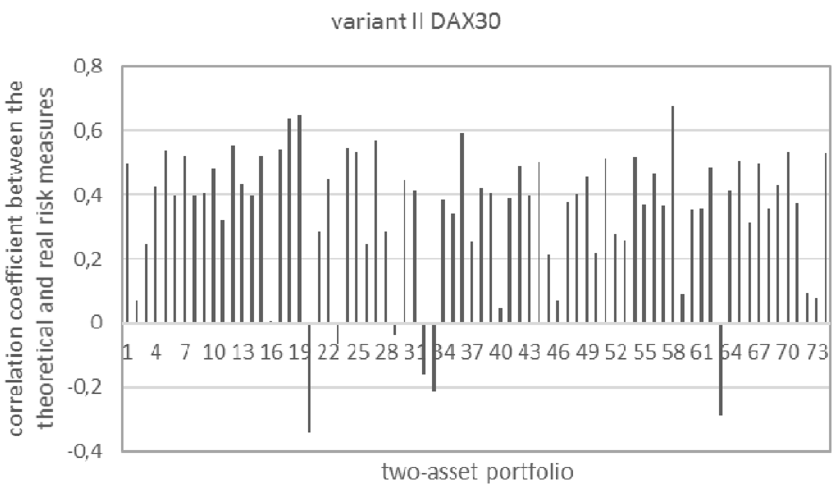
b) for four-asset portfolios in variant II (n = 36, k = 26) for the Paris Stock Exchange

Figure 3. **Comparison between theoretical (theor) and real (real) risk measures),** *authors' own research*

Figure 4a presents the values of the correlation coefficient between theoretical and real risk measures in variant I for the Frankfurt Stock Exchange for all two-asset portfolios that were analysed. Figure 4b presents the same but for variant II.



a) for all 64 two-asset portfolios in variant I (n = 26, k = 36) for the Frankfurt Stock Exchange



b) for all 74 two-asset portfolios in variant II (n = 36, k = 26) for the Frankfurt Stock Exchange

Figure 4. **Correlation coefficients between theoretical and real risk measures,** *authors' own research*

The values of the correlation coefficients between theoretical and real rates of return and between theoretical and real risk measures observed in Figures 2a, 2b, 2c, 4a and 4b are small. Moreover, a negative correlation can be observed for certain portfolios (Figures 2a, 2b, 2c, 4a and 4b).

Even for those portfolios for which the value of the correlation coefficient between theoretical and real rates of return is maximal, there is a mismatch between theoretical and real rates of return (Figure 1a). Likewise, a mismatch can be observed between theoretical and real risk measures (Figure 3a).

Figures 1a, 1b, 3a and 3b show the examples of mismatches between theoretical and real rates of return and between theoretical and real risk measures, but this result can be observed for all the European and US markets analysed (see the results for the values of the correlation coefficients presented in Tables 5–8).

4.3. Results concerning the average rate of return. The average rate of return has been analysed for all companies from table 1 because it is one of the two main measures of the quality of portfolio construction under Markowitz's theory.

In the analysis, the average return rate was determined for different numbers of observations. The question is: does there exist a $t_{min} < t_{max}$ such that $a_t < a_{t_{min}}$ or $t_{min} < t \le t_{max}$, hile the average rate of return in the analysed period from t_{min} to t_{max} is positive? Three cases were considered: $t_{max} = N = 936$, $t_{max} = t_{min} + 500$, $t_{max} = t_{min} + 250$.

First case: $t_{max} = N = 936$.

For each of the 18 companies presented in Table 9, out of the 100 companies presented in Table 1, there exists such a t_{min}. The number of observations for these periods from t_{min} to $t_{max} = N = 936$ for these companies is larger than 150, and for 15 companies of the 18 the number of observations is larger than 500. The results for the New York Stock Exchange are not presented here because t_{min} exists for only 3 companies of the 20 presented in Table 1 and the number of observations from t_{min} to $t_{max} = N = 936$ is less than 100.

Table 9. **Companies and corresponding time values of t_{min} on the basis of the periods presented in Table 2 for** $t_{max} = N = 936$, *authors' own research*

	DAX30		EURONEXT100		WIG30		NASDAQ100	
No	Companies	t_{min}	Companies	t_{min}	Companies	t_{min}	Companies	t_{min}
1	BMW.DE	778	ACA.PA	343	ACP	639	ADSK	344
2	DBK.DE	341	ALU.PA	421	BRS	758	GMCR	433
3	IFX.DE	349	BNP.PA	294	PEO	218	NVDA	360
4	SIE.DE	322	CA.PA	326	TVN	342	SHLD	334
5			LG.PA	10				
6			VIV.PA	277				

The second case: $t_{max} = t_{min} + 500$.

Such t_{min} exists for 21 companies, as presented in Table 10, out of the 100 companies that are presented in Table 1. For 14 companies of those 21, there exists more than one value of t_{min}. For the New York Stock Exchange no such t_{min} exists.

The third case: $t_{max} = t_{min} + 250$.

In this case it emerges that for each of 37 companies (Table 11) out of the 100 presented in Table 1 there exists such a t_{min}. For 13 companies of those 37, there exists more than one value of t_{min}.

The circle corresponds to the result presented in Table 9, the square corresponds to the result presented in Table 10 and the triangle corresponds to the result presented in Table 11.

Figure 5 presents the results of t_{min} for LG.PA. For this company, in the period from $t_{min} = 10$ to $t_{max} = N = 936$ the average rate of return is positive, but for each $t > 10$ the inequality $a_t < a_{10}$ is true. We can observe a similar result in the period from $t_{min} = 236$ to $t_{max} = 763$ and in the period from $t_{min} = 409$ to $t_{max} = 659$.

Table 10. **Companies and corresponding time values of t_{min} on the basis of the periods presented in Table 2 for $t_{max} = t_{min} + 500$,** *authors' own research*

	DAX30		EURONEXT100		WIG30		NASDAQ100	
No	Companies	t_{min}	Companies	t_{min}	Companies	t_{min}	Companies	t_{min}
1	DB1.DE	390, 395, 400	ACA.PA	408–411	BHW	328, 330, 333	ADBE	57, 78, 79
2	DBK.DE	300-302, 304, 385-387, 389, 390	ALU.PA	433	PEO	252, 256, 326	ADSK	344, 345, 349, 356
3	LHA.DE	269	BNP.PA	294–297	PKN	324	GMCR	433, 435
4			CA.PA	370, 376–377, 388–393	PKO	401	MU	333–336, 339, 340
5			CS.PA	74			NVDA	366
6			DEC.PA	327–329, 345, 352, 354				
7			LG.PA	263				
8			RNO.PA	279, 280, 288–292, 294				
9			VIV.PA	277–280, 287				

5. Conclusions. The calculation was performed for the multi-asset portfolios for two variants connected with the number of observations: N = 936, n = 26 and k = 36 in variant I and N = 936, n = 36 and k = 26 in variant II. The results obtained for both variants confirm the shortcomings of the Markowitz model mentioned in the Introduction. The results presented in the graphs and tables show there is no satisfactory correlation between real and theoretical average rates of return, and there is no satisfactory correlation between real and theoretical risk measures, regardless which market (European or American) is selected. The correlation coefficient values are greater for the correlation between theoretical and real risk measures represented by the standard deviations than they are for the correlation between theoretical and real average return rates.

Table 11. **Companies and corresponding time values of t_{min} on the basis of the periods presented in Table 2 for $t_{max} = t_{min} + 250$,** *authors' own research*

	DAX30		EURONEXT100		WIG30		DJIA		NASDAQ100	
No	Companies	t_{min}	Companies	t_{min}	Companies	t_{min}	Companies	t_{min}	Companies	t_{min}
1	CON.DE	389, 390	ACA.PA	75, 474	ACP	639	AXP	78	ADBE	12, 78, 81
2	DAI.DE	569	ALU.PA	626	ATT	357	BA	342, 344	DLTR	634
3	DB1.DE	551	CS.PA	78	BHW	380, 387, 393-395, 398	DIS	295	GMCR	547, 548–551
4	DTE.DE	409	DEC.PA	327-329, 345	GTC	2, 8, 14	GE	292, 295	INTC	82
5	EOAN.DE	400, 402, 404, 405	LG.PA	409	KGH	474	JPM	319, 320	MSFT	581
6	IFX.DE	351	RI.PA	257	PKO	410	PFE	12	MU	544
7	RWE.DE	404	RNO.PA	305, 309			V	89	SPLS	597
8			VIV.PA	404					VIAB	382, 383, 392, 393
9									YHOO	458

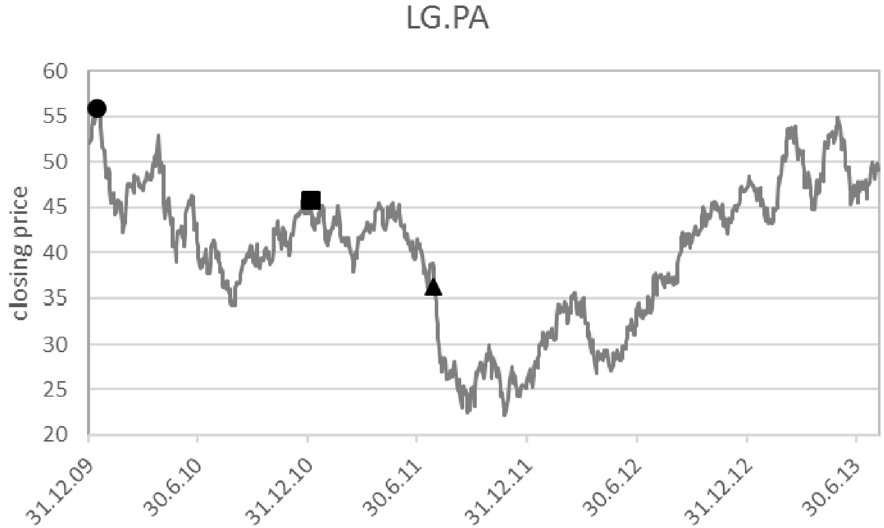


Figure 5. **Closing prices for LG.PA at the Paris Stock Exchange for the period from 31.12.2009 to 6.08.2013,** *authors' own research*

Additionally, in many cases the correlation coefficients are negative for both the average rate of return and the measure of risk.

We hope that the results we obtained were not just caused by the global number of observations N and the number of observations in each period n.

Moreover, the results obtained in the final section confirm the previous observation in (Rzymowski and Surowiec, 2013) that the average rates of return determined on the basis of historical data should not be used for portfolio construction.

References:

Bai, Z., Liu, H., Wong, W.K. (2009). On the Markowitz mean – variance analysis of self-financing portfolios. Risk and Decision Analysis, 1: 35–42.

Bernstein, P.L. (1992). Capital Ideas: The Improbable Origins of Modern Wall Street. The Free Press, New York, NY.

Brodie, J., Daubechies, I., De Mol, C., Giannone, D., Loris, I. (2009). Sparse and stable Markowitz portfolios. Proceedings of the National Academy of Science, 106(30) // www.pnas.org.

Jensen, M.C., Black, F., Scholes, M.S. (1972). The capital asset pricing model: Some empirical tests. In: Jensen, M.C. Studies in the theory of capital markets. Praeger Publishers, Inc.

Koponen, T.M. (2003), Commodities in action: measuring embeddedness and imposing values. The Sociological Review, 50(4): 543–569.

Kosowski, R., Timmermann, A., Werners, R., White, H. (2006). Can mutual fund "Stars" really pick stocks? New evidence from a bootstrap analysis. Journal of Finance, LXI(6): 2551–2595.

Markowitz, H.M. (1952), Portfolio selection. The Journal of Finance, 7(1): 77–91.

Markowitz, H.M. (1991), Portfolio selection – efficient diversification of investment. Blackwell, Cambridge, MA.

Rzymowski, W., Surowiec, A. (2013): Two-asset portfolio. In: Zielinski, Z.E. Rola informatyki w naukach ekonomicznych i spolecznych. Innowacje i implikacje interdyscyplinarne (Vol. 2, pp. 183–192). Wydawnictwo Wyzszej Szkoly Handlowej, Kielce.

Scherer, B. (2002). Portfolio resampling: Review and critique. Financial Analysts Journal, 58(6): 98–109.

Shleifer, A. (2000). Inefficient Markets: An Introduction to Behavioral Finance. Clarendon Lecture in Economics.

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