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MATHEMATICAL MODEL OF FLAT-VERTICAL PROFILE MOISTURE TRANSFER UNDER TRICKLE IRRIGATION IN CONDITIONS OF INCOMPLETE SATURATION

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Aim. To develop an efficient method of building a numerical model for the process of moisture transfer under trickle irrigation, with the mathematical modeling of the method involving the system of differential equations in partial derivatives of Klute-Richards, and to perform computing experiments regarding flat-vertical profile moisture transfer with point sources. **Methods.** The mathematical apparatus of the theory of differential schemes of solving differential equations in partial derivatives, and Newton's method of iterative approximate solving of non-linear equations. **Results.** A stable differential two-step symmetrized algorithm (TS-algorithm) along with the corresponding scheme of the method of numerical solution for initially-boundary task for Richards' equation was created. The method was realized in the form of a computer program in C++ language, the computing experiments were performed with three deeper points, the humidity zones for volume moisture and potential were obtained. **Conclusions.** The numerical method was suggested, ensuring the efficient solution to Richards' non-linear equation in conditions of several deep point sources. The algorithm structure allows reducing the system of non-linear algebraic equations with many unknowns to solving independent non-linear equations with one unknown. The presented method may easily be expanded for three-dimensional cases. The results of computing experiments are in agreement with natural observations.

Keywords: trickle irrigation, humidity zone, moisture transfer, numerical modeling, equations of Klute-Richards.

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INTRODUCTION

Trickle irrigation is one of the most efficient ways of microirrigation, under which water is supplied locally in dosed portions into the pre-root zone of plants using the dispensing droppers. On the one hand, it raises the problem of sustaining the soil humidity level, optimal for plants, and on the other hand – the minimization of water use. In particular, the solution to this problem is related to forecasting the moisture distribution depending on the capacity of droppers, their position (deeper or on the surface), as well as the density of their placing.

The current work is the continuation of the studies [1] on numerical modeling of the moisture transfer process under trickle irrigation, where mathematical modeling was based on the system of differential equations in partial derivatives of Klute-Richards [2], which considers the conditions of moisture transfer process in the most complete way. The researchers either linearize the model and reduce the task to solving the system of linear algebraic equations at any moment of time [1, 3], or solve the non-linear system iterationally at any moment of time [4], which requires considerable computing resources, especially in the three-dimensional case. The suggested numerical method ensures efficient solving of Richards' non-linear equation in conditions of deeper point sources due to the splitting of the non-

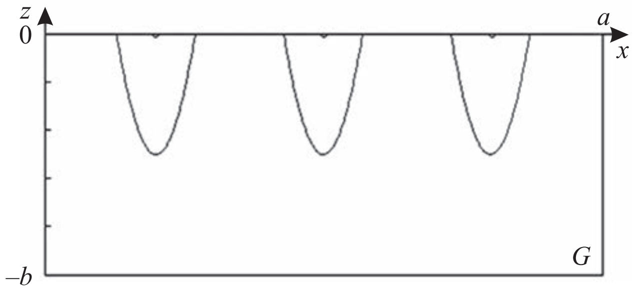

Fig. 1. Region G

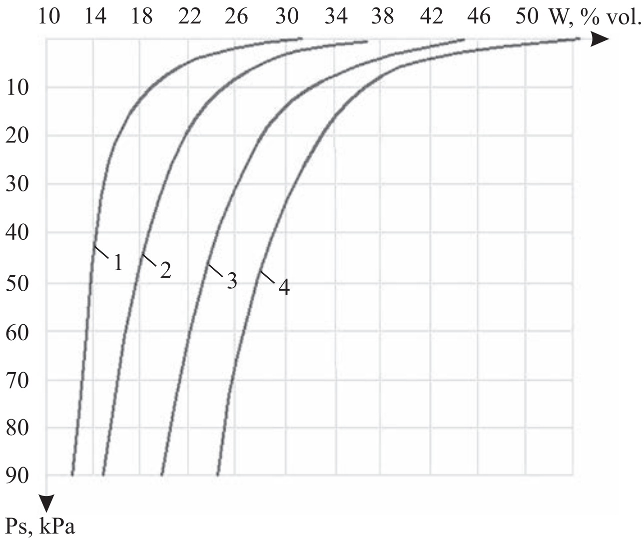

Fig. 2. Generalized dependences $P_S = (W)$ for different types of soils: 1 – sandy; 2 – light loamy; 3 – medium loamy; 4 – heavy loamy

linear system, which easily and efficiently covers the three-dimensional case.

The work was based on the assumption that there are no dissolved salts in the irrigation water, the process of moisture spreading is isothermal, the soil structure is not deformed, the pressure of soil air equals the atmospheric pressure, the moisture in soil is not condensed, moisture transfer occurs under the impact of capillary, gravitational forces, moisture gradients, evaporation, and the intake force of the plant root system.

This work was aimed at developing an efficient method for numerical modeling of the moisture transfer process under trickle irrigation, and at performing computing experiments for flat-vertical profile moisture transfer with point sources.

MATERIALS AND METHODS

Taking axial symmetry into consideration, let us reduce the three-dimensional task of moisture transfer to the task of flat-vertical moisture transfer.

In the rectangular region $G = \{(x, z)|\ 0 < x < a, -b < z < 0\}$(Fig. 1) let us consider the non-linear differential equation:

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial t}\left(k(W)\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial z}\left(k(W)\frac{\partial U}{\partial z}\right) + f(x, z, t), \quad 0<t<T_1. \qquad (1)$$

Here the unknowns are: W – volume humidity; $U = P + z$ – piezometric head; P – hydrodynamic potential, dependent on W: $P = \varphi(W)$; φ – a known function, which depends on the soil type and may be built rather accurately using tenziometric method (Fig. 2) [5]; $f(x, z, t) = f_k(x, z, t) - f_\tau(x, z, t)$ – the function of sources (droppers) and drains (moisture intake by plant roots). In conditions of point sources (droppers) the function $f_k(x, z, t)$ presents a linear combination of δ-functions of Dirac.

It should be noted that the equation (1) may be written in the terms of the head [6]:

$$C(W)\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(K(U)\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial z}\left(K(U)\frac{\partial U}{\partial z}\right) + f(x,z,t), \qquad (1')$$

where $C(W)\frac{\partial U}{\partial t}$ – the function of water saturation from the head, defined in an experimental way.

The coefficient of moisture transfer is a given dependence which may be experimentally defined. The Averianov's formula [7, 8] was used in this work as a possible variant of approximation:

$$k(W) = K_\Phi\left(\frac{W - W^*}{m - W^*}\right)^{3.5}, \qquad (2)$$

where K_f – filtration coefficient; W – volume humidity of soil; W^* – maximal molecular moisture capacity (MMMC) according to Lebedev; m – complete moisture capacity of soil.

The equation (1) is supplemented as follows

With the initial condition:

$$W|_{t=0} = W_0(x, z) \text{ – a given function;} \qquad (3)$$

and boundary conditions:

on the surface $z = 0$:

$U = 0$ (deeper sources – absent from the surface) or , $U = h_k$ (4)

where h_k – the height of capillary rising;

on the lower boundary $z = -b$:

$$\frac{\partial W}{\partial z} = 0; \qquad (5)$$

on the right and left vertical boundaries:

$$\left.\frac{\partial W}{\partial z}\right|_{x=0} = 0, \quad \left.\frac{\partial W}{\partial x}\right|_{x=a} = 0. \qquad (6)$$

Solving th∈e initial-boundary task (1)–(6), (8) will allow forecasting the dynamics of the change in the volume humidity and the head in the given time interval under given parameters of droppers in the given conditions.

RESULTS AND DISCUSSION

Model analysis. Taking into account the formula (2), let us write the equation (1) in a non-divergent form:

$$\frac{\partial W}{\partial t} = k'(W)\left(\frac{\partial U}{\partial x}\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}\frac{\partial W}{\partial z}\right) + k(W)\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2}\right) + f \qquad (1'')$$

where $k'(W) = 3.5K_{\Phi}\left(\frac{W - W^*}{m - W^*}\right)^{2.5}\frac{1}{m - W^*}$.

To use the main hydrophysical characteristic – dependence $P_S = f(W)$ in the presented mathematical model, it is more convenient to apply smooth approximations of dependences from the charts in Fig. 2 in the analytic form [9, 10]:

$$P_s(W) = \mu h_k\left(-\ln\left|\frac{W - W^*}{m - W^*}\right|\right)^{1/n}, \qquad (7)$$

where μ and n – empiric parameters, selected experimentally or by Newton's method.

Here we receive the formula of dependence of the unknown piezometric head on the unknown volume humidity:

$$U(W) = \mu h_k\left(-\ln\left|\frac{W - W^*}{m - W^*}\right|\right)^{1/n} + z \equiv \varphi(W, z). \qquad (8)$$

The numerical discrete model. The numerical solution of the presented initial-boundary task (1′′), (2)–(6), (8) was efficient via the method of finite differences, namely, a two-step symmetrized algorithm (TS-algorithm) [11] in combination with Newton's method for non-linear equations.

The idea of a TS-algorithm is as follows. The region of G is applied an even net,

$$\Omega_{h\tau}\ \{(x_i, z_j, t_n) | x_i = ih_x,\ z_j = -jh_z,\ t_n = n\tau,\ i = \overline{0,M_1},\ j = \overline{0,M_2},\ n = \overline{0,N}\}$$

divided into two subsets (Fig. 3).

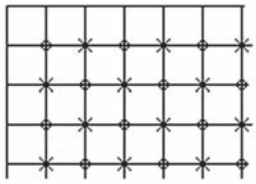

Fig. 3. Net $\Omega_{h\tau}$

$\Omega^{(1)}_{h\tau}$ $\{(x_i, z_j, t_n) | i + j + n$ – even$\}$ (×) and
$\Omega^{(2)}_{h\tau}$ $\{(x_i, z_j, t_n) | i + j + n$ – even$\}$ (○).

Let us define the initials on the net:

$$W^0_{ij} = W_0(ih_x, -jh_z),\ i = \overline{0, M_1},\ j = \overline{0, M_2} \qquad (9)$$

and boundary conditions:

$$D(W^{n+1}_{i0})\frac{W^{n+1}_{i1} - W^{n+1}_{i0}}{h_z} - k(W^{n+1}_{i0}) = \alpha(W^{n+1}_{i0} - W^B_0). \qquad (10)$$

$$-\frac{W^{n+1}_{iM_2} - W^{n+1}_{iM_2-1}}{h_x} = 0 \qquad (11)$$

$$\frac{W^{n+1}_{1j} - W^{n+1}_{0j}}{h_x} = 0, \quad \frac{W^{n+1}_{M_1j} - W^{n+1}_{M_1-1j}}{h_x} = 0. \qquad (12)$$

To find the solution of W^{n+1}_{ij}, at each subsequent $(n + 1)$ time step in the inner joints we shall use the schemes with central differences: at first we shall find the solution of W^{n+1}_{ij} in joints $(x_i, z_j, t_{n+1}) \in \Omega^{(1)}_{h\tau}$ by the evident differential scheme:

$W^{n+1}_{ij} = W^n_{ij} + \tau L_{h\tau}(W^{n+1}_{ij})$, $i = \overline{1,M_1-1}$, $j = \overline{1,M_2-1}$, where (13)

$$L_{h\tau}(W^n_{ij}) = k'_W(W^n_{ij})\left(\frac{U^n_{i+1j} - U^n_{i-1j}}{2h_x}\frac{W^n_{i+1j} - W^n_{i-1j}}{2h_x} + \frac{U^n_{ij+1} - U^n_{ij-1}}{2h_z}\frac{W^n_{ij+1} - W^n_{ij-1}}{2h_z}\right) + k(W^n_{ij})\left(\frac{U^n_{i+1j} - 2U^n_{ij} + U^n_{i-1j}}{h_x^2} + \frac{U^n_{ij+1} - 2U^n_{ij} + U^n_{ij-1}}{h_x^2}\right) + f^n_{ij},$$

then in joints $(x_i, z_j, t_{n+1}) \in \Omega^{(2)}_{h\tau}$ by the non-evident differential scheme

$$W^{n+1}_{ij} = W^n_{ij} + \tau L_{h\tau}(W^{n+1}_{ij}). \qquad (14)$$

Taking into consideration the fact that the values in neighboring joints are found by the evident scheme (13), the system M_1M_2 of non-linear algebraic equations (9)−(14) with M_1M_2 unknowns is split into $M_1M_2/2$ independent non-linear equations with one unknown:

$$F^{n+1}_{ij}(W^{n+1}_{ij}) = 0$$

where (15)

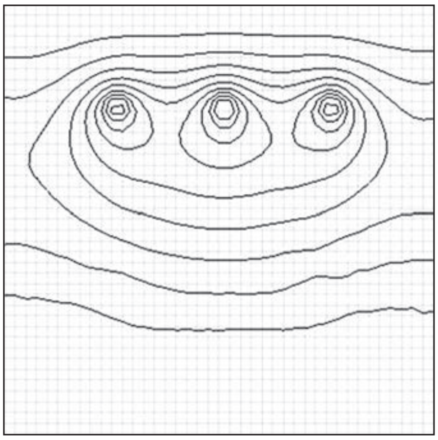

Fig. 4. Isolines of solution W at a moment of time t = 0.066 for all three deeper sources (droppers)

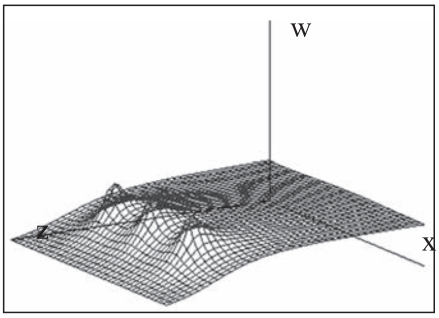

Fig. 5. Calculated values of humidity W at a moment of time t = 0.066 for three deeper sources (droppers)

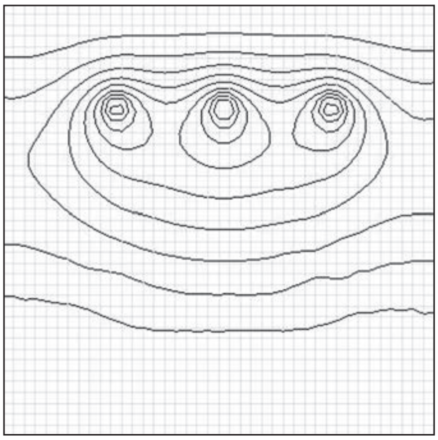

Fig. 6. Isolines of solution P at a moment of time t = 0.066 for three deeper sources (droppers)

$$F_{ij}^{n+1}(w) = W_{ij}^{n} - \tau k'(w)\left(\frac{U_{i+1j}^{n+1} - U_{I-1j}^{n+1}}{2h_x}\frac{W_{I+1j}^{n+1} - W_{I-1j}^{n+1}}{2h_x} + \right.$$
$$\left. + \frac{U_{ij+1}^{n+1} - U_{ij-1}^{n+1}}{2h_z}\frac{W_{ij+1}^{n+1} - W_{ij-1}^{n+1}}{2h_z}\right) - \tau k(w)\left(\frac{U_{I+1j}^{n+1} + U_{I-1j}^{n}}{h_x^2} + \right.$$
$$\left. + \frac{U_{ij+1}^{n+1} + U_{ij-1}^{n+1}}{h_z^2}\right) + 2\tau k(w)\varphi(w, z_j)\left(\frac{1}{h_x^2} + \frac{1}{h_z^2}\right) - f_{ij}^{n+1},$$
$$z_j = d + jhz.$$

The obtained non-linear equations are solved by Newton's method:

$$w_{s+1} = w_s - \frac{F(w_s)}{F'(w_s)}, \ \mathrm{s} = 0, 1, 2, ... \quad (16)$$

$$(F_{ij}^{n+1})'(w) = 1 - \tau k''(w)\left(\frac{U_{i+1j}^{n+1} - U_{i-1j}^{n+1}}{2h_x}\frac{W_{i+1j}^{n+1} - W_{i-1j}^{n+1}}{2h_x} + \right.$$
$$\left. + \frac{U_{ij+1}^{n+1} - U_{ij-1}^{n+1}}{2h_z}\frac{W_{ij+1}^{n+1} - W_{ij-1}^{n+1}}{2h_z}\right) + 2\tau k'(w)\varphi(w, z_j)\left(\frac{1}{h_x^2} + \frac{1}{h_z^2}\right) +$$
$$+ 2\tau k(w)\varphi'(w, z_j)\left(\frac{1}{h_x^2} + \frac{1}{h_z^2}\right) - \tau k'(w)\left(\frac{U_{i+1j}^{n+1} + U_{i-1j}^{n+1}}{h_x^2} + \frac{U_{ij+1}^{n+1} + U_{ij-1}^{n+1}}{h_z^2}\right),$$
$$\varphi'(w, z_j) = \frac{\mu h_k}{n}\left(-\ln\left|\frac{W - W^*}{m - W^*}\right|\right)^{\frac{1}{n}-1}\frac{1}{|W - W^*|}.$$

Let us consider the condition of completing the iterative process to be

$$|w_{s+1} - w_s| \le \varepsilon,$$

where ε – given accuracy.

The algorithm is locally stable by its initial data [11].

The analysis of computing experiments. The presented numerical method was realized in C/C++ language in Microsoft Visual Studio in the form of two program modules:

1) a computing module, forming approximate solutions W^{n+1}_{ij} and U^{n+1}_{ij}, $\overline{i = 0, M_1}$, $\overline{j = 0, M_2}$, $\overline{n = 0, N}$ in the form of binary files;

2) a graphic module – a module of graphic processing of the obtained binary files, used to obtain Fig. 4–6.

The computing experiments were performed using the following parameters:

– filtration coefficient K_f;

– bound moisture (MMMC) W^* = 0.17 (sandy soils);

– poriness m = 0.31;

– height of capillary uprise h_k = 1.25;

– width a = 1 and depth b =1 of the region; final moment of time T_1 = 1;

– three point sources with intensities 1;

– $h_x = h_z$= 1/40, τ = 1/500

– accuracy of Newton's iterative method: $\varepsilon = 10^{-7}$

Taking into account the fact that accurate analytical solution may be found only in case of stable coefficients, a posteriori evaluation of a deviation is impossible. Here the results of the experiments correspond to natural observations and confirm the efficiency of the suggested method.

CONCLUSIONS

The efficient numerical method of solving the initial-boundary equation for Richards' equation in flat-vertical case R^2 was developed, describing the process of moisture distribution in soil under trickle irrigation with point sources. The algorithm may be extended to the natural case R^3 of three-dimensional space. The structure of the method, namely, the splitting of a non-linear system into independent non-linear equations with one unknown allows for parallel calculations, which may be especially efficient for case R^3. The method was realized and successfully applied for numerical modeling of moisture transfer process in case of three deep point sources (Fig. 4–6). There is an observed alignment error of the humidity contours on sources one and three towards the second source (Fig. 4, 6). In our opinion, this phenomenon may be explained by the interference of point sources: the middle source gives additional moisture to the humidity zone of sources one and three, their boundaries (contours) "merge".

Математична модель плоско-вертикального профільного вологоперенесення за краплинного зрошення в умовах неповного насичення

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Мета. Розробка ефективного методу для чисельного моделювання процесу вологоперенесення за краплинного зрошення, в основі математичного моделювання якого використано систему диференційних рівнянь у частинних похідних Клюта–Річардса, та провести обчислювальні експерименти для випадку плоско-вертикального профільного вологоперенесення з точковими джерелами. **Методи.** Математичний апарат теорії різницевих схем розв'язання диференціальних рівнянь в частинних похідних, а також метод Ньютона ітераційного наближеного розв'язання нелінійних рівнянь. **Результати.** Створено стійкий різницевий двокроковий симетризований алгоритм (ДС-алгоритм) та відповідну схему методу чисельного розв'язання початково-крайової задачі для рівняння Річардса. Метод реалізовано у вигляді комп'ютерної програми мовою С++, проведено обчислювальні експерименти з трьома заглибленими джерелами, отримано зони зволоження для об'ємної вологості і потенціалу. **Висновки.** Запропоновано чисельний метод, який дозволяє ефективно розв'язувати нелінійне рівняння Річардса за умов кількох заглиблених точкових джерел. Структура алгоритму дозволяє звести розв'язання системи нелінійних алгебраїчних рівнянь з багатьма невідомими до розв'язання незалежних нелінійних рівнянь з одним невідомим. Наведений метод може бути легко поширений на тривимірний випадок. Результати обчислювальних експериментів узгоджуються з натуральними спостереженнями.

Ключові слова: краплинне зрошення, зона зволоження, вологоперенесення, чисельне моделювання, рівняння Клюта-Річардса.

Математическая модель плоско-вертикального профильного влагопереноса при капельном орошении в условиях неполного насыщения

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Цель. Разработать эффективный метод для численного моделирования процесса влагопереноса при капельном орошении, в основе математического моделирования которого использована система дифференциальных уравнений в производных частей Клюта–Ричардса, и провести вычислительные эксперименты для случая плоско-вертикального профильного влагопереноса с точечными источниками. **Методы.** Использован математический аппарат теории разностных схем решения дифференциальных уравнений в производных частей, а также метод Ньютона итерационного приближенного решения нелинейных уравнений. **Результаты.** Разработан стойкий разностный двушаговый симметризованный алгоритм (ДС-алгоритм) и соответствующая схема метода численного решения изначально краевой задачи для уравнения Ричардса. Метод реализован в виде компьютерной программы на языке С++, проведены вычислительные эксперименты с тремя углубляющими источниками, получены зоны увлажнения для объемной влажности и потенциала. **Выводы.** Предложен численный метод, позволяющий эффективно решать нелинейное уравнение Ричардса в условиях нескольких углубляющих точечных источников. Структура алгоритма дает возможность свести решение системы нелинейных уравнений алгебраизма со многими неизвестными к решению независимых нелинейных уравнений с одним неизвестным. Представленный метод может быть легко распространен на трехмерный случай. Результаты вычислительных экспериментов согласуются с натуральными наблюдениями.

Ключевые слова: капельное орошение, зона увлажнения, влагоперенос, численное моделирование, уравнение Клюта-Ричардса.

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