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Generalization of the Weierstrass $\wp$, ζ and σ functions

Побудовано аналоги $\wp$, ζ і σ функцій Вейєрштрасса для подвійно p-еліптичних функцій, тобто мероморфних в $\mathbb{C}$ функцій g, що задовольняють умову $g(u+m\omega_1+n\omega_2)=p^{m+n}g(u)$ для деяких ω_1,ω_2, деякого p і для всіх $m,n\in\mathbb{Z}$.

For double p-elliptic functions, i. e. meromorphic in $\mathbb{C}$ functions g satisfying the condition $g(u+m\omega_1+n\omega_2)=p^{m+n}g(u)$ for some ω_1,ω_2 and p and for all $m,n\in\mathbb{Z}$, analogues of $\wp$, ζ and σ Weierstrass functions are constructed.

Denote $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$. Let ω_1,ω_2 be complex numbers such that $Im\frac{\omega_2}{\omega_1}>0$. A meromorphic in $\mathbb{C}$ function g is called **elliptic** [1] if for every $u\in\mathbb{C}$

$$g(u+\omega_1)=g(u),\ \ g(u+\omega_2)=g(u).$$

Elliptic functions were first discovered by Niels Henrik Abel as inverse functions of elliptic integrals, and their theory was improved by Carl Gustav Jacobi. A more complete investigation of elliptic functions was later undertaken by Karl Theodor Wilhelm Weierstrass, who found a simple elliptic function ($\wp$) in terms of which all the others could be expressed.

Definition 1. *Let ω_1,ω_2 be complex numbers such that $Im\frac{\omega_2}{\omega_1}>0$. A meromorphic in $\mathbb{C}$ function g is called* **double p-elliptic**, *if there exists $p\in\mathbb{C}^*$, such that for every $u\in\mathbb{C}$*

$$g(u+\omega_1)=pg(u),\ \ g(u+\omega_2)=pg(u).$$

Denote the class of double p-elliptic functions by $\mathcal{DE}_p$.

Let $\omega=m\omega_1+n\omega_2,\ m,n\in\mathbb{Z}$. If $f\in\mathcal{DE}_p$ then Definition 1 implies

$$g(u+\omega)=p^{m+n}g(u).$$

Remark. *If $p=1$ in Definition 1, we obtain classic elliptic function.*

The classic Weierstrass $\wp$-function has the form ([1], [2])

$$\wp(u)=\frac{1}{u^2}+\sum_{\omega\neq0}\left(\frac{1}{(u-\omega)^2}-\frac{1}{\omega^2}\right). \qquad (1)$$

The Weierstrass $\wp$-function is elliptic [1] of periods ω_1,ω_2. Representations of classic Weierstrass ζ and σ functions are also well known [1], [2]:

$$\zeta(u)=\frac{1}{u}+\sum_{\omega\neq0}\left(\frac{1}{u-\omega}+\frac{1}{\omega}+\frac{u}{\omega^2}\right), \qquad (2)$$

$$\sigma(u)=u\prod_{\omega\neq0}\left(1-\frac{u}{\omega}\right)e^{\frac{u}{\omega}+\frac{u^2}{2\omega^2}}. \qquad (3)$$

It should be noted, next equalities are valid

$$\wp(u)=-\zeta'(u),\ \ \zeta(u)=\frac{\sigma'(u)}{\sigma(u)},$$

$$\wp(u)=-\left(\frac{\sigma'(u)}{\sigma(u)}\right)'.$$

Let us remark that each elliptic function can be represented using (1), (2), (3). So, these functions play an important role for representation of elliptic functions.

The purpose of this article is to construct double p-elliptic function $\widetilde{\wp}_\alpha(u)$, which is an analogue of $\wp(u)$, as well as corresponding analogues of ζ and σ functions.

Let $p=e^{i\alpha}$, $\alpha\neq2\pi l$, $l\in\mathbb{Z}$. Consider the function

$$G_\alpha(u)=\frac{1}{u^2}+\sum_{\omega\neq0}\left(\frac{1}{(u-\omega)^2}-\frac{1}{\omega^2}\right)e^{i(m+n)\alpha}, \qquad (4)$$

where $\omega_1,\omega_2\in\mathbb{C}$, $Im\frac{\omega_2}{\omega_1}>0$, $\omega=m\omega_1+n\omega_2$, $m,n\in\mathbb{Z}$.

Since the double series $\sum\limits_{\omega\neq0}\frac{1}{|\omega|^3}$ is convergent ([1], [2]), then the series in the right hand side of (4) is uniformly convergent on every compact subset of $\mathbb{C}$.

Let us note, that if $\alpha = 2\pi l$, $l \in \mathbb{Z}$, then $G_\alpha(u)$ coincides with $\wp(u)$.

We will show that there exists a unique constant C_α such that $(G_\alpha + C_\alpha) \in \mathcal{DE}_p$, i. e.

$$G_\alpha(u+\omega_j) + C_\alpha = e^{i\alpha}(G_\alpha(u) + C_\alpha),\ j = 1,2.$$

The last property is called multi p-periodicity of ω_j.

Let us consider the derivative of G_α

$$G'_\alpha(u) = -2\sum_\omega \frac{e^{i(m+n)\alpha}}{(u-\omega)^3}.$$

Hence, we have

$$\begin{aligned} G'_\alpha(u+\omega_1) &= \\ &= -2\sum_{m,n\in\mathbb{Z}} \frac{e^{i(m+n)\alpha}}{(u+\omega_1 - m\omega_1 - n\omega_2)^3} = \\ &= -2\sum_{m,n\in\mathbb{Z}} \frac{e^{i(m+n)\alpha}}{(u-(m-1)\omega_1 - n\omega_2)^3} = \\ &= -2e^{i\alpha}\sum_{m,n\in\mathbb{Z}} \frac{e^{i(m-1+n)\alpha}}{(u-(m-1)\omega_1 - n\omega_2)^3} = \\ &= e^{i\alpha}G'_\alpha(u). \end{aligned}$$

Thus, we obtain

$$G'_\alpha(u+\omega_1) - e^{i\alpha}G'_\alpha(u) = 0. \tag{5}$$

Note that a function $(G_\alpha + C)$ for any $C \in \mathbb{C}$ satisfies (5). Put

$$C = C_\alpha = \frac{G_\alpha\left(\frac{\omega_1}{2}\right) - e^{i\alpha}G_\alpha\left(-\frac{\omega_1}{2}\right)}{e^{i\alpha}-1}. \tag{6}$$

We also define here $C_0 = 0$. Then the relation (5) implies

$$G_\alpha(u+\omega_1) + C_\alpha - e^{i\alpha}\big(G_\alpha(u) + C_\alpha\big) = A,$$

where A is a constant. If we set $u = -\frac{\omega_1}{2}$, it is easy to obtain

$$G_\alpha\left(\frac{\omega_1}{2}\right) - e^{i\alpha}G_\alpha\left(-\frac{\omega_1}{2}\right) + (1-e^{i\alpha})C_\alpha = A.$$

Taking into account the choice of C_α by equality (6), we deduce that $A = 0$. Therefore, we have

$$G_\alpha(u+\omega_1) + C_\alpha = e^{i\alpha}\big(G_\alpha(u) + C_\alpha\big), \tag{7}$$

that is we have shown that the function $\big(G_\alpha + C_\alpha\big)$ is multi p-periodic of ω_1.

It remains to prove the uniqueness of C_α. Suppose that there exists a constant C different from C_α such that a function $\big(G_\alpha + C\big)$ is multi p-periodic of ω_1 too. So we get

$$G_\alpha(u+\omega_1) + C = e^{i\alpha}\big(G_\alpha(u) + C\big).$$

Subtracting this equality from (7), we obtain $C_\alpha - C = e^{i\alpha}\big(C_\alpha - C\big)$. Since $\alpha \neq 2\pi l$, $l \in \mathbb{Z}$, then $C = C_\alpha$.

Similarly, for period ω_2 we have

$$G_\alpha(u+\omega_2) + C_\alpha = e^{i\alpha}\big(G_\alpha(u) + C_\alpha\big) + B, \tag{8}$$

where B is some constant. Let us find B. Using equalities (7) and (8), we obtain

$$\begin{aligned} G_\alpha(u+\omega_1+\omega_2) + C_\alpha &= \\ = e^{i\alpha}(G_\alpha(u+\omega_1) + C_\alpha) + B &= \\ = e^{i2\alpha}(G_\alpha(u) + C_\alpha) + B&, \end{aligned}$$

and

$$\begin{aligned} G_\alpha(u+\omega_1+\omega_2) + C_\alpha &= \\ = e^{i\alpha}(G_\alpha(u+\omega_2) + C_\alpha) &= \\ = e^{i2\alpha}(G_\alpha(u) + C_\alpha) + Be^{i\alpha}&. \end{aligned}$$

Equating the right hand sides of these relations, we get

$$B = Be^{i\alpha}.$$

Since $\alpha \neq 2\pi l$, $l \in \mathbb{Z}$, then the previous equality implies that $B = 0$. Therefore,

$$G_\alpha(u+\omega_2) + C_\alpha = e^{i\alpha}\big(G_\alpha(u) + C_\alpha\big).$$

Thus, function $G_\alpha + C_\alpha$ is multi p-periodic of ω_j, $j = 1,2$.

Hence, we can write the obtained results as the following theorem.

Theorem 1. *A function of the form*

$$\widetilde{\wp}_\alpha(u) = G_\alpha(u) + C_\alpha,$$

where

$$C_\alpha = \frac{G_\alpha\left(\frac{\omega_1}{2}\right) - e^{i\alpha}G_\alpha\left(-\frac{\omega_1}{2}\right)}{e^{i\alpha}-1},$$

belongs to $\mathcal{DE}_p$ *with* $p = e^{i\alpha}$.

Remark. *It is easy to see that C_α can be also expressed in the form*

$$C_\alpha = \frac{G_\alpha\left(\frac{\omega_2}{2}\right) - e^{i\alpha}G_\alpha\left(-\frac{\omega_2}{2}\right)}{e^{i\alpha}-1}.$$

Now consider the function

$$\widetilde{\zeta}_\alpha(u) = \frac{1}{u} + \sum_{k\in\mathbb{Z}} e^{ik\alpha} \sum_{m+n=k} \left(\frac{1}{u-\omega} + \frac{1}{\omega} + \frac{u}{\omega^2}\right),$$

where $\omega_1, \omega_2 \in \mathbb{C}$, $Im\frac{\omega_2}{\omega_1} > 0$, $\omega = m\omega_1 + n\omega_2$, $m^2+n^2 \neq 0$, $m,n \in \mathbb{Z}$. The remainders of the series converge uniformly on the compact subsets of $\mathbb{C}$ [2]. Differentiating $\widetilde{\zeta}_\alpha$, we obtain

$$G_\alpha(u) = -\widetilde{\zeta}'_\alpha(u).$$

Hence,

$$\widetilde{\wp}_\alpha(u) = G_\alpha(u) + C_\alpha = C_\alpha - \widetilde{\zeta}'_\alpha(u).$$

For $k \in \mathbb{Z}\backslash\{0\}$ denote

$$\widetilde{\chi}_k(u) = \sum_{m+n=k} \left(\frac{1}{u-\omega} + \frac{1}{\omega} + \frac{u}{\omega^2}\right).$$

Also for $k=0$ and $m^2+n^2\neq 0$

$$\widetilde{\chi}_0(u) = \frac{1}{u} + \sum_{m+n=0} \left(\frac{1}{u-\omega} + \frac{1}{\omega} + \frac{u}{\omega^2}\right).$$

Then $\widetilde{\zeta}_\alpha$ can be rewritten as follows

$$\widetilde{\zeta}_\alpha(u) = \sum_{k\in\mathbb{Z}} e^{ik\alpha}\widetilde{\chi}_k(u). \qquad (9)$$

Denote the complex plane $\mathbb{C}$ with radial slits from ω to ∞ by A^*. Integrating $\left(\widetilde{\chi}_0(t) - \frac{1}{t}\right)$ and $\widetilde{\chi}_k(t)$ along a path in A^* which connects points 0 and u, we obtain

$$\int_0^u \left(\widetilde{\chi}_0(t) - \frac{1}{t}\right)dt =$$

$$= \sum_{m+n=0}\left(\log\left(1-\frac{u}{\omega}\right) + \frac{u}{\omega} + \frac{u^2}{2\omega^2}\right), \qquad (10)$$

where $m^2+n^2\neq 0$ and

$$\int_0^u \widetilde{\chi}_k(t)dt = \sum_{m+n=k}\left(\log\left(1-\frac{u}{\omega}\right) + \frac{u}{\omega} + \frac{u^2}{2\omega^2}\right). \qquad (11)$$

Let us consider entire functions

$$\widetilde{\sigma}_0(u) = u\prod_{m+n=0}\left(1-\frac{u}{\omega}\right)e^{\frac{u}{\omega}+\frac{u^2}{2\omega^2}}, \quad m^2+n^2\neq 0,$$

$$\widetilde{\sigma}_k(u) = \prod_{m+n=k}\left(1-\frac{u}{\omega}\right)e^{\frac{u}{\omega}+\frac{u^2}{2\omega^2}}, \quad k\in\mathbb{Z}\backslash\{0\}.$$

Using these functions, we can rewrite (10) and (11) in the form

$$\int_0^u \left(\widetilde{\chi}_0(t) - \frac{1}{t}\right)dt = \log\frac{\widetilde{\sigma}_0(u)}{u},$$

$$\int_0^u \widetilde{\chi}_k(t)dt = \log\widetilde{\sigma}_k(u).$$

If we differentiate these relations, we obtain

$$\widetilde{\chi}_0(u) = \frac{\widetilde{\sigma}'_0(u)}{\widetilde{\sigma}_0(u)}, \quad \widetilde{\chi}_k(u) = \frac{\widetilde{\sigma}'_k(u)}{\widetilde{\sigma}_k(u)}.$$

Taking into account such representations of $\widetilde{\chi}_k(u)$, $k\in\mathbb{Z}$, we can rewrite (9) as follows

$$\widetilde{\zeta}_\alpha(u) = \sum_{k\in\mathbb{Z}} e^{ik\alpha}\frac{\widetilde{\sigma}'_k(u)}{\widetilde{\sigma}_k(u)}.$$

Hence, $\widetilde{\wp}_\alpha$ can be rewritten in the next form

$$\widetilde{\wp}_\alpha(u) = C_\alpha + \sum_{k\in\mathbb{Z}} e^{ik\alpha}\frac{\widetilde{\sigma}'^2_k(u) - \widetilde{\sigma}''_k(u)\widetilde{\sigma}_k(u)}{\widetilde{\sigma}^2_k(u)}.$$

Remark. *If we consider a product $\prod_{k\in\mathbb{Z}}\widetilde{\sigma}_k(u)$, we obtain the classic Weierstrass σ-function. If $\alpha = 2\pi l, l\in\mathbb{Z}$, then $\widetilde{\zeta}_0$ is the classic Weierstrass ζ-function.*

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