Nataliia B. Andreishina[1]

RESEARCH ON STABILITY OF SUPPLY AND DEMAND EQUILIBRIUM MODEL USING THE D-PARTITION METHOD

The article builds a continuous dynamic model of supply and demand equilibrium that takes into account the delay of demand price, in which analytic expression for the function of the equilibrium price is found. Using the D-partition method, the limits of stability of the model are determined.

Keywords: *demand; supply; economic equilibrium; delay; D-partition method.*

Наталія Б. Андрейшина

ДОСЛІДЖЕННЯ СТІЙКОСТІ МОДЕЛІ РІВНОВАГИ ПОПИТУ ТА ПРОПОЗИЦІЇ МЕТОДОМ D-РОЗБИТТІВ

У статті побудовано неперервну динамічну модель рівноваги попиту та пропозиції з урахуванням запізнення ціни попиту, в якій знайдено аналітичний вираз для функції рівноважної ціни. Використовуючи метод D-розбиттів визначено границі стійкості побудованої моделі.

Ключові слова: *попит; пропозиція; економічна рівновага; запізнення; метод D-розбиттів; стійкість.*

Форм. 14. Рис. 1. Літ. 12.

Наталья Б. Андрейшина

ИССЛЕДОВАНИЕ УСТОЙЧИВОСТИ МОДЕЛИ РАВНОВЕСИЯ СПРОСА И ПРЕДЛОЖЕНИЯ МЕТОДОМ D-РАЗБИЕНИЙ

В статье построена непрерывная динамическая модель равновесия спроса и предложения с учетом запаздывания цены спроса, в которой найдено аналитическое выражение для функции равновесной цены. Используя метод D-разбиений определены границы устойчивости построенной модели.

Ключевые слова: *спрос; предложение; экономическое равновесие; запаздывание; метод D-разбиений.*

Problem setting. Supply and demand are the main categories of economics, and successful functioning of economy in general, as well as that of individual companies, depends on understanding their nature. There are different views on the nature of macro- and microeconomic equilibrium of supply and demand. This research is based on the assumption that there are mechanisms that push market economy to an equilibrium of cumulative demand and cumulative supply, but enterprises must have their own mechanisms with similar functions that can be used to improve the efficiency of individual companies.

After dynamic equilibrium models of supply and demand were first developed, it became necessary to consider the effect of delay that may occur due to delays in deliveries to market, belated response of consumers to a new product or advertising of this product etc. However, for some time, differential market models were developed without considering delays, due to their complicated mathematics. Later, with the development of the theory and methods for solving differential equations with a delay, continuous dynamic market models taking the delay into account began to develop.

[1] Institute for Entrepreneurship "Strategy", Zhovti Vody, Ukraine.

In this context, the task of building dynamic models which would take the delay into account to find the equilibrium price and examining their stability gains special practical usefulness.

Recent research and publications analysis. A lot of scientists have studied the categories of supply and demand and modeled their interactions, and their works are dedicated to special features and directions of interactions between market players, to research the specifics and the mechanisms aimed to achieve economic equilibrium at the macro and microlevels.

Continuous and discrete dynamic models taking the delay of demand price into account were built by D. Brown and F. Kubler (2008), S. Didur (2005), V. Poddubnyi (2004), N. Obrosova (1998), L. Sergeeva (2001), P. Samuelson (2002) etc.

Despite the large number of scientific works on the topic of supply and demand equilibrium, models which would describe the process of achieving equilibrium that take into account the delay of demand price and give their stability analysis have not been developed enough.

The research objective. Based on the above, the purpose of this paper is to study the stability of the supply and demand equilibrium model that takes delay into account with the help of the D-partition method.

Key research findings. The mechanism of achieving equilibrium under market economy has a rather well-worked out theory, in the basis of which the classical authors of economic thought put understanding of the basic concept "economic equilibrium" as a correlation between the limited available resources and the growing needs of society (Cournot, 1938; Valras, 2000). This means that equilibrium is achieved either by limiting the needs or by increasing production volumes. Equilibrium at the microlevel implies such a state of subjects that ensures the equality of supply and demand of goods, services and resources. At the same time, resource suppliers, manufacturers and consumers maximize their economic benefits.

The price mechanism allowing to achieve the equilibrium of supply and demand for specific goods, suppliers and other market players at the microlevel contributes to reaching the equilibrium.

In the study of economic interactions between consumers and producers, it is important to explore the existence of equilibrium and analyze its stability. These tasks are integrated into the task of finding the equilibrium state of economy by calculating and using equilibrium prices. The main tools of solving these problems are the methods of economic and mathematical modelling, which are used to build different models of supply and demand equilibrium. The model of supply and demand equilibrium refers to mathematical or non-mathematical scheme that describes the formation of equilibrium prices.

P. Samuelson (2002) was the first to offer a clear classification and analysis methods of market equilibrium systems. The most significant contribution of this scientist to the development of the market equilibrium theory was proving the need for a dynamic approach to market equilibrium. P. Samuelson used the definition of stability spread widely in mathematics: equilibrium is stable if the system returns to it in accordance with the laws of dynamics after the equilibrium has been disturbed.

The cause of instability in discrete models is an implicitly present inertia in the reaction of consumers and producers to price changes. In continuous models there are two methods to take this inertia into consideration. In the first method, proposed by H. Lorenz (1995), inertia is modeled as follows: supply does not depend on the price directly but its derivatives do. In this case, the process of finding the equilibrium price is described by a system of differential equations.

The other method of taking the inertia into account is introducing a delay into the continuous model of supply and demand equilibrium. The delay in the reaction of a producer to price changes reflects the processes taking place at the market and allows for economic interpretation, for example, the length of production cycle.

We have constructed a mathematical model of supply and demand equilibrium which takes into account the delay in demand price. The presence of such delay is due to the fact that the values of the price p(t) at a given time t change the demand only after a certain period of time delay τ. For example, if one needs to advertise the product or analyze its quality and so on. In turn, the availability of a product and its price form the supply precisely at a given time.

The functions of supply and demand will be selected to be linear and depending on product price. Price is the determining factor in the formation of supply and demand and serves, on the one hand, as an indicator that reflects the policy and market conditions and, on the other, as a marketing regulator of the market that is used to affect the supply and demand, market structure and capacity, purchasing power of a currency, inventory turnover etc. P. Allen (1963) introduced an additional element to account for price changes expressed by the derivative $p'(t)$. Then the functions of supply and demand will be as:

$$\begin{aligned} D(t) &= a_1 p(t) + b_1 p'(t) + c_1; \\ S(t) &= a_2 p(t) + b_2 p'(t) + c_2, \end{aligned} \tag{1}$$

where a_1, b_1, c_1, a_2, b_2, c_2 are the coefficients.

Let us consider a situation where a price is set in such a way that demand is fully satisfied by supply, on the one hand, and supply is fully absorbed by demand, on the other, at each moment of time t. Such situation is possible, for example, at a market of perishable products which make keeping merchandise stock impossible. Then the functions of supply and demand $D(t)$ and $S(t)$ must satisfy the equation:

$$D(p(t), p'(t)) = S(p(t), p'(t)). \tag{2}$$

Considering the delay of demand price, we obtain:

$$\begin{aligned} D(t) &= D(p(t-\tau), p'(t)); \\ S(t) &= S(p(t), p'(t)), \end{aligned} \tag{3}$$

where τ is the delay.

From the supply and demand balance ratio, considering the formulas (1) and (3), we get to the following differential equation with the argument which is delayed:

$$a_1 p(t-\tau) + b_1 \frac{dp}{dt} + c_1 = a_2 p(t) + b_2 \frac{dp}{dt} + c_2, \tag{4}$$

after simplification we get:

$$\frac{dp}{dt} = \frac{-a_1}{b_1 - b_2} p(t-\tau) + \frac{a_2}{b_1 - b_2} p(t) + \frac{c_2 - c_1}{b_1 - b_2}, \tag{5}$$

or

$$\frac{dp}{dt} = k_1 \times p(t-\tau) + k_2 \times p(t) + k_3, \tag{6}$$

where $k_1 = \frac{a_1}{b_2 - b_1}$, $k_2 = \frac{a_2}{b_1 - b_2}$, $k_3 = \frac{c_2 - c_1}{b_1 - b_2}$.

Note that, in order to obtain a clear solution of the equation (6), it is necessary to define some initial function $p_0(t)$ on the delay interval $[0, \tau)$, which distinguishes this problem from the Cauchy problem for ordinary differential equations.

Thus, the model of supply and demand equilibrium taking into account the delay of demand price leads to the problem of solving differential equations with a delay, but there are no general analytical methods of solution for such problems. The analytical solution of a linear differential equation with one delay and constant coefficients was offered by L. Elsholtsem (1964). The equation is solved using the Laplace transform, and the solution is expressed as a sequence which consists of an infinite number of characteristic quasipolynomial roots.

Among the many methods for differential equations with a delay, the most common is the method of steps, which can be considered universal for numerical solutions of these equations and which replaces a differential equation with a delay with a series of differential equations without a delay. At the same time, any numerical solution algorithms must include solution derivatives' break points that follow one after another at the distance of the delay value into integration points. Because of that, numerical methods are rather cumbersome to implement. Also, numerical methods are not used in cases where delay is smaller than the step that ensures the accuracy of solution. Additionally, when solving the differential equation of supply and demand equilibrium numerically, for the equilibrium price function, we obtain a discrete set of values at the given moment of time, which is not very convenient for forecasting. This makes the task of finding an analytical solution of differential equations with a delay extremely important.

The solution of the equation (6) with the initial condition $t \in [0, \tau)$, $p(t) = p_0(t)$ can be represented as follows:

$$p(t) = \sum_{m=0}^{\infty} \chi_m(t) \times p_m(t), \tag{7}$$

where

$$p_m(t) = P\big(m\times\tau,(m+1)\times\tau,t,p_{m-1}\big) = p_{m-1}(m\times\tau)\times e^{k_2\cdot(t-m\cdot\tau)} + \\ + \frac{k_3}{k_2}\left[e^{k_2\cdot(t-m\cdot\tau)} - 1\right] + k_1 \int_{m\cdot\tau}^{t} e^{k_2\cdot(t-\xi)} p_{m-1}(\xi - m\times\tau)d\xi; \tag{8}$$

$\chi_m(t)$ – is the characteristic function of the half-interval $[m \times \tau,(m + 1) \times \tau)$, i.e.:

$$\chi_m(t) = \begin{cases} 1, & t \in [m\times\tau,(m+1)\times\tau) \\ 0, & t \notin [m\times\tau,(m+1)\times\tau). \end{cases} \tag{9}$$

Note that, since for any fixed value of the variable $t \geq 0$ there is only one half-interval containing it in the sequence $[m \times \tau,(m + 1) \times \tau)$, then the ratio of (7), for each fixed value of $t \geq 0$, contains only one non-zero addend.

Let us consider the question of stability of the equation (6), with $t \geq 0$. Let us rewrite it in the following form:

$$p'(t) - k_1 \times p(t-\tau) - k_2 \times p(t) = k_3. \tag{10}$$

To examine the stability of (10), we use the D-partition method (Neymark, 2010).

Let us consider the characteristic quasipolynomial of a homogeneous equation corresponding to (10):

$$R(z) = z - k_1 e^{-\tau z} - k_2 = 0. \tag{11}$$

Each root z_k of the characteristic equation (10) corresponds to the solution e^{zk} of (10).

The quasipolynomial $R(z)$ has the root $z = 0$ if $k_1 + k_2 = 0$. Let us assume that this quasipolynomial has an imaginary root $z = iy$ with $y \in (0, \infty)$. Thus,

$$iy - k_1 e^{-\tau iy} - k_2 = 0. \tag{12}$$

Using Euler's formula: $e^{i\varphi} = \cos\varphi + i\sin\varphi$, we obtain: $iy - k_1 \cos\tau y + ik_1 \sin\tau y - k_2 = 0$.

Dividing real and imaginary parts, we obtain the system

$$\begin{cases} k_1 \cos \tau y + k_2 = 0 \\ y + ik_1 \sin\tau y = 0 \end{cases}, \tag{13}$$

k_1 i k_2 are used as parameters, then:

$$\begin{cases} k_1 = \dfrac{-y}{\sin\tau y} \\ k_2 = y\, ctg\, \tau y \end{cases}, \tag{14}$$

where $0 < y < \dfrac{\pi}{\tau}$, because in points of $y = \dfrac{\pi n}{\tau}$, $n = \overline{1,\infty}$, the denominator (14) turns to zero.

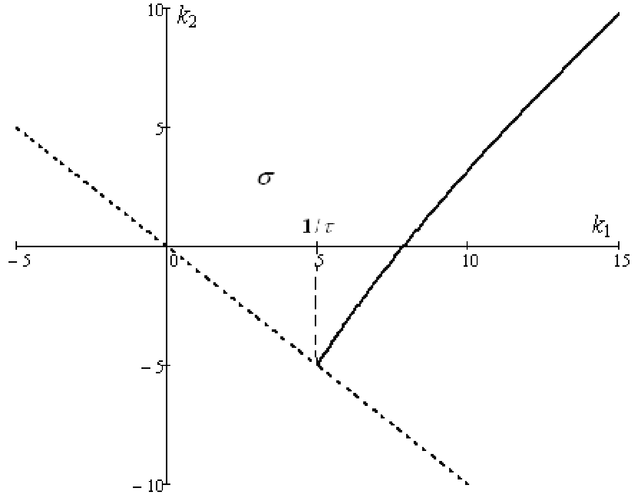

Figure 1. **The domain of stability of the equation (10) when τ = 0.2,** *authors*

Thus, the parametric equations (14) and the line $k_1 + k_2 = 0$ form the boundaries of D-partitions for the quasipolynomial $R(z)$.

The statement that the equation (10) is asymptotically stable means that the point with the coordinates (k_1, k_2) belongs to the domain σ (Figure 1). At the same time, the boundary $k_1 = -k_2$ up to the point $k_1 = -1 / \tau$ (including this point) means simple stability.

Conclusions:

1. Examining the microeconomic equilibrium, it can be stated that the balance between producers and consumers can exist, and it can be regulated with specially designed tools, one of which is equilibrium prices.
2. We have constructed a continuous supply and demand equilibrium model taking into account the delay of demand price that is represented by a differential equation with a delay, for which an analytical solution is found.
3. Using the D-partition method, we have defined the boundaries of stability for the developed model, which will allow determining stability or instability of a model just by looking at the values of its coefficients, without solving differential equations with a delay.

References:

Аллен Р. Математическая экономия / Пер. с англ.; Под ред. и со вступит. статьей А. Вайнштейна. – М.: Изд-во иностранной лит-ры, 1963.– 670 с.

Вальрас Л. Элементы чистой политической экономии или теория собственного богатства / Пер. на рус. яз. И. Егорова, А. Белянина. – М.: Экономика, 2000. – 304 с.

Дідур С.В. Регулювання попиту і пропозиції на макрорівні: Монографія. – К.: Наук. світ, 2005. – 319 с.

Неймарк Ю.И. Динамические системы и управляемые процессы. – М.: КД Либроком, 2010. – 336 с.

Обросова Н.К. Потеря устойчивости равновесной цены в моделях ценообразования вальрасовского типа // Математическое моделирование.– 1998. – Т. 10, №5. – С. 47–57.

Поддубный В.В. Оптимальная стабилизация рынка, описываемого модифицированной моделью Вальраса-Маршалла // Обработка данных и управление в сложных системах. – Вып. 6. – Томск: Томский ун-т, 2004. – С. 161–171.

Самуэльсон П. Экономика: В 2-х т. – М.: Алгоп, 2002. – Т. 1. – 527 с.

Сергеева Л.Н. Методы моделирования рыночного взаимодействия // Экономическая кибернетика.– 2001.– №5–6. – С. 36–39.

Эльсгольц Л.Э. Введение в теорию дифференциальных уравнений с отклоняющимся аргументом. – М.: Наука, 1964. – 128 с.

Brown, D., Kubler, F. (2008). Computational Aspects of General Equilibrium Theory: Refutable Theories of Value. LNEMS. Springer: Berlin-Heidelberg. 604 p.

Cournot, A. (1938). Recherches sur les principles mathematique de la theorie des richesses. Paris. Ch. VII.

Lorenz, H.-W. (1995). Nonlinear Dynamical Economics and Chaotic Motion. Springer, XV.

Стаття надійшла до редакції 14.04.2015.