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Modeling the default spread for bank loans

Abstract

In this paper, we propose a discrete time model to measure the default spread for bank loans. The model provides a closed-form solution for the short- and medium-term default spread, which we assume to be dependent on the default probabilities, the losses given default, the risk grades transition probabilities, seen in a Markov chain, the prime rate and the economic cycle phases. The model is tested with real data provided by a bank, and allows one to conclude that the actual spread is, on the one hand, insufficient to cover the whole credit risk for the low-risk clients and, on the other hand, excessive for the high-risk clients. We believe that this study may contribute to improving the pricing for bank loans.

Keywords: bank loans, default spread, pricing.
JEL Classification: G12, G21.

Introduction

The events of August 2007, caused by the increasing defaults in the subprime mortgage market, have forced the Federal Reserve and the European Central Bank to provide the financial system with large amounts of money as an attempt to control the liquidity crisis that arose. This fact suggests that in certain cases the banks are not properly evaluating the risks inherent to their activity. This situation reinforces the importance of the credit risk models in assessing the credit risk and pricing the loans.

Altman et al. (2004) classify the credit risk models into two main categories: the pricing models and the value-at-risk models. The pricing models are in turn divided into three main approaches: the two structural-form approaches (of first and second generation), and the reduced-form approach. While structured-form models are inspired by the work developed by Merton (1974), which is based on the Black and Scholes (1973) theory for options valuation, the reduced-form models are inspired by the work of several authors, such as Litterman and Iben (1991), Kim et al. (1993), Jarrow and Turnbull (1995), Longstaff and Schwartz (1995), Jarrow et al. (1997), Lando (1998), Duffie and Singleton (1999), among others.

Despite their different approaches, both the structural and the reduced-form models have as central objective the estimation of the default probabilities (Allen and Saunders, 2003). The main factor that differentiates the two categories is that in the case of the structural-form models, the default process of a firm is directly connected to its asset value, and the default occurs when the assets market price becomes lower than the liability value, while in the case of the reduced-form models some hypotheses are made regarding the dynamics of several variables, which are not directly connected to the firm's asset value. For instance, the construction of a function for the default intensity is done by using the credit spreads of risky bonds (Allen and Saunders, 2003). In these models, the credit spread is perceived as the product between the default probabilities and the losses given default $(p \times u)$, and serves as input to infer the behavior of the default probabilities, since the credit spread is observable at the market by comparing the prices of risky bonds and default risk-free bonds.

In the case of bank loans the situation is the opposite, despite the involved variables are the same. For the banks that have a reasonable set of information about their clients, the variables related to the default can be measured and, for that reason, the main issue is to determine the credit spread, which is seen as a function of other variables, where the default probabilities are included. In this case, the credit spread is not directly observable due to the absence of a market for these products.

In this study, we intend to model one of the two components of the credit spread: the default spread (also known as default premium). Like in the reduced-form models, we do not take into account the firm's asset value. We use five exogenous variables for the default spread calculation, four explicit and one implicit. Two out of the four explicit variables correspond to the main variables mentioned by Altman et al. (2004): the default probability (p) and the probability of a loss given default (u), which is equal to one minus the recovery rate. The third variable measures the risk grade transition probabilities (a). Jarrow et al. (1997) are among the first who considered the transition probabilities in bond pricing models.

The fourth variable is the prime rate (r). This is the rate for the bank best clients and it is obtained by adding a spread (the risk premium) to the risk-free rate, which main purpose is to compensate the lender for investing in a risky loan. The risk premium together with the default spread (s) form the two components of the credit spread (Volkart and Mettler, 2004). While the risk premium intends to compensate the bank for unexpected losses, the default spread aims to compensate the bank for losses that may occur due to the existence of default

The opinions stated in this study are personal views of the author.

risk. Volkart and Mettler (2004) show how the credit spread can be divided into the risk premium and the default spread components.

When a bank lends money at the prime rate it is assuming that its clients comply with their entire obligations, and so they are seen as default risk-free clients. For the clients that exhibit some risk, the bank expects on average to suffer some losses, and tackles this situation by adding the default spread to the prime rate. To simplify we assume the prime rate as fixed.

Finally, the fifth variable that we use is the impact of the economic cycle on the default spread. Since we consider this variable as implicit, it is not shown in the functional form of the model, but its relevance can be seen in the estimation procedure of the other variables. The consideration of the economic cycle in the credit pricing is a central issue. Allen and Saunders (2003) refer that if the models do not take into account the different phases of the economic cycle, they can amplify the procyclical tendencies of banking, with potential negative impact on the macroeconomic stability.

In fact, if the default risk estimates are too optimistic in boom times, the banks may have the tendency to overlend, leading the economy to a possible overheating and to the consequent inflationary pressure. On the other hand, if the banks are too pessimistic in estimating the default risk during recessions, a possible expansionary monetary policy may have not enough strength to encourage the banks to lend to clients that they perceive to be risky.

As we show later, the variables that measure the risk grade transition probabilities are correlated with the economic cycle phases. On the other hand, the literature have shown that the default probabilities (p) and the losses given default (u) are also influenced by the economic cycle. Carey (1998), Schuermann (2004) or Altman and Brady (2001), for instance, show that the losses given default assume higher values in economic downturns, while Fama (1986) and Wilson (1997) find empirical evidence that the default probabilities grow dramatically during downturns.

The rest of the study is organized as follows: in section 1, we propose a closed-form solution for the simple short-term default spread in discrete time, without differentiating the borrowers according to their credit risk; in section 2, we improve the model, taking into account the risk grade transition probabilities; in section 3, we generalize the model for any term of the loan; in section 4, we test the model by estimating the parameters with real data[1], taking into consideration the economic cycle; finally, in the last section we present the main conclusions.

[1] Due to the sensitivity of the subject, the data used in this study, which was provided by a bank, is old enough to avoid confidentiality issues.

1. The short-term default spread

When a bank lends an amount C of money to a default risk-free client, the loan is charged at the prime rate r. After a certain term (one year, for instance), the bank expects to receive the principal and the respective interests, i.e. $C(1+r)$. If we consider that for a certain group of clients, the default probability is positive, and given by p, and if we also consider that there exists a positive probability u of a loss given default, then if the bank lends an amount C to a client within this group, it should charge the loan at a rate higher than r (let us say $r+s$), in order to assure that the amount of money the bank expects to receive is, on average, equal to $C(1+r)$. To find the spread that makes the bank indifferent to lend money to clients belonging to both groups, we must solve the following equation with respect to s:

$$C(1+r) = (1-p)C(1+r+s) + pC(1-u);$$

$$u, p \in [0,1]; \quad C,\ r > 0. \tag{1}$$

The left-hand side of the equation represents the amount the bank expects to receive if the client does not exhibit any risk of default. The right-hand represents the amount the bank expects to receive if the client exhibits some risk. For the latter, the interests come from the rate $r+s$ that charges only the "live" capital (i.e. the proportion of the principal that is not subject to default, and is paid by the borrower), which is $(1-p)C$. The remaining portion of the principal, pC, which we assume that does not pay interests, defaults. However, one part of the principal, $pC(1-u)$, is recovered; and the rest, pCu, is lost forever. The solution is given by:

$$s = \frac{r + p.u}{1-p} - r. \tag{2}$$

It can be seen that the spread increases whenever the default risk increases. In fact, when $p \to 1$ we get $s \to +\infty$. If the client complies with his entire obligations ($p = 0$), the spread is null.

Considering that the parameters p and u depend on the risk grade, we tackle this situation in the following section.

2. The augmented short-term default spread

If the borrower has organized financial statements at his disposal, and if he provides the bank with the necessary elements to perform a credit risk assessment, then the bank is able to attribute a rating to the borrower according to a certain scoring model. Let us consider that the grades or levels of risk exposure are measured through the following sorted elements of a set R, where $R = \{1,2+,2,3+,3,4,5\}$. The element "1" represents the lowest level of risk, the element "5" represents the highest, and the remaining are intermediate grades[1].

Equation (2) assumes that the client keeps the same level of risk throughout the loan term. In fact, it is common the risk level of a client varying through time, according to the economic cycle. We deal with this situation adding the risk grade transition probabilities to the default spread calculation.

Let a_{ij} be the probability of a client, randomly chosen, moving from risk grade i to grade j; p_i, the default probability of a grade i client; and u_i, the probability of a loss given default for a grade i client. If we want to calculate the spread for a grade 5 client, for instance, we have to solve the following equation with respect to s_5:

$$C(1+r) = \sum_{j=1}^{5}\left[(1-p_j)C(1+r+s_5) + p_j C(1-u_j)\right]a_{5j},$$

$$\text{subject to } \sum_i a_{5i} = 1,\ i \in R. \tag{3}$$

When $a_{55} = 1$ we obtain the simple case presented in equation (1), where movements between risk grades are not considered. When $a_{55} \neq 1$ (and so $\exists\, a_{5j} \neq 0,\ j \in R \setminus \{5\}$), we assume that a proportion of the principal changes to another risk grade according to the transition probability for that grade, and the part of the principal that changes must be evaluated according to the destination grade parameters, p_j and u_j. In other words, it is like we divide the original loan into several smaller loans to lend to clients with different risk grade, and for each of these clients we would not consider the possibility of grade transitions, using equation (2) to perform the calculation of the several smaller spreads. The spread for the original loan would be given by the sum of the smaller spreads.

The solution, for a generic risk grade z: $z \in R$ is given by:

$$s_z = \frac{(1+r) - \sum_{i \in R}((1-u_i)p_i)a_{zi}}{\sum_{i \in R}(1-p_i)a_{zi}} - (1+r). \tag{4}$$

This equation states that the default spread is also a function of the movements between risk grades.

3. The medium-term default spread

In this section, we generalize the expression (4) for a generic term n of the loan. In the context of economic stability, when a bank lends an amount C for a n year term, if the client accomplishes his obligations the bank expects to receive, after n years, the amount $C(1+r)^n$. If the client has a rating of 5, for instance, then there is a default probability of p_5, and a u_5 probability of a loss given default.

If the loan term is 1 year (n = 1), the part of the principal that pays interests is $(1-p_5)C$. After 1 year the bank expects to receive, assuming a null recovery rate, the amount $(1-p_5)C(1+r+s_5^{(1)})$, where $s_5^{(1)}$ represents the spread for a 1 year loan term. If the term is 2 years (n = 2), the bank, in the second year, only gets interests from the principal that does not default in the first year. Therefore, the expected value to be received is:

$$\left[(1-p_5)C(1+r+s_5^{(2)})\right](1-p_5)(1+r+s_5^{(2)}) = (1-p_5)^2 C(1+r+s_5^{(2)})^2.$$

For a loan term of n, the bank expects to accumulate the amount $(1-p_5)^n C(1+r+s_5^{(n)})^n$ at the end of the n^{th} year, under the hypothesis of a null recovery rate.

However, the bank is able to recover some part of the principal that defaults during the n years. In the first year, the bank expects a default of $p_5 C$. In the second year, the bank expects a default of $p_5(1-p_5)C$. In the n^{th} year, the bank expects a default of $p_5(1-p_5)^{n-1}C$. Putting these defaults together, we get a total of: $p_5C + p_5(1-p_5)C + \ldots + p_5(1-p_5)^{n-1}C$, which is equivalent to $p_5C\left[1+(1-p_5)+\ldots+(1-p_5)^{n-1}\right]$ = $C\left[1-(1-p_5)^n\right]$. The last factor of the right-hand side is the sum of a geometric sequence with common ratio $(1-p_5)$. This is the expression for the expected defaults throughout the loan term. When the term is over, the bank expects to recover the amount $C\left[1-(1-p_5)^n\right](1-u_5)$, since $(1-u_5)$ represents the recovery rate.

Moreover, since there is a transition probability from risk grade 5 to other grades, after n periods, which we call $a_{5j}^{(n)}, j \in R$, we calculate the medium-term spread taking into consideration the transition probabilities for n periods, as it was done for equation (3). The spread for a client with grade 5 is given by solving the following equation with respect to $s_5^{(n)}$:

$$C(1+r)^n = \sum_{i=1}^{5} C a_{5i}^{(n)}\left(\left[(1-p_i)^n(1+r+s_5^{(n)})^n\right] + \left[(1-(1-p_i)^n)(1-u_i)\right]\right). \tag{5}$$

[1] These are the grades used by the bank that provided the data.

The solution, for a generic risk grade $z: z \in R$, is:

$$s_z^{(n)} = \left[\frac{(1+r)^n - \sum_{i \in R} (1-(1-p_i)^n)(1-u_i) a_{zi}^{(n)}}{\sum_{i \in R} (1-p_i)^n a_{zi}^{(n)}} \right]^{\frac{1}{n}} - (1+r). \quad (6)$$

This is the most general closed-form solution that we present in this study. As the reader may notice, equations (2) and (4) are particular cases of equation (6). In fact, if we have $n = 1$, we get equation (4), and if we additionally have $a_{zz} = 1$, we get equation (2).

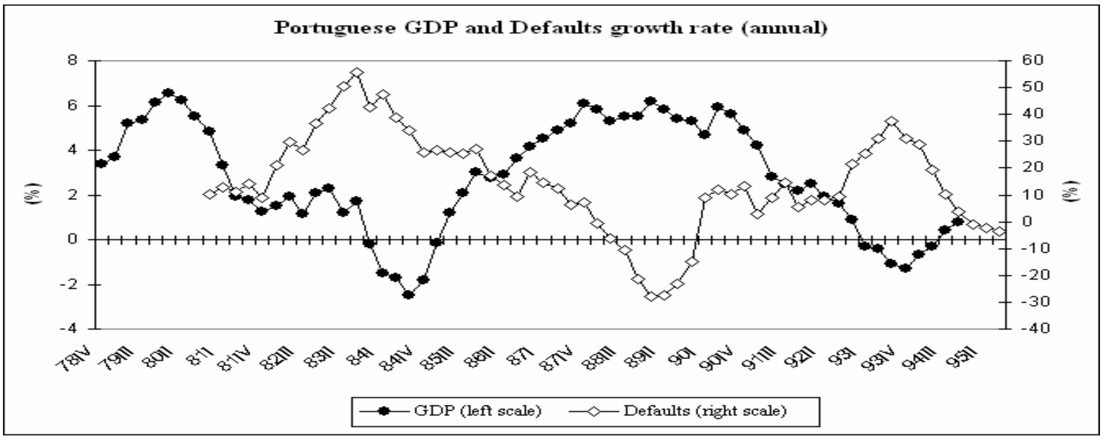
3.1. Some hypotheses. The model assumes that the defaults occur at the beginning of the loan term. This is a conservative hypothesis that causes an increase in the spread; this deviation from the "optimum spread" is somehow compensated by the hypothesis that states that the recoveries, $p_i C(1-u_i)$, occur at the end of the loan term, when in fact this happens some time ahead.

Another hypothesis is that the amounts the bank recovers do not pay interests. This hypothesis is justified by the fact that the firms usually default due to economic complications, and therefore, it is very hard for the bank to recover the interests in many cases.

Finally, the model does not include the expenses related to the recovery attempts (such as the judicial expenses or the costs of having staff assigned to the recovery tasks). We could deal with this situation by considering these expenses as a portion of the principal C and subtract them from the recovered amount at the end of the loan term.

4. Data and model testing

As we mentioned before, there is empirical evidence in the literature that the credit risk variables are correlated with the economic cycle. The following chart shows the inverse relation existing between the Portuguese GDP and the defaults in Portuguese credit institutions.



Source: Instituto Nacional de Estatistica; Banco de Portugal.

Fig. 1. Economic cycle and defaults in Portugal

We assume the existence of two phases in the economic cycle: a phase of increases in economic growth (upturn) and a phase of decreases in economic growth (downturn). The model is tested for the upturn phase.

The data was provided by a bank and refers to non financial firms with organized financial statements. It comprises information about the monthly defaults, the clients' monthly obligations, per risk grade, the monthly recoveries and the clients' annual ratings.

The data goes from 1991 until September 1995, and includes the two phases of the economic cycle, as depicted in chart 1: the downturn phase from 1991 to 1993, and the upturn phase from 1994 to September 1995.

4.1. The insolvency curve. The empirical insolvency curve allows the estimation of the losses given default per risk grade (u_i). It makes possible to observe, on a monthly basis, the behavior of the defaults. If for a certain month the bank recovers any defaulted money, the curve assumes a decrescent behavior; if no money is recovered, the curve becomes constant. Moreover, the data permitted to observe that the second derivative of u_i with respect to time is negative, which means that in the course of time it becomes more difficult to recover the defaults. We consider as an adequate estimate for the losses given default, the value corresponding to the asymptote to the insovency curve for all clients of the bank[1].

[1] This includes the clients that are non financial firms, even if they have no attributed rating. In most cases these clients present defaults and losses given default higher than the clients with attributed rating. Notice that from the 12th month the curve assumes an atypic behavior that could be explained by the usage of provisions. This fact is a difficulty for us since the files provided by the bank do not distinguish between the effective recoveries and the usage of provisions. If we exclude the last two points of the curve, the asymptote corresponds approximately to 60%.

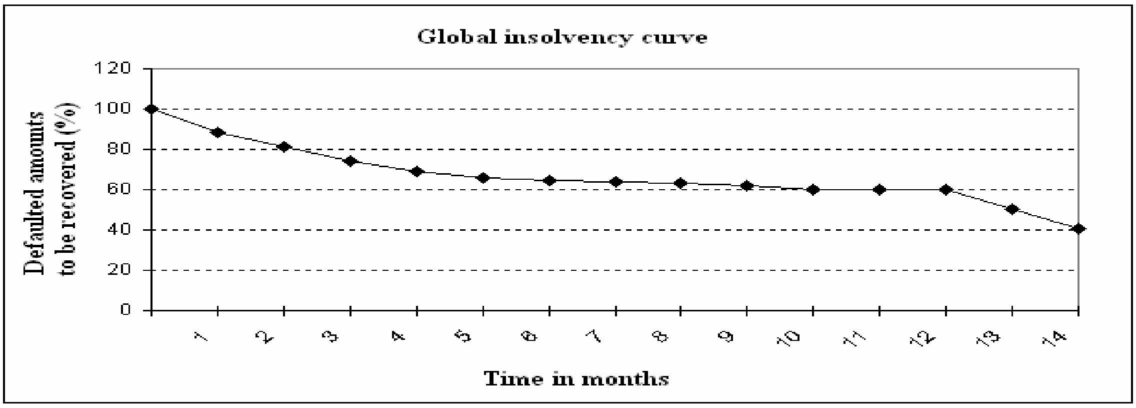


Fig. 2. Insolvency curve for the non-financial firms

As an example, we explain how we built the insolvency curve for the risk grade 5 clients for an economic upturn phase.

Table 1. Monthly evolution of defaults for the risk grade 5 clients (%)

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Jul-94	100	99	84	68	59	48	47	47	46	46	42	38	38	37	37
Aug-94	100	88	84	84	71	70	70	69	69	63	63	63	63	63	
Sep-94	100	86	82	77	75	75	73	73	72	55	55	55	54		
Oct-94	100	92	86	86	83	81	81	80	78	77	77	77			
Now-94	100	65	64	56	55	55	55	50	50	50	50				
Dec-94	100	94	93	89	86	86	83	83	82	82					
Jan-95	100	91	87	86	78	75	75	75	75						
Feb-95	100	84	82	81	65	65	65	64							
Mar-95	100	89	88	82	81	81	81								
Apr-95	100	76	56	56	56	53									
May-95	100	84	80	80	77										
Jun-95	100	96	95	92											
Jul-95	100	100	86												
Aug-95	100	93													
Sep-95	100														

If we trace the defaults occurred in July 1994, for instance, we notice that after 10 months it remains to be recovered 42% of the defaulted amount.

The calculation of the average value for each column allows us to have an idea of the curve behavior.

Table 2. Monthly evolution of the defaults for an economic upturn phase (%, risk grade 5)

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
%	100	88	82	78	72	69	70	68	67	62	57	58	52	50	37

The estimates that refer to the last months have lower confidence levels because of the scarce number of observations. The irregularities that happen from month 5 to month 6, and from month 10 to month 11, that cause an increasing behavior to the curve, could probably be eliminated if a larger number of observations was used.

If we force the curve to be not crescent (replacing the value of month 6 by the value of month 5, and the value of month 11 by the value of month 10, and doing the same thing for the curves related to the other risk grades), and if we admit that the value of month 10 is a good indicator for the asymptote to the curve (as we assumed in the case of Figure 2), then the estimates for the probabilities of losses given default per risk grade, for an upturn phase are the following:

Table 3. Estimates for the probabilities of losses given default per risk grade for an economic upturn phase

u_1	u_{2+}	u_2	u_{3+}	u_3	u_4	u_5
0%	52%	53%	41%	45%	48%	57%

Except for the grades 2 and 2+, the percentage of losses given default becomes lower when the risk becomes lower.

4.2. The default probabilities. In this section, we estimate the p_i, the probability of a grade i client defaults.

The information used corresponds to 15 months of observations. We compare the monthly defaults with the monthly obligations per risk grade and construct a ratio that represents the proportion of monthly defaults:

Table 4. Percentage of monthly defaults per risk grade for an economic upturn phase

%	Jul-94	Aug	Sep	Oct	Now	Dec	Jan-95	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep-95
Rating 5	0.57	0.16	1.24	1.59	1.38	0.73	0.76	0.71	1.21	0.27	0.42	0.43	1.47	0.30	0.33
Rating 4	0.85	0.10	0.45	0.35	1.01	0.22	0.77	0.98	0.73	0.82	0.91	0.49	0.500	0.22	1.04
Rating 3	0.27	0.16	0.31	0.33	0.32	0.68	0.26	0.50	0.67	0.22	0.47	0.50	0.14	0.51	0.26
Rating 3+	0.07	0.03	0.12	0.28	0.11	0.19	0.20	0.34	0.36	0.13	0.13	0.07	0.37	0.03	0.08
Rating 2	0.03	0.07	0.36	0.09	0.05	0.07	0.05	0.08	0.42	0.11	0.11	0.05	0.03	0.01	0.21
Rating 2+	0.00	0.00	0.17	0.03	0.32	0.02	0.04	0.00	0.00	0.00	0.01	0.15	0.01	0.01	0.00
Rating 1	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.01	0.11	0.00	0.00	0.00	0.00

In order to annualize the percentage of defaults we calculate the sum in rows and obtain the proportion correspondent to 12 months. The result is an estimate for the default probabilities for an economic upturn phase:

Table 5. Estimates for the default probabilities per risk grade for an economic upturn phase

p_1	p_{2+}	p_2	p_{3+}	p_3	p_4	p_5
0.10%	0.62%	1.36%	2.00%	4.48%	7.55%	9.26%

It can be seen that the probability of default is higher for the highest risk grades, which shows the efficacy of the bank scoring model.

4.3. Risk grade transition matrices. This subsection deals with the estimation of a_{ij} : the annual probability of transition from risk grade i to risk grade j .

We build two transition matrices, one referring to the economic downturn phase, based on the ratings calculated by the bank using the clients' financial statements of 1991[1], 1992 and 1993; and the other referring to the economic upturn phase, based on the ratings calculated using the financial statements of 1993 and 1994.

Table 6. Portuguese GDP growth rate

1989	1990	1991	1992	1993	1994
5.7%	4.1%	2.3%	1.7%	-1.2%	1.0%

Source: Banco de Portugal.

Using the financial statements of 1991 and 1992, the bank attributed ratings for the years 1992 and 1993, respectively. Since from 1991 to 1992 and from 1992 to 1993, the economy experienced a downturn phase, shown in Table 6 and Figure 1, then the risk grade transition matrices, from 1992 to 1993 and from 1993 to 1994, are therefore, two matrices referring to the downturn phase.

[1] The files provided by the bank include information on the ratings for the years 1992 to 1995. Notice that the rating attributed for a certain year is calculated with financial information of the previous year.

Table 7. Risk grade transition matrices for the years 1991, 1992 and 1993 of the economic cycle

%		Rating 1993						
		1	2+	2	3+	3	4	5
Rating 1992	1	42	31	9	5	9	1	3
	2+	7	34	33	8	10	4	4
	2	2	14	41	23	13	3	4
	3+	0	2	21	36	24	9	8
	3	1	2	8	20	42	14	13
	4	0	0	4	12	26	31	27
	5	0	1	3	7	14	19	56

%		Rating 1994						
		1	2+	2	3+	3	4	5
Rating 1993	1	46	26	15	5	7	1	0
	2+	9	36	28	9	12	3	3
	2	2	13	39	22	14	5	5
	3+	0	3	18	33	29	8	9
	3	0	3	6	15	46	14	16
	4	0	1	3	10	23	34	29
	5	0	1	3	6	16	14	60

Notice that the elements of the two matrices are relatively close. We perform a simple average to obtain one matrix representative of the economic downturn phase.

Table 8. Transition matrix representative of the economic downturn phase

%		Rating year 1						
		1	2+	2	3+	3	4	5
Rating year 0	1	44	29	12	5	7	1	2
	2+	8	35	31	8	11	3	4
	2	2	14	40	22	14	4	4
	3+	0	3	20	34	27	8	8
	3	1	2	7	17	45	14	14
	4	0	1	3	11	25	32	28
	5	0	1	3	6	15	17	58

The element $a_{12+} = 29\%$, for instance, means that out of the firms with risk grade 1 in a certain year, 29% changes to grade 2+ in the next year. The main diagonal (shaded) shows the percentage of firms that keep the same risk grade after 1 year. As the reader may notice, the elements of the diagonals on the right of the main diagonal are almost always higher than the elements of the diagonals on the left of the main diagonal, which means that in downturn phases there are more firms worsening their risk grades instead of improving.

Based on the clients' financial statements of 1994, the bank attributed ratings for 1995. From 1993 to 1994 the economy started to recover, the following matrix corresponds to the economic upturn phase.

Table 9. Risk grade transition matrix for an economic upturn phase

%		Rating 1995						
		1	2+	2	3+	3	4	5
Rating 1994	1	61	24	5	4	4	1	1
	2+	20	46	23	5	4	1	1
	2	3	23	46	16	8	2	2
	3+	2	8	29	34	17	5	5
	3	1	4	12	23	42	10	8
	4	1	2	8	16	25	31	17
	5	0	2	7	10	18	16	47

In this turn the tendency is reverted: the elements of the diagonal on the right of the main diagonal are in general lower than the elements of the diagonals on the left of the main diagonal. Moreover, if we compare the main diagonals of the two previous matrices, we verify that in economic upturns, the maintenance in low-risk grades is higher than the maintenance in the same risk grades in downturns. On the other hand, the maintenance in high-risk grades in economic upturns is lower than the maintenance in the same risk grades in economic downturns.

These results are in line with the works of Bangia, Diebold and Schuermann (2000) and Nickell, Perraudin and Varotto (2000) who find similar evidence of macroeconomic impact on rating transitions.

In order to test the model, we use as estimates for the transition probabilities the matrix of Table 9, since it represents an economic upturn phase and the model is being tested for this phase.

Table 10. Risk grade transition probabilities for an economic upturn phase (%)

$a_{11} = 61$	$a_{12+} = 24$	$a_{12} = 5$	$a_{13+} = 4$	$a_{13} = 4$	$a_{14} = 1$	$a_{15} = 1$
$a_{2+1} = 20$	$a_{2+2+} = 46$	$a_{2+2} = 23$	$a_{2+3+} = 5$	$a_{2+3} = 4$	$a_{2+4} = 1$	$a_{2+5} = 1$
$a_{21} = 3$	$a_{22+} = 23$	$a_{22} = 46$	$a_{23+} = 16$	$a_{23} = 8$	$a_{24} = 2$	$a_{25} = 2$
$a_{3+1} = 2$	$a_{3+2+} = 8$	$a_{3+2} = 29$	$a_{3+3+} = 34$	$a_{3+3} = 17$	$a_{3+4} = 5$	$a_{3+5} = 5$
$a_{31} = 1$	$a_{32+} = 4$	$a_{32} = 12$	$a_{33+} = 23$	$a_{33} = 42$	$a_{34} = 10$	$a_{35} = 8$
$a_{41} = 1$	$a_{42+} = 2$	$a_{42} = 8$	$a_{43+} = 16$	$a_{43} = 25$	$a_{44} = 31$	$a_{45} = 17$
$a_{51} = 0$	$a_{52+} = 2$	$a_{52} = 7$	$a_{53+} = 10$	$a_{53} = 18$	$a_{54} = 16$	$a_{55} = 47$

4.4. The empirical short-term default spread. In the previous sub-sections we followed with estimates for the variables that are relevant to perform the calculation of the default spread, for an economic upturn phase. Using the bank's prime rate as of September 1995 ($r = 11\%$), and using equation (4), we are able to determine the spread per risk grade, which is shown in the second column of the following table. The third column shows the spreads that are effectively used by the bank to price the loans.

Table 11. Spread estimate vs. actual spread (for 1 year term)

Rating	(i) Spread given by the model	(ii) Actual spread	(i) – (ii)
1	0.41%	0.00%	0.41%
2+	0.67%	0.50%	0.17%
2	1.13%	1.00%	0.13%
3+	1.61%	1.75%	-0.14%
3	2.41%	2.75%	-0.34%
4	3.31%	4.00%	-0.69%
5	4.5%	7.25%	-2.75%

According to the model, the ultimate rate that the bank should apply to cover the credit risk of a risk grade 5 client, for instance, should be 15.5% ($11\% + 4.5\% = r + s_5$).

The last column of Table 11 shows that the actual spreads are, on the one hand, insufficient to cover the whole credit risk for the low-risk clients (1,2+ and 2) and, on the other hand, excessive for the high-risk clients (3, 4 and 5). This situation is particularly evident for the grade 5 clients, who face an excessive spread of 2.75%.

Moreover, since the model suggests a positive spread of 0.41% for the grade 1 clients, it captures the fact that these clients exhibit a positive probability of default ($p_1 = 0.10\%$, see Table 5). By considering the actual spread as null, it seems that the bank neglected the existence of a positive (although small) default risk for grade 1 clients.

4.5. The long-term distribution of the risk grades seen in a Markov chain. In this section, we analyze the long-term distribution of the risk grades. This distribution can be seen as an indicator of the risk tendency for the bank clients.

When the loan term is greater than 1 year we can use equation (6) to compute its value. If the loan term is large enough to include economic upturns and downturns, then the input parameters should be re-estimated to address the different phases of the economic cycle. For instance, the transition matrix to use in equation (6) should not be the matrix of Table 9, because it only considers one phase of the economic cycle.

A possible matrix to use could be the resultant of the average between the first matrix of Table 7 and the matrix presented in Table 8, since these two matrices correspond to a downturn and an upturn phase, respectively:

Table 12. Transition matrix for a complete economic cycle

%		Rating year 1						
		1	2+	2	3+	3	4	5
Rating year 0	1	49	27	10	5	6	1	2
	2+	12	38	29	7	8	3	3
	2	2	17	42	20	12	3	4
	3+	1	4	23	34	24	7	7
	3	1	3	9	19	44	13	11
	4	0	1	5	13	25	32	24
	5	0	1	4	8	16	17	54

The element $a_{2+,1} = 12\%$, for instance, means that out of the firms with risk grade 2+ in a certain year, 12% on average moves to grade 1 in the next year.

To determine the limit distribution according to a Markov chain we must first establish some hypotheses. First, we assume that $X(t)$ is a random variable representative of a client's risk grade at discrete time t. The sequence of various variables of this type is a discrete stochastic process: $\{X(t); t = 0,1,2,...\}$.

A Markov chain is a stochastic process with short memory, i.e. the state assumed by the process in a certain period only depends on the state the process assumes in the previous period (the state space is finite and given by R = {1,2+,2,3+,3,4,5}). For instance, the probability of a firm having risk grade 5 in the next year only depends on the risk grade that the firm exhibits in the current year. The knowledge about the grades assumed by the firm two or more years ago is negligible.

We also suppose that the Markov chain is time-homogeneous, i.e. the transition probabilities are stationary, and therefore, only depend on the amplitude of the time interval, not depending on the specific period of time where the process is. For instance, the probability of moving from grade 5 at time 0, to grade 4 at time 1 is equal to the probability of moving from grade 5 at time 8, to grade 4 at time 9. In both cases the amplitude of the time interval is the same (1-0 = 9-8).

Under these conditions a Markov chain is completely defined if we know, on the one hand, the one-step transition probability matrix and, on the other hand, the specification of the probability distribution on the state of the process at time 0 (Taylor and Karlin, 1984).

As transition probability matrix we choose the matrix of Table 12, because it represents a complete economic cycle. To determine the probability distribution we look at the risk grade data available for the downturns and upturns, and calculate the average percentage of clients that lie in each risk category. This procedure permits to obtain:

$$\begin{cases} \mathrm{P}[\mathrm{X}(0)=1]=3.64\% \\ \mathrm{P}[\mathrm{X}(0)=2+]=7.98\% \\ \mathrm{P}[\mathrm{X}(0)=2]=15.01\% \\ \mathrm{P}[\mathrm{X}(0)=3+]=16.28\% \\ \mathrm{P}[\mathrm{X}(0)=3]=23.47\% \\ \mathrm{P}[\mathrm{X}(0)=4]=13.40\% \\ \mathrm{P}[\mathrm{X}(0)=5]=20.22\% \end{cases}$$

$P[X(0)=1]=3.64\%$ means that a firm, randomly chosen, has risk grade 1 at time 0 with 3.64% of probability, regardless the risk grade that the firm moves to in the next year.

Since the n-step ($n > 1$) transition probabilities ultimately depend on the one-step transition probability matrix (Taylor and Karlin, 1984), then if we want to find the probability of a firm having risk grade j, after k years, given that at time 0 the firm has grade i, all we need to do is to raise the one-step matrix of Table 12 to the power of k. Using $k = 3$, for instance, we obtain the following results:

Table 13. Transition probabilities for 3 years

%		Rating year 3						
		1	2+	2	3+	3	4	5
Rating year 0	1	17	21	22	13	14	6	7
	2+	9	18	24	16	17	7	9
	2	5	13	24	19	20	8	11
	3+	3	9	20	19	24	11	14
	3	2	7	16	19	26	13	17
	4	1	5	13	17	26	15	23
	5	1	4	12	16	24	16	27

The first element of the matrix shows that only 17% of the firms with grade 1 at a certain year keep their risk level after 3 years. To calculate the default spread for a 3 year term we could use the elements of this matrix in equation (6).

This Markov chain is regular, i.e. has a limit distribution. If we represent the limit distribution with π_j, $j \in R$, then π_1, for instance, gives the probability of a firm reaching the risk grade 1 in the long term, regardless of the risk grade it has now. The limit distribution is obtained by solving the following system with respect to each π_j (Taylor and Karlin, 1984):

$$\pi_j = \sum_{i \in R} \pi_i a_{ij}, j \in R,$$

subject to $\sum_{i \in R} \pi_i = 1$.

The solution is:

$$\begin{cases} \pi_1 = 3.7\% \\ \pi_{2+} = 9.3\% \\ \pi_2 = 18\% \\ \pi_{3+} = 17.6\% \\ \pi_3 = 23.1\% \\ \pi_4 = 11.6\% \\ \pi_5 = 16.7\% \end{cases}$$

In the long term, the risk grade 3 will be the grade of the majority of the bank clients. On the other hand, the model suggests that the most part of the clients will be high-risky. In fact, if we sum the proportion of clients with risk grade 3, 4 or 5, we obtain a value of 51.4%, higher than the value correspondent to the percentage of clients with grades 1,2+ or 2, which is only 31% of the total. This suggests that if the bank maintains its actual client prospecting policy as it is, then the most part of the clients will probably be high-risky in the long term.

Conclusions

In this study we developed a time discrete model to measure the default risk for bank loans.

The model was built in three stages. In the first, we tackled the short-term default spread, ignoring the possibility of the bank clients moving between different risk grades; in the second stage, we considered that possibility and, in the third, we generalized the model for any term of the loan.

The main input variables that we considered – the default probabilities, the loss given default probabilities and the probabilities of risk grade transitions – were estimated according to the phases of the economic cycle. Due to the existing relation between the credit risk variables and the economic cycle, evidenced by Fama (1986), Wilson (1997), Carey (1998), Altman and Brady (2001) or Schuermann (2004), among others, we considered very important to subject the credit risk model to the phases of the economy, as referred by Allen and Saunders (2003).

The model was tested for an economic upturn phase, using real data provided by a bank. The results suggest that the spreads used by the bank to price the loans are, on the one hand, insufficient to cover the whole credit risk of the low-risk clients (grades 1, 2+ or 2) and, on the other hand, excessive for high-risk clients (grades 3, 4 or 5), especially in the case of grade 5 clients, who face an actual spread of about 61% higher than the spread given by the model.

On the other hand, since the model indicates a spread of 0.41% for the grade 1 clients, it captures the fact that these clients exhibit a positive probability of default. By considering the actual spread for grade 1 clients as null, it seems that the bank neglected the existence of a positive (although small) default risk for this group of clients.

The risk grade transition matrices built for the different phases of the economic cycle show that during economic downturn phases there are more clients worsening their risk grades instead of improving. This situation is reverted during economic upturns, where the most part of the clients improve their risk grades. These results are in line with the works of Bangia, Diebold and Schuermann (2000) and Nickell, Perraudin and Varotto (2000), who find similar evidence of macroeconomic impact on rating transitions.

Moreover, we conclude that for economic upturns, the percentage of maintenance in low-risk grades is higher than the percentage of maintenance in the same risk grades during downturns. On the other hand, the maintenance in high-risk grades for economic upturns is lower than the maintenance in the same risk grades for economic downturns.

Finally, we built the risk grades limit distribution according to a time-homogeneous Markov chain. This distribution can be seen as an indicator of the risk tendency for the bank clients, and suggests that if the bank maintains its actual client prospecting policy as it is, then the most part of the clients will probably be high-risky in the long term.

References

1. Allen, L. and Saunders, A. (2003). A Survey of Cyclical Effects in Credit Risk Measurement Models. *BIS Working Papers*, No. 126.
2. Altman, E. and Brady, B. (2001). Explaining Aggregate Recovery Rates on Corporate Bond Defaults. *Salomon Center working paper*, November.
3. Altman, E., Resti, A. and Sironi, A. (2004). Default Recovery Rates in Credit Risk Modeling: A Review of the Literature and Empirical Evidence. *Economic Notes,* 33 (2), 183-208.
4. Bangia, A., Diebold, F. and Schuermann (2000). Ratings Migration and the Business Cycle, With Applications to Credit Portfolio Stress Testing. *Wharton Financial Institutions Center working paper*, No. 26 (April).
5. Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economics*. May, 637-659.

6. Carey, M. (1998). Credit Risk in Private Debt Portfolios. *Journal of Finance*, 53 (4), 1363-1387.
7. Duffie, D. and Singleton, K. (1999). Modeling the Term Structures of Defaultable Bonds. *Review of Financial Studies*, 12, 687-720.
8. Jarrow, R., Lando, D. and Turnbull, S. (1997). A Markov Model for the Term Structure of Credit Risk Spreads. *Review of Financial Studies*, 10, 481-523.
9. Jarrow, R. and Turnbull, S. (1995). Pricing Derivatives on Financial Securities Subject to Credit Risk. *Journal of Finance*, 50, 53-86.
10. Kim, I., Ramaswamy, K. and Sundaresan, S. (1993). Does Default Risk in Coupons Affect the Valuation of Corporate Bonds?:A Contingent Claims Model. *Financial Management*, 22 (3), 117-131.
11. Lando, D. (1998). On Cox Processes and Credit Risky Securities. *Review of Derivatives Research*, 2, 99-120.
12. Litterman, R. and Iben, T. (1991). Corporate Bond Valuation and the Term Structure of Credit Spreads. *Financial Analysts Journal*, Spring, 52-64.
13. Longstaff, F. and Schwartz, E. (1995). A Simple Approach to Valuing Risky Fixed and Floating Rate Debt. *Journal of Finance*, 50, 789-819.
14. Merton, R. (1974). On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance*, 2, 449-471.
15. Nickell, P., Perraudin, W. and Varotto, S. (2000). Stability of Rating Transitions. *Journal of Banking and Finance*, 24 (1/2), 203-228.
16. Schuermann, T. (2004). What Do We Know About Loss Given Default?. In Shimko (Eds.), *Credit Risk Models and Management*, 2nd edition, Risk Books.
17. Taylor, H. and Karlin, S. (1984). An Introduction to Stochastic Modeling, New York: Revised edition, Academic Press.
18. Volkart, R. and Mettler, A. (2004). Option Pricing Theory, Credit Risk Spreads and Risk-Adjusted Pricing: An Integrative Approach. 2004 FMA European Conference, Zurich, Switzerland.